Topological model for h”-vectors of simplicial manifolds

Anton Ayzenberg

In memory of Professor Samuel Gitler Hammer.

Abstract. Given a simplicial poset S whose geometrical realization is a closed orientable homology manifold, Novik and Swartz introduced a Poincare duality algebra $(\mathcal{R}[S]/(l.s.o.p.))/I_{NS}$, which is a quotient of the face ring of the poset S. The ranks of graded components of this algebra are now called h''-numbers of S and can be computed from face-numbers and Betti numbers of S. We introduce a topological model for this Poincare duality algebra. Given an $(n-1)$-dimensional simplicial homology manifold S we construct a $2n$-dimensional homology manifold with boundary $\widehat{X}$ carrying the action of a compact n-torus. The Poincare–Lefschetz duality on $\widehat{X}$ is used to reconstruct the algebra $(\mathcal{R}[S]/(l.s.o.p.))/I_{NS}$.

1. Introduction

A finite poset S is called simplicial if it contains a unique minimal element (denoted $\varnothing$) and for each element $I \in S$ the subset $S_{\leqslant I} = \{J \in S \mid J \leqslant I\}$ is isomorphic to a boolean lattice (that is the poset of faces of a simplex). The rank of the lattice is called the rank of the element I and is denoted $|I|$. The elements $I \in S$ are called simplices and the number $|I|-1$ is called the dimension of I. Dimension of S is the maximal dimension of its simplices. The elements of rank 1 are called the vertices of S. The number of simplices of fixed dimension j is called the f-number and is denoted f_j. A simplicial poset S is called pure if all simplices maximal by inclusion have the same dimension.

2010 *Mathematics Subject Classification.* Primary 13F55, 57P10; Secondary 57N65, 55R20, 55N45, 13F50, 05E45, 06A07, 16W50, 13H10, 55M05.

Key words and phrases. Poincare duality algebra, manifold with boundary, Buchsbaum simplicial complex, simplicial manifold, face ring, h”-numbers, locally standard torus action, face submanifold, intersection product.

The author was supported by the JSPS postdoctoral fellowship program during his work in Japan and supported in part by Young Russian Mathematics award and by Simons-IUM fellowship in Russia.

Every simplicial poset S can be turned into a topological space $|S|$ by associating a topological simplex of the same dimension to any element of S and gluing these simplices together using order relation in S. One of the basic questions in this area is the following: what can be said about f-numbers of S, if $|S|$ belongs to a given class of topological spaces (say $|S|$ is a homology sphere or $|S|$ is a homology manifold)?

Let S be a pure simplicial poset of dimension $n-1$ and $[m] = \{1,\ldots,m\} = \operatorname{Vert}(S)$ be the set of its vertices. Let $\mathcal{R}$ be a ground ring which is either a field or $\mathbb{Z}$. A map $\lambda\colon [m] \to \mathcal{R}^n$ is called a *(homological) characteristic function* if, for any maximal simplex $I \in S$, the set of vertices of I maps to a basis of $\mathcal{R}^n$. We assume that there is a fixed basis in $\mathcal{R}^n$, and, for any vertex $i \in [m]$, the value $\lambda(i)$ has coordinates $(\lambda_{i,1},\ldots,\lambda_{i,n})$ in this basis.

Let $\mathcal{R}[S]$ be the face ring of S (see **[11, 3]**). By definition, $\mathcal{R}[S]$ is a commutative associative graded algebra over $\mathcal{R}$ generated by formal variables v_I, one for each simplex $I \in S$, with relations

$$v_{I_1}\cdot v_{I_2} = v_{I_1\cap I_2}\cdot \sum_{J\in I_1\vee I_2} v_J, \qquad v_{\hat{0}} = 1.$$

Here $I_1 \vee I_2$ denotes the set of least upper bounds of $I_1, I_2 \in S$, and $I_1 \cap I_2 \in S$ is the intersection of simplices (it is well-defined and unique when $I_1 \vee I_2 \neq \hat{0}$). The summation over an empty set is assumed to be 0. For topological reasons we take the doubled grading on the ring: the generator v_I has degree $2|I|$. The natural map $\mathcal{R}[m] = \mathcal{R}[v_1,\ldots,v_m] \to \mathcal{R}[S]$ defines the structure of $\mathcal{R}[m]$-module on $\mathcal{R}[S]$.

Any characteristic function $\lambda\colon [m] \to \mathcal{R}^n$ determines the set of linear elements:

$$\theta_1 = \sum_{i\in[m]} \lambda_{i,1}v_i, \quad \theta_2 = \sum_{i\in[m]} \lambda_{i,2}v_i, \quad \cdots, \quad \theta_n = \sum_{i\in[m]} \lambda_{i,n}v_i \in \mathcal{R}[S]$$

(these elements have degree 2, but we will use the term "linear" when its meaning is clear from the context). The definition of characteristic function implies that $\theta_1,\ldots,\theta_n$ is a *linear system of parameters* in $\mathcal{R}[S]$ (see e.g.[**4**, Lm.3.5.8]). Moreover, any linear system of parameters arises from some characteristic function in this way. Let Θ be the ideal in $\mathcal{R}[S]$ generated by the elements $\theta_1,\ldots,\theta_n$.

The quotient $\mathcal{R}[S]/\Theta$ is a finite-dimensional vector space. The standard reasoning in commutative algebra implies that, whenever S is Cohen–Macaulay, the dimension of the homogeneous component $(\mathcal{R}[S]/\Theta)_{2k}$ is h_k, the h-number of S **[11]**.

When S is Buchsbaum, the additive structure of $\mathcal{R}[S]/\Theta$ is still independent of the choice of characteristic function but dimensions of homogeneous components have more complicated description. By Schenzel's theorem **[10, 8]**, the dimension of $(\mathcal{R}[S]/\Theta)_{2k}$ is

$$h'_k \stackrel{\text{def}}{=} h_k + \binom{n}{k}\left(\sum_{j=1}^{k-1}(-1)^{k-j-1}\widetilde{\beta}_{j-1}(S)\right),$$

where $\widetilde{\beta}_{j-1}(S) = \operatorname{rk}\widetilde{H}_{j-1}(S;\mathcal{R})$.

Recall that the socle of an $\mathcal{R}[m]$-module $\mathcal{M}$ is an $\mathcal{R}$-subspace

$$\operatorname{Soc}\mathcal{M} \stackrel{\text{def}}{=} \{y \in \mathcal{M} \mid \mathcal{R}[m]^{+} \cdot y = 0\},$$

where $\mathcal{R}[m]^{+}$ is the maximal graded ideal of the ring $\mathcal{R}[m]$. Since the products with polynomials of positive degrees are trivial, the socle is an $\mathcal{R}[m]$-submodule of $\mathcal{M}$. In [**8**] Novik and Swartz proved the existence of certain submodules in $\operatorname{Soc}(\mathcal{R}[S]/\Theta)$ for any Buchsbaum simplicial poset. Namely, in degree $2k < 2n$ there exists a vector subspace

$$(I_{NS})_{2k} \subseteq \operatorname{Soc}(\mathcal{R}[S]/\Theta)_{2k},$$

isomorphic to $\binom{n}{k}\widetilde{H}^{k-1}(S;\mathcal{R})$, the direct sum of $\binom{n}{k}$ copies of $\widetilde{H}^{k-1}(S;\mathcal{R})$. Let I_{NS} denote the direct sum of $(I_{NS})_{2k}$ over all k, where we assume $(I_{NS})_{2n} = 0$. Since I_{NS} lies in the socle, it is an $\mathcal{R}[m]$-submodule. Moreover, I_{NS} is an ideal in $\mathcal{R}[S]/\Theta$ (for simplicial complex this fact easily follows from the surjectivity of the map $\mathcal{R}[m] \to \mathcal{R}[S]$, and for simplicial poset, whose geometrical realization is a homology manifold, this was checked in [**2**, Rem. 8.3]). Therefore we may consider the quotient ring $(\mathcal{R}[S]/\Theta)/I_{NS}$. The dimension of its homogeneous component of degree $2k$ is equal to h''_k where

$$h''_k \stackrel{\text{def}}{=} h'_k - \binom{n}{k}\widetilde{\beta}_{k-1}(S) = h_k + \binom{n}{k}\left(\sum_{j=1}^{k}(-1)^{k-j-1}\widetilde{\beta}_{j-1}(S)\right),$$

for $0 \leqslant k \leqslant n-1$, and $h''_n = h'_n$. In particular, $h''_k \geqslant 0$ for any Buchsbaum simplicial poset.

Now we restrict to the case when the ground ring is either $\mathbb{Z}$ or $\mathbb{Q}$. The class of Cohen–Macaulay simplicial posets contains an important subclass of sphere triangulations. By abuse of terminology we call simplicial poset a homology sphere (resp. manifold) if its geometrical realization is a homology sphere (resp. manifold).

Every homology sphere is Cohen–Macaulay. For homology spheres the ring $\mathcal{R}[S]/\Theta$ is a Poincare duality algebra (this is not surprising in view of Danilov–Jurkiewicz and Davis–Januszkiewicz theorems). In general one can prove this by the following topological argument. Consider the cone over $|S|$ endowed with a dual simple face stratification and consider the identification space $X_S = (\operatorname{Cone}|S| \times T^n)/{\sim}$, similar to the construction of quasitoric manifolds [**6**]. Using the same ideas as in [**6**], one can prove that the cohomology algebra of X_S over $\mathcal{R}$ is isomorphic to $\mathcal{R}[S]/\Theta$ (see e.g. [**7**]). When $\mathcal{R} = \mathbb{Z}$, the space X_S is a homology manifold over integers. In case $\mathcal{R} = \mathbb{Q}$, this space is a homology manifold over $\mathbb{Q}$. In both cases the Poincare duality over the corresponding ring implies that $\mathcal{R}[S]/\Theta$ is a Poincare duality algebra. In particular, this proves Dehn–Sommerville relations for homology spheres: $h_k = h_{n-k}$.

The goal of this paper is to construct a topological model for the algebra $(\mathcal{R}[S]/\Theta)/I_{NS}$, where S is a homology manifold. Any homology manifold S is a Buchsbaum simplicial poset. Thus the ring $(\mathcal{R}[S]/\Theta)/I_{NS}$ is well-defined. Novik and Swartz [**9**] proved that $(\mathcal{R}[S]/\Theta)/I_{NS}$ is a Poincare duality algebra if S is an oriented connected homology manifold. This implies $h''_j = h''_{n-j}$ (generalized Dehn–Sommerville relations) for oriented connected homology manifolds. In the case $\mathcal{R} = \mathbb{Q}$ or $\mathbb{Z}$ we recover this result by exploiting a Poincare–Lefschetz duality on a certain $2n$-dimensional homology manifold with boundary associated with the simplicial poset S.

The idea of our construction is the following. In Section 2 we associate a Poincare duality algebra with any manifold with boundary $(M, \partial M)$, either smooth, topological, or homological. This algebra will be denoted $\mathtt{PD}^*_{(M,\partial M)}$. Given any homology manifold S, instead of taking the cone (as in the case of spheres) we consider the cylinder $\widehat{Q} = |S| \times [0,1]$. This space is a manifold with two boundary components: $\partial_0\widehat{Q}$ and $\partial_1\widehat{Q}$. Consider the identification space $\widehat{X} = (\widehat{Q} \times T^n)/\sim$ where the identification collapses certain torus subgroups over the points of $\partial_0\widehat{Q}$ (similarly to a quasitoric case), and does nothing over $\partial_1\widehat{Q}$. The space $\widehat{X}$ is a homology manifold with boundary; its boundary consists of points over $\partial_1\widehat{Q}$. Then we have

Theorem 1. *The algebra $\mathtt{PD}^*_{(\hat{X},\partial\hat{X})}$ is isomorphic to $(\mathcal{R}[S]/\Theta)/I_{NS}$ if $\mathcal{R} = \mathbb{Q}$ or $\mathbb{Z}$.*

The only place in the argument, where we need the restriction on a ground ring, is the construction of the torus space. The relation $\sim$ collapses certain compact subgroups of the compact torus T^n, and this identification cannot be defined for characteristic functions over general fields. Nevertheless, if the characteristic function λ over $\mathcal{R}$ can be represented as $\lambda' \otimes \mathcal{R}$ for some characteristic function λ' over $\mathbb{Z}$ (or $\mathbb{Q}$), then the statements hold true over a field $\mathcal{R}$ and this particular choice of characteristic function.

2. Poincare duality algebras

Definition 2.1. A finite-dimensional, graded, associative, graded-commutative, connected algebra $A^* = \bigoplus_{k=0}^{d} A^k$ over $\mathcal{R}$ is called *Poincare duality algebra* of formal dimension d, if

(1) $A^d \cong \mathcal{R}$;
(2) The product map $A^k \otimes A^{d-k} \xrightarrow{\times} A^d$ is a non-degenerate pairing for all $k = 0, \dots, d$. Over integers the finite torsion should be mod out.

While the motivating examples of Poincare duality algebras are cohomology of connected orientable closed manifolds, there exist another natural source of duality algebras.

Construction 2.2. Let $(M, \partial M)$ be a compact connected orientable homology manifold with boundary, $\dim M = d$. As a technical requirement we will also assume that M contains a neighborhood of ∂M of the form $\partial M \times [0, \varepsilon]$. Consider the $\mathcal{R}$-module $A^* = \bigoplus_{k=0}^{d} A^k$, where

$$A^k = \begin{cases} H^0(M), \text{ if } k = 0; \\ \text{image of } \imath^*\colon H^k(M, \partial M) \to H^k(M), \text{ if } 0 < k < d; \\ H^d(M, \partial M), \text{ if } k = d. \end{cases}$$

The homomorphism $\imath^*\colon H^k(M, \partial M) \to H^k(M)$ is induced by the inclusion $\imath\colon (M, \varnothing) \hookrightarrow (M, \partial M)$.

There is a well-defined product on A^* induced by the cup-products in cohomology. Indeed, let $a_1 \in A^{k_1}$ and $a_2 \in A^{k_2}$. If either k_1 or k_2 is zero, then there is nothing to define, since A^0 is spanned by the unit of the ring. If $k_1 + k_2 < d$ then $a_1 \cdot a_2$ is just the product of two elements in the ring $H^*(M)$. This product lies in the image of $H^*(M, \partial M)$ since the factors do. If $k_1 + k_2 = d$, then we may consider the elements $b_1, b_2 \in H^*(M, \partial M)$ such that $\imath^*(b_\epsilon) = a_\epsilon$, and take their product in the ring $H^*(M, \partial M)$. This gives an element in $H^d(M, \partial M) = A^d$ which we call the product of a_1 and a_2. It is easily seen that this element does not depend on the choice of representatives b_1, b_2 for the elements a_1, a_2.

The Poincare–Lefschetz duality [**5**, Th.9.2] implies that the pairing between A^k and A^{d-k} is non-degenerate. Thus A^* is a Poincare duality algebra. We denote it by $\mathtt{PD}_{(M,\partial M)}$ and call *the Poincare duality algebra of a manifold with boundary.*

Remark 2.3. By Poincare–Lefchetz duality, instead of cohomology we can work with homology. We have

$$\mathtt{PD}^k_{(M,\partial M)} \cong \begin{cases} H_d(M, \partial M), \text{ if } k = 0; \\ \text{image of } \imath_*\colon H_{d-k}(M) \to H_{d-k}(M, \partial M), \text{ if } 0 < k < d; \\ H_0(M), \text{ if } k = d, \end{cases}$$

and the product is given by the intersection product in homology.

3. Collar model

3.1. Buchsbaum simplicial posets. Let S' be the barycentric subdivision of a simplicial poset S. For each proper simplex $I \in S\backslash\hat{0}$ consider the following subsets of the geometrical realization $|S| \cong |S'|$:

$$G_I = |\{(I_0 < I_1 < \ldots) \in S' \text{ such that } I_0 \geqslant I\}|,$$

$$\partial G_I = |\{(I_0 < I_1 < \ldots) \in S' \text{ such that } I_0 > I\}|.$$

The subset G_I is called the face of $|S|$ dual to I. A simplicial poset S is called *Buchsbaum* (over $\mathcal{R}$) if $H_j(G_I, \partial G_I; \mathcal{R}) = 0$ for any $I \in S\backslash\hat{0}$ and $j \neq \dim G_I$. In particular, any homology manifold is Buchsbaum, since in this case G_I are homological cells.

3.2. Collar model. Consider the compact n-torus with a fixed coordinate representation $T^n = \{(t_1, \dots, t_n) \mid t_s \in \mathbb{C}, |t_s| = 1\}$. If $\mathcal{R}$ is either $\mathbb{Z}$ or $\mathbb{Q}$, the vector $w = (w_1, \dots, w_n) \in \mathcal{R}^n$ determines a compact 1-dimensional subgroup $t^w = \{(e^{2\pi\sqrt{-1}w_1 t}, \dots, e^{2\pi\sqrt{-1}w_n t}) \mid t \in \mathbb{R}\}$. Let $[m] = \mathrm{Vert}(S)$ be the set of vertices of S and let $\lambda\colon [m] \to \mathcal{R}^n$ be a characteristic function over $\mathbb{Z}$ or $\mathbb{Q}$. Let $T_i \subset T^n$ denote the one dimensional subgroup $t^{\lambda(i)}$. For a simplex $I \in S$, $I \neq \hat{0}$ let T_I denote the product of the one-dimensional subgroups T_i corresponding to the vertices of I where the product is taken inside T^n. The definition of characteristic function implies that T_I is a compact subtorus of T^n of dimension $|I|$.

Consider the space $\widehat{Q} = |S| \times [0,1]$ which will be called the *collar* of $|S|$. Let $\partial_\epsilon \widehat{Q}$ denote the subset $|S| \times \{\epsilon\}$ for $\epsilon = 0, 1$. The faces G_I can be considered as the subsets of $\partial_0 \widehat{Q} \subset \widehat{Q}$. To make the notation uniform, we set $G_{\hat{0}} = \widehat{Q}$ and $T_{\hat{0}} = \{1\} \subset T^n$.

Construction 3.1. Consider the identification space $\widehat{X} = (\widehat{Q} \times T^n)/\sim$, where the points (x,t), (x',t') are identified whenever $x = x' \in G_I$ and $t^{-1}t' \in T_I$ for some simplex $I \in S$. Let $f\colon \widehat{Q} \times T^n \to \widehat{X}$ denote the quotient map, and μ denote the projection to the first factor, $\mu\colon \widehat{X} \to \widehat{Q}$. The preimage $\mu^{-1}(G_I)$ is denoted by X_I. Let $\partial_1 \widehat{X}$ denote the subset $\partial_1 \widehat{Q} \times T^n \subset \widehat{X}$. Note that T^n acts on $\widehat{X}$, and $\widehat{Q}$ is the orbit space of this action. The j-dimensional orbits of the action are the interior points of the faces G_I, $\dim G_I = j$

3.3. Absolute and relative spectral sequences. The dual face structure on $|S|$ induces the topological filtration

$$Q_0 \subset Q_1 \subset \ldots \subset Q_{n-1} = \partial_0 \widehat{Q} \subset Q_n = \widehat{Q}, \qquad Q_j = \bigcup_{\dim G_I \leqslant j} G_I$$

which lifts to the orbit type filtration on $\widehat{X}$:

$$X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_n = \widehat{X}.$$

Let $(E_{\widehat{Q}})^1_{p,q} = H_{p+q}(Q_p, Q_{p-1}) \Rightarrow H_{p+q}(\widehat{Q})$ and $(E_{\widehat{X}})^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(\widehat{X})$ be the homological spectral sequences associated with these filtrations. By the result of [**1**, Th.5.2], whenever S is Buchsbaum, the map

$$f_*^2\colon \bigoplus_{q_1+q_2=q} (E_{\widehat{Q}})^2_{p,q_1} \otimes H_{q_2}(T^n) \to (E_{\widehat{X}})^2_{p,q}$$

is an isomorphism for $p > q$ and injective for $p = q$ (and zero for $p < q$, since target groups are trivial by dimensional reasons).

Nontrivial higher differentials $d^{\geqslant 2}$ in both spectral sequences for $\widehat{Q}$ and $\widehat{X}$ can only originate in the rightmost column $(E_{\widehat{Q}})^*_{n,*}$ (resp. $(E_{\widehat{X}})^*_{n,*}$). For $\widehat{Q}$ this can be easily seen from the shape of $(E_{\widehat{Q}})^*_{*,*}$: all other differentials either originate in zero or hit zero, since $S = Q_{n-1}$ is a manifold. For $\widehat{X}$ it follows from the properties of f_*^2, mentioned in the previous paragraph. Indeed, let $d^r\colon (E_{\widehat{X}})^r_{p,q} \to (E_{\widehat{X}})^2_{p-r,q+r-1}$ be a nontrivial differential. Then $d^r \circ f_* = f_* \circ (d^r_Q \otimes \mathrm{id}_{H^*(T^n)})$ is also nontrivial. This contradicts to the facts that $d^r_Q\colon (E_{\widehat{Q}})^r_{p,q} \to (E_{\widehat{Q}})^2_{p-r,q+r-1}$ is zero and f_* is injective (if its target is nonzero).

On the other hand, the rightmost column of the spectral sequence vanishes: $(E_{\widehat{Q}})^1_{n,*} \cong H_{n+*}(\widehat{Q}, \partial_0\widehat{Q}) = 0$, since the collar $\widehat{Q}$ collapses to $\partial_0\widehat{Q}$. Similar vanishing occurs for $(E_{\widehat{X}})^*_{*,*}$: we have $(E_{\widehat{X}})^1_{n,q} \cong H_{n+q}(\widehat{X}, \partial_0\widehat{X}) = 0$. Thus there are no higher differentials $d^{\geqslant 2}$ in both spectral sequences.

We also need the homological spectral sequences for the relative homology:

$$(E_{(\widehat{Q},\partial_1\widehat{Q})})^r_{p,q} \Rightarrow H_{p+q}(\widehat{Q}, \partial_1\widehat{Q}) = 0, \qquad (E_{(\widehat{X},\partial_1\widehat{X})})^r_{p,q} \Rightarrow H_{p+q}(\widehat{X}, \partial_1\widehat{X}).$$

The first pages are the following:

$$(E_{(\widehat{Q},\partial_1\widehat{Q})})^1_{p,q} = \begin{cases} H_{p+q}(Q_p, Q_{p-1}), & \text{if } p < n; \\ H_{n+q}(\widehat{Q}, Q_{n-1} \sqcup \partial_1\widehat{Q}), & \text{if } p = n, \end{cases}$$

$$(E_{(\widehat{X},\partial_1\widehat{X})})^1_{p,q} = \begin{cases} H_{p+q}(X_p, X_{p-1}), & \text{if } p < n; \\ H_{n+q}(\widehat{X}, X_{n-1} \sqcup \partial_1\widehat{X}), & \text{if } p = n. \end{cases}$$

Note that the rightmost terms $(E_{(\widehat{Q},\partial_1\widehat{Q})})^1_{n,q}$ have the form:

$$H_{n+q}(\widehat{Q}, \partial_0\widehat{Q} \sqcup \partial_1\widehat{Q}) \cong H_{n+q}(|S| \times [0,1], |S| \times \{0,1\}) \cong H_{n+q-1}(S),$$

and the higher differentials

$$(d_{(\widehat{Q},\partial_1\widehat{Q})})^r\colon (E_{(\widehat{Q},\partial_1\widehat{Q})})^*_{n,-r+1} \to (E_{(\widehat{Q},\partial_1\widehat{Q})})^*_{n-r,0} \cong H_{n-r}(S)$$

are isomorphisms (so that the spectral sequence for $(\widehat{Q}, \partial_1\widehat{Q})$ collapses to zero). Similar to the non-relative case, the induced map

$$f^2_*\colon \bigoplus_{q_1+q_2=q} (E_{(\widehat{Q},\partial_1\widehat{Q})})^2_{p,q_1} \otimes H_{q_2}(T^n) \to (E_{(\widehat{X},\partial_1\widehat{X})})^2_{p,q}$$

is an isomorphism for $p > q$ and injective for $p = q$ (this follows from the general method developed in [**1**]).

3.4. Proof of Theorem 1. The proof essentially relies on calculations made in [**2**]. If S is a connected orientable homology manifold, then $\widehat{X}$ is a connected orientable homology manifold with the boundary $\partial_1\widehat{X} \cong |S| \times T^n$. The boundary admits a collar neighborhood as required in Construction 2.2. For $I \neq \hat{0}$ the subset X_I is a closed submanifold of codimension $2|I|$ lying in the interior of $\widehat{X}$. It is called the *face submanifold*, and its homology class $[X_I] \in H_{2n-2|I|}(\widehat{X})$ is called the *face class*. Note that for $|I| \neq 0$ the classes $[X_I]$ appear in the spectral sequence $(E_{\widehat{X}})^*_{*,*}$ as the free generators of the group: $(E_{\widehat{X}})^1_{q,q}$ with $q = n - |I|$. The relations on these classes in $H_{2q}(\widehat{X})$ are the elements in the image of the first differential, hitting the group $(E_{\widehat{X}})^1_{q,q}$ (since all higher differential vanish). In [**2**, Prop.4.3 and

Lm.8.2] we checked that these relations are the same as the linear relations on v_I in the ring $(\mathcal{R}[S]/\Theta)_{2(n-q)}$ when $q \leqslant n-2$. If $q = n-1$, there are no relations on $[X_I] \in H_{2n-2}(\widehat{X})$ since there are no differentials hitting the group $(E_{\widehat{X}})^1_{n-1,n-1}$.

In addition to face classes, there exist other homology classes in $H_*(\widehat{X})$, namely the classes coming from the part of the spectral sequence below the diagonal. They lie in the groups $(E_{\widehat{X}})^2_{p,q} \cong H_p(\widehat{Q}) \otimes H_q(T^n)$ for $q < p < n$. In [**2**] we called them *spine classes*.

Let us keep track on the behavior of homology classes, when they map to the relative homology by the homomorphism $\imath_*\colon H_*(\widehat{X}) \to H_*(\widehat{X}, \partial_1\widehat{X})$. Again, we may look at their representatives in the spectral sequence $(E_{(\hat{X},\partial_1\hat{X})})^*$. At this time, higher differentials are nontrivial. All spine classes of $H_*(\widehat{X})$ are killed by higher differentials. Indeed, they lie in the part of the relative spectral sequence which is isomorphic to $(E_{(\hat{Q},\partial_1\hat{Q})})^*_{*,*} \otimes H_*(T^n)$, and the latter sequence collapses to 0.

On the other hand, the diagonal terms $(E_{(\hat{X},\partial_1\hat{X})})^*_{q,q}$ are hit by higher differentials as well. Thus there are more relations on $[X_I]$ in the group $H_*(\widehat{X}, \partial_1\widehat{X})$ than in the group $H_*(\widehat{X})$. The higher differential

$$(d_{(\hat{X},\partial_1\hat{X})})^r\colon (E_{(\hat{X},\partial_1\hat{X})})^1_{n,n-2r+1} \to (E_{(\hat{X},\partial_1\hat{X})})^1_{n-r,n-r}$$

is injective, as follows from [**2**, Prop.2.7(5)], and gives an inclusion of $H_{n-r}(S) \otimes H_{n-r}(T^n)$ into $(E_{(\hat{X},\partial_1\hat{X})})^1_{n-r,n-r}$. Under the degree reversing identification $[X_I] \leftrightarrow v_I$ (and by Poincare duality in S), this inclusion gives the Novik–Swartz submodule $(I_{NS})_{2r} \cong \binom{n}{r}\widetilde{H}^{r-1}(S)$ inside $(\mathcal{R}[S]/\Theta)_{2r}$, for $r \geqslant 2$ (see details in [**2**, Th.4.6 and Sect.8]). When $r = 1$, only the first differential $(d_{(\hat{X},\partial_1\hat{X})})^1$ hits the cell $(E_{(\hat{X},\partial_1\hat{X})})^1_{n-1,n-1}$. Its image corresponds to the linear span of $\theta_1, \ldots, \theta_n$ in $\mathcal{R}[S]_2$. The Novik–Swartz submodule I_{NS} in degree 2 vanishes, since S is connected.

These considerations prove that the image of the map

$$\imath_*\colon H_{2n-2r}(\widehat{X}) \to H_{2n-2r}(\widehat{X}, \partial_1\widehat{X})$$

is isomorphic to the homogeneous component of $(\mathcal{R}[S]/\Theta)/I_{NS}$ of degree $2r$ for each $0 < r < n$.

When $r = n$, the submodule $(I_{NS})_{2n}$ is trivial. Thus $H_0(\widehat{X})$ coincides with $(\mathcal{R}[S]/\Theta)_{2n} = ((\mathcal{R}[S]/\Theta)/I_{NS})_{2n}$. The group $H_{2n}(\widehat{X}, \partial_1\widehat{X}) \cong \mathcal{R}$ is obviously identified with $((\mathcal{R}[S]/\Theta)/I_{NS})_0 \cong \mathcal{R}$.

Theorem 1 now follows from Remark 2.3 and the fact that the correspondence $[X_I] \leftrightarrow v_I$ translates the intersection product on $\widehat{X}$ to the product in the face ring.

Acknowledgements

The author thanks the referee for his valuable remarks and suggestions which helped to improve the exposition.

References

[1] A. Ayzenberg, *Locally standard torus actions and sheaves over Buchsbaum posets*, preprint arXiv:1501.04768.

[2] A. Ayzenberg, *Homology cycles in manifolds with locally standard torus actions*, preprint arXiv:1502.01130v2, to appear in Homology, Homotopy Appl.

[3] V. M. Buchstaber, T. E. Panov, *Combinatorics of Simplicial Cell Complexes and Torus Actions*, Proc. of the Steklov Institute of Mathematics, Vol. 247, 2004, pp. 1–17.

[4] V. Buchstaber, T. Panov, *Toric Topology*, preprint arXiv:1210.2368.

[5] G. E. Bredon, *Topology and Geometry.* Grad. Texts in Math. 139 (1993).

[6] M. Davis, T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J., 62:2 (1991), 417–451.

[7] M. Masuda, T. Panov, *On the cohomology of torus manifolds*, Osaka J. Math. 43 (2006), 711–746.

[8] I. Novik, Ed Swartz, *Socles of Buchsbaum modules, complexes and posets*, Adv. Math., 222 (2009), 2059–2084.

[9] I. Novik and E. Swartz, *Gorenstein rings through face rings of manifolds*, Composit. Math. 145 (2009), 993–1000.

[10] P. Schenzel, *On the Number of Faces of Simplicial Complexes and the Purity of Frobenius*, Math. Zeitschrift 178, 125–142 (1981).

[11] R. P. Stanley, *f-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra 71 (1991), 319–331.

Faculty of Mathematics, Higher School of Economics, Vavilova 7, Moscow, 117312, Russia

E-mail address: ayzenberga@gmail.com