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Colored Graphs and Matrix Integrals

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Abstract—We discuss applications of generating functions for colored graphs to asymptotic expansions of matrix integrals. The described technique provides an asymptotic expansion of the Kontsevich integral. We prove that this expansion is a refinement of the Kontsevich expansion, which is the sum over the set of classes of isomorphic ribbon graphs. This yields a proof of Kontsevich's results that is independent of the Feynman graph technique.

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1. INTRODUCTION

In this article we discuss two different asymptotic expansions of matrix integrals. The original approach using the so-called Feynman diagram technique, which is described, for instance, in [3], leads to sums over isomorphism classes of ribbon graphs (see [4]). Asymptotic expansions of more general Gaussian integrals are sums over isomorphism classes of colored graphs without ribbon structure. Here we derive the former expansion from the latter one. This provides an independent proof of the expansion used by Kontsevich in [4]. It might be very interesting to compare the algebra arising in these two approaches. The asymptotic expansion using ribbon graphs in [4] leads to the tau function of the KdV hierarchy, while the sums over colored graphs satisfy simple partial differential equations that generalize the Burgers equation (see [1] or [2] for details).

We describe the general approach using colored graphs in Section 2. In Section 3 we specialize the results of Section 2 for the matrix integral discussed in [4]. In this section we also derive the expansion over ribbon graphs. The proof is based on simple topological considerations, which are contained in Section 4. In the last Section 5 we give an explicit calculation of the first term of the expansion using colored graphs.

2. COLORED GRAPHS

The notion of colored graph was introduced in [1]. Informally speaking, a colored graph is a combinatorial object that may be constructed using a finite number of components of the following two types.

1. *Vertices* having a finite number of *tails* (i.e., half-edges) each of which is colored with one of $r \geq 1$ given colors. Denote the color set by Ω; in this section it would be convenient to number the colors and assume that

$$\Omega = \{1, \dots, r\}. \tag{2.1}$$

For a vertex v we define its *valency multi-index* $\nu(v) = (n_1, \dots, n_r)$, where $n_i \geq 0$ is the number of tails of color i adjacent to v. The total valency of the vertex is defined as $|\nu(v)| = \sum n_i$. Generally, colored graphs are by definition weighted graphs; this means that each vertex v is assigned a nonnegative integer $g(v)$, which is called the *genus* of the vertex v. However, sometimes there is no

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need to consider weighted graphs; in this case we may simply assume that the genera of all vertices are zero. Corresponding simplifications of our notations are described in Remark 2.1.

2. In order to connect two tails of colors i and j into one edge, we need a component of the second type, which we will call an (ij)-*connector*. So connectors may be considered as two-valent vertices of a different type, but probably it is more natural to think of them as "middles of edges."

So by definition a colored graph Γ is any finite connected graph constructed using the described vertices and connectors; some tails may remain free, but each connector must be adjacent to exactly two half-edges. For a colored graph Γ we define its *multi-index of tails* $N(\Gamma)=(N_1,\dots,N_r)$, where $N_i\ge 0$ is the number of free tails of color i. The vertex set of Γ is denoted by $V(\Gamma)$ or simply by V if no confusion is possible, and the edge set of Γ is denoted by $E(\Gamma)$ or simply by E. The *genus* of a connected graph Γ is defined as

$$g(\Gamma)=b_1(\Gamma)+\sum_{v\in V(\Gamma)} g(v), \tag{2.2}$$

where $b_1(\Gamma)$ is the first Betti number of the graph Γ.

Next consider a set of independent variables $a_{m,N}$ for all nonnegative integers m and all multi-indices $N=(n_1,\dots,n_r)$ with all $n_i\ge 0$ and a symmetric $r\times r$ matrix $\Sigma=(\sigma_{ij})$. The variables $a_{m,N}$ may be considered as coefficients of a formal Taylor power series expansion

$$U(Y,\hbar)=\sum_{m,N} a_{m,N}\frac{Y^N}{N!}\hbar^{m-1}\in\frac{1}{\hbar}\mathbb{C}[[y_1,\dots,y_r,\hbar]], \tag{2.3}$$

where $Y=(y_1,\dots,y_r)$, $Y^N=y_1^{n_1}\dots y_r^{n_r}$, and $N!=n_1!\dots n_r!$. We will also use the following notations: the multi-index $(0,\dots,0,1,0,\dots,0)$ (the ith component is 1 and all the other components are zero) will be denoted by $\{i\}$; the multi-index $\{i\}+\{j\}$ will be denoted by $\{ij\}$; the multi-index $\{i\}+\{j\}+\{k\}$ will be denoted by $\{ijk\}$, and so on.

Consider the expansion

$$U(Y,\hbar)=\sum_m U_m(Y)\hbar^{m-1} \tag{2.4}$$

and define the gradient formal power series vector

$$F(Y)=\nabla_Y U_0(Y)=\left(\frac{\partial U_0(Y)}{\partial y_1},\dots,\frac{\partial U_0(Y)}{\partial y_r}\right) \tag{2.5}$$

and the Hessian power series matrix

$$H(Y)=\nabla_Y F(Y),\qquad H(Y)_{ij}=\frac{\partial^2 U_0(Y)}{\partial y_i\partial y_j}. \tag{2.6}$$

Consider the following generating power series:

$$\Psi(\{a_{m,N}\},Y,\Sigma,\hbar)=\sum_{\substack{\text{All colored}\\ \text{graphs }\Gamma}}\frac{Y^{N(\Gamma)}\hbar^{g(\Gamma)-1}}{|\operatorname{Aut}\Gamma|}\prod_{v\in V(\Gamma)} a_{g(v),\nu(v)}\prod_{e\in E(\Gamma)}\sigma_{ij} \tag{2.7}$$

and

$$\mathcal{P}(\{a_{m,N}\},\Sigma,\hbar)=\sum_{\substack{\text{All stable colored}\\ \text{graphs }\Gamma\text{ without tails}}}\frac{\hbar^{g(\Gamma)-1}}{|\operatorname{Aut}\Gamma|}\prod_{v\in V(\Gamma)} a_{g(v),\nu(v)}\prod_{e\in E(\Gamma)}\sigma_{ij}. \tag{2.8}$$

In the last products in (2.7) and (2.8), the indices i and j are the colors of the connector formed by the edge e.

Recall that a graph is called *stable* if it has no genus 0 vertices of valency less then 3 and no genus 1 isolated (i.e., 0-valent) vertices. Therefore, there are no genus 0 and genus 1 stable graphs without tails. Consider the expansions

$$\Psi(\{a_{m,N}\}, Y, \Sigma, \hbar) = \sum_{g\geq 0} \Psi_g(\{a_{m,N}\}, Y, \Sigma)\hbar^{g-1} \tag{2.9}$$

and

$$\mathcal{P}(\{a_{m,N}\}, \Sigma, \hbar) = \sum_{g\geq 2} \mathcal{P}_g(\{a_{m,N}\}, \Sigma)\hbar^{g-1}. \tag{2.10}$$

Note that all the Ψ_g are formal power series, while $\mathcal{P}_g$ are polynomials depending only on a finite number of $a_{m,N}$ with $m \leq g$ and satisfying certain homogeneity conditions described in [1]. These polynomials are called *stable graph polynomials* in [1]. (We will not need the explicit form of the homogeneity conditions for stable graph polynomials in this paper.)

In [1] we prove that the main generating power series Ψ can be expressed in terms of the series F and H (see (2.5) and (2.6)) and the stable graph polynomials (2.10). To state these results, let us consider the gradient formal power series vector

$$\Phi(\{a_{m,N}\}, Y, \Sigma) = \nabla_Y \Psi_0(\{a_{m,N}\}, Y, \Sigma) = \left(\frac{\partial \Psi_0}{\partial y_1}, \dots, \frac{\partial \Psi_0}{\partial y_r}\right) \tag{2.11}$$

and

$$\Theta(\{a_{m,N}\}, Y, \Sigma) = Y + \Sigma\Phi(\{a_{m,N}\}, Y, \Sigma). \tag{2.12}$$

Next we present (without proofs) the results of [1]. To make the expressions shorter, we will omit the arguments of Θ defined in (2.12).

1. The vector power series Φ satisfies the functional equation

$$\Phi(\{a_{m,N}\}, Y, \Sigma) = F(\Theta), \tag{2.13}$$

or, equivalently, the formal power series

$$Y - \Sigma F(Y) \qquad \text{and} \qquad \Theta = Y + \Sigma\Phi(\{a_{m,N}\}, Y, \Sigma) \tag{2.14}$$

are inverse to each other. Inverting $Y - \Sigma F(Y)$ or solving the functional equation (2.13) and then integrating Φ, we determine the series Ψ_0.

2. For $g = 1$

$$\Psi_1(\{a_{m,N}\}, Y, \Sigma) = U_1(\Theta) - \frac{1}{2}\operatorname{tr}\log(E - \Sigma H(\Theta)), \tag{2.15}$$

where E is the identity matrix.

3. For $g > 1$

$$\Psi_g(\{a_{m,N}\}, Y, \Sigma) = \mathcal{P}_g\left(\left\{\frac{\partial^N U_m}{\partial Y^N}(\Theta)\right\}, (E - \Sigma H(\Theta))^{-1}\Sigma\right). \tag{2.16}$$

The most important property of the generating series Ψ (2.7) is that it provides a formal asymptotic expansion of a certain Gaussian integral. For this purpose we set Σ to be a real positive definite symmetric matrix and consider the Gaussian measure on $\mathbb{R}^r$

$$d\mu_{Y,\Sigma}(X) = \frac{1}{(2\pi\hbar\det\Sigma)^{r/2}} \exp\left(-\frac{(Y-X)^{\mathrm{T}}\Sigma^{-1}(Y-X)}{2\hbar}\right) dX \tag{2.17}$$

with mean value Y and covariance matrix $\hbar\Sigma$. By analogy with [2], one can easily check that the formal power series (2.7) is the asymptotic expansion of the logarithm of the following Gaussian integral:

$$\log \int \exp(U(X))\, d\mu_{Y,\Sigma}(X) \sim \Psi(\{a_{g,N}\}, Y, \Sigma, \hbar). \tag{2.18}$$

The series U is given by the Taylor expansion (2.3); therefore, an asymptotic expansion of any integral of this kind may be interpreted as a generating function with appropriate coefficients $a_{g,N}$. In the most common case, when

$$a_{0,N} = 0 \quad \text{for} \quad |N| \le 2 \qquad \text{and} \qquad a_{1,(0,\dots,0)} = 0, \tag{2.19}$$

nonstable graphs contribute zero summands to the generating function Ψ (2.7) and therefore we obtain $\mathcal{P}$ substituting 0 for Y in Ψ:

$$\mathcal{P}(\{a_{g,N}\}, \Sigma, \hbar) = \Psi(\{a_{g,N}\}, 0, \Sigma, \hbar). \tag{2.20}$$

So, in this case, the generating series for stable graph polynomials (2.8) may also be interpreted as an asymptotic expansion of the Gaussian integral

$$\log \int \exp(U(X))\, d\mu_{0,\Sigma}(X) \sim \mathcal{P}(\{a_{g,N}\}, \Sigma, \hbar), \tag{2.21}$$

where

$$d\mu_{0,\Sigma}(X) = \frac{1}{(2\pi\hbar \det \Sigma)^{r/2}} \exp\biggl(-\frac{X^{\mathrm{T}}\Sigma^{-1}X}{2\hbar}\biggr)\, dX. \tag{2.22}$$

Remark 2.1. The expression (2.3) for the series U is sometimes too general. For instance, for matrix integrals discussed in this paper and in certain other interesting cases it is not necessary to study weighted graphs. For this purpose we may simply put

$$a_{g,N} = 0 \qquad \text{for} \quad g > 0. \tag{2.23}$$

In this case $U = \frac{U_0}{\hbar}$ and $U_1 = U_2 = \ldots = 0$ and we may shorten the notations and write a_N instead of $a_{0,N}$.

If in addition U_0 is a homogeneous polynomial of degree $d \ge 3$, then

$$a_N = 0 \qquad \text{for} \quad |N| \ne d, \tag{2.24}$$

which means that in the sum (2.7) it is sufficient to consider the summation only over d-valent graphs. For the matrix integrals discussed in this paper $d = 3$.

3. MATRIX INTEGRALS

Now we are going to apply the described approach to matrix integrals discussed in [4]. Now X is a Hermitian $n \times n$ matrix; denote

$$X = Z + \sqrt{-1}\,V, \tag{3.1}$$

where Z is symmetric and V is skew-symmetric. So X depends on

$$r = n^2 \tag{3.2}$$

real parameters; in [4] the Gaussian measure on $\mathbb{R}^r$ is given by

$$d\mu_\Lambda(X) = c_\Lambda \exp\left(-\operatorname{tr}\frac{X\Lambda X}{2}\right) dX \tag{3.3}$$

and the matrix integral is

$$\log \int \exp\left(\frac{\sqrt{-1}}{6}\operatorname{tr} X^3\right) d\mu_\Lambda(X). \tag{3.4}$$

By the homogeneous substitution

$$X \to \hbar^{-1/3}X, \qquad \Lambda \to \hbar^{-1/3}\Lambda \tag{3.5}$$

we can transform this integral to the form corresponding to (2.21):

$$\log \int \exp\left(\frac{\sqrt{-1}}{6\hbar}\operatorname{tr} X^3\right) d\widetilde{\mu}_\Lambda(X), \tag{3.6}$$

where

$$d\widetilde{\mu}_\Lambda(X) = \widetilde{c}_\Lambda \exp\left(-\operatorname{tr}\frac{X\Lambda X}{2\hbar}\right) dX. \tag{3.7}$$

This is exactly the integral (2.21) for the case $r = n^2$ and

$$U(X) = \frac{U_0(X)}{\hbar} = \frac{\sqrt{-1}}{6\hbar}\operatorname{tr} X^3. \tag{3.8}$$

Note that we have just the case described in Remark 2.1: $a_{g,N}$ are nonzero only for $g = 0$ and $|N| = 3$.

Next let us present explicit expressions for the cubic polynomial U_0 and the covariance matrix Σ in this case.

To make all the formulas more clear, let us fix the following notations for the coordinates in $\mathbb{R}^r$. There are exactly $\frac{n(n+1)}{2}$ coordinates z_{ij} with $i \le j$ corresponding to the elements of the symmetric matrix Z from (3.1); let us denote them by $x_{\underline{ij}}$. The remaining $\frac{n(n-1)}{2}$ coordinates v_{ij} with $i < j$ correspond to the nondiagonal elements of the skew-symmetric matrix V from (3.1); let us denote them by $x_{\overline{ij}}$. It would be convenient for us to define the symbols $\underline{ij}$ and $\overline{ij}$ to be symmetric: $\underline{ij} = \underline{ji}$ and $\overline{ij} = \overline{ji}$, but in any case $x_{\underline{ij}}$ (or $x_{\overline{ij}}$) will mean the corresponding element of the matrix Z (respectively V) located above the main diagonal. Let us denote the corresponding subsets of the index set Ω by

$$\Omega_0 = \{\underline{ii},\ 1 \le i \le n\}, \qquad \Omega_+ = \{\underline{ij},\ 1 \le i < j \le n\}, \qquad \Omega_- = \{\overline{ij},\ 1 \le i < j \le n\}, \tag{3.9}$$

so that

$$\Omega = \Omega_0 \cup \Omega_+ \cup \Omega_-. \tag{3.10}$$

Now it is easy to determine the $n^2 \times n^2$ matrix Σ for the case of a diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \tag{3.11}$$

Proposition 3.1. *For Λ defined by* (3.11) *Σ is a diagonal matrix with*

$$\sigma_{\underline{ii}\,\underline{ii}} = \frac{1}{\lambda_i}, \qquad \sigma_{\underline{ij}\,\underline{ij}} = \frac{1}{\lambda_i + \lambda_j}, \qquad \sigma_{\overline{ij}\,\overline{ij}} = \frac{1}{\lambda_i + \lambda_j}. \tag{3.12}$$

Also it is not hard to present an explicit form of the function $U_0(X) = \frac{\sqrt{-1}}{6}\operatorname{tr} X^3$.

Proposition 3.2.

$$U_0(X) = \sqrt{-1}\Biggl(\sum_{1\le i\le j\le k\le n} \frac{1}{\{ijk\}!} x_{\underline{ij}} x_{\underline{jk}} x_{\underline{ik}} + \sum_{1\le i\le j\le n}\ \sum_{k\ne i,\ k\ne j}\Bigl(\pm\frac{1}{\{ij\}!} x_{\underline{ij}} x_{\overline{jk}} x_{\overline{ik}}\Bigr)\Biggr), \tag{3.13}$$

where the sign in the second sum is minus if k is located between i and j and plus otherwise.

Corollary 3.1. 1. *$a_{g,N} = 0$ if $g \ne 0$ or $|N| \ne 3$.*

2. *For $g = 0$ and $|N| = 3$ all nonzero values of the coefficients $a_{g,N}$ are as follows:*

$$a_{0,N} = \begin{cases} \sqrt{-1} & \textit{for}\ \ N = \{\underline{ij}\ \underline{jk}\ \underline{ik}\} \quad \forall i,j,k, \\ \pm\sqrt{-1} & \textit{for}\ \ N = \{\underline{ij}\ \overline{jk}\ \overline{ik}\} \quad \forall i,j,k,\ i\ne k,\ j\ne k, \end{cases} \tag{3.14}$$

where the sign in the second line is minus if k is located between i and j and plus otherwise.

From now on we will use the shortened notations suggested in Remark 2.1: we will write $a_{\{\alpha\beta\gamma\}}$ instead of $a_{0,\{\alpha\beta\gamma\}}$, $\alpha,\beta,\gamma \in \Omega$.

The fact that Σ is diagonal means that in our case both half-edges of any edge have the same color and therefore we obtain a coloring of edges with the same set Ω:

$$\mu\colon\ E(\Gamma) \to \Omega. \tag{3.15}$$

According to Corollary 3.1, such a coloring must satisfy the following condition: any three edges meeting at one vertex should be colored either with

$$\underline{ij},\quad \underline{jk},\quad \text{and}\quad \underline{ik} \qquad \text{for certain}\ \ i,j,k \tag{3.16}$$

or with

$$\underline{ij},\quad \overline{jk},\quad \text{and}\quad \overline{ik} \qquad \text{for certain}\ \ i,j,k,\ i\ne k\ne j. \tag{3.17}$$

We will also use *reduced colorings*

$$\varkappa\colon\ E(\Gamma) \to \Omega_0 \cup \Omega_+ \tag{3.18}$$

that should satisfy condition (3.16).

For any coloring μ (3.15) we define a reduced coloring $\varkappa_\mu$ as

$$\varkappa_\mu(e) = \begin{cases} \mu(e) & \text{if}\ \ \mu(e) \in \Omega_0 \cup \Omega_+, \\ \underline{ij} & \text{if}\ \ \mu(e) = \overline{ij} \in \Omega_-. \end{cases} \tag{3.19}$$

Note that according to (3.12) $\sigma_{\mu(e)} = \sigma_{\varkappa_\mu(e)}$.

A reduced coloring $\varkappa$ defines a subgraph $\Delta(\varkappa) \subset \Gamma$ that has the same set of vertices $V(\Delta(\varkappa)) = V(\Gamma)$ and all edges that are colored by Ω_+: $E(\Delta(\varkappa)) = \varkappa^{-1}(\Omega_+)$. Of course, $\Delta(\varkappa)$ is not necessarily connected; according to (3.16) and (3.17), Δ may have three-valent, two-valent, and null-valent (isolated) vertices.

Using a coloring μ let us define a subgraph $\eta(\mu) \subset \Delta(\varkappa_\mu)$ as a minimal subgraph containing all the edges from $\mu^{-1}(\Omega_-)$. According to (3.17) all vertices of $\eta(\mu)$ are two-valent, and therefore $\eta(\mu)$ consists of a finite number of disjoint cycles and thus represents an element of $H_1(\Delta(\varkappa_\mu),\mathbb{Z}_2)$. Evidently the coloring $\varkappa_\mu$ and the cycle $\eta(\mu)$ define μ uniquely.

Proposition 3.3. *The mapping* $\mu \mapsto (\varkappa_\mu, \eta(\mu))$ *is a natural bijection between the set of all colorings* μ (3.15) *satisfying* (3.16) *and* (3.17) *and the set of all pairs* $(\varkappa, \eta)$, *where* $\varkappa$ *is a reduced coloring* (3.18) *satisfying* (3.16) *and* η *is an arbitrary cycle in* $H_1(\Delta, \mathbb{Z}_2)$.

Let us denote the coloring μ defined by a reduced coloring $\varkappa$ and a cycle $\eta \in H_1(\Delta(\varkappa), \mathbb{Z}_2)$ by $\mu(\varkappa, \eta)$.

Therefore, the asymptotic expansion (3.6) looks as follows:

$$\log \int \exp\left(\frac{\sqrt{-1}}{6\hbar} \operatorname{tr} X^3\right) d\widetilde{\mu}_\Lambda(X) \sim \sum_{\substack{\text{All trivalent} \\ \text{graphs } \Gamma}} \sum_{\substack{\text{All colorings} \\ \mu \text{ of } E(\Gamma)}} \frac{\hbar^{g(\Gamma)-1}}{|\mathrm{Aut}(\Gamma, \mu)|} \prod_{e \in E(\Gamma)} \sigma_{\mu(e)} \prod_{v \in V(\Gamma)} a_{\nu(v)}$$

$$= \sum_{\substack{\text{All trivalent} \\ \text{graphs } \Gamma}} \sum_{\substack{\text{All reduced} \\ \text{colorings } \varkappa \\ \text{of } E(\Gamma)}} \prod_{e \in E(\Gamma)} \sigma_{\varkappa(e)} \sum_{\substack{\text{All colorings } \mu \\ \text{of } E(\Gamma) \text{ such} \\ \text{that } \varkappa_\mu = \varkappa}} \frac{\hbar^{g(\Gamma)-1}}{|\mathrm{Aut}(\Gamma, \mu)|} \prod_{v \in V(\Gamma)} a_{\nu(v)}. \quad (3.20)$$

For our further considerations it would be more convenient to use the corrected values

$$\bar{a}_N = \frac{a_N}{\sqrt{-1}}, \quad (3.21)$$

so that according to Corollary 3.1

$$\bar{a}_N = \begin{cases} 1 & \text{for } N = \{\underline{ij}\ \underline{jk}\ \underline{ik}\} \quad \forall i, j, k, \\ \pm 1 & \text{for } N = \{\underline{ij}\ \overline{jk}\ \overline{ik}\} \quad \forall i, j, k,\ i \neq k,\ j \neq k, \end{cases} \quad (3.22)$$

where the sign in the second line is minus if k is located between i and j and plus otherwise. Since a genus g trivalent graph has $2g-2$ vertices, the expansion can be expressed in terms of the values $\bar{a}_N$ as follows:

$$\log \int \exp\left(\frac{\sqrt{-1}}{6\hbar} \operatorname{tr} X^3\right) d\widetilde{\mu}_\Lambda(X)$$

$$\sim \sum_{\substack{\text{All trivalent} \\ \text{graphs } \Gamma}} (-\hbar)^{g(\Gamma)-1} \sum_{\substack{\text{All reduced} \\ \text{colorings } \varkappa \\ \text{of } E(\Gamma)}} \prod_{e \in E(\Gamma)} \sigma_{\varkappa(e)} \sum_{\substack{\text{All colorings } \mu \\ \text{of } E(\Gamma) \text{ such} \\ \text{that } \varkappa_\mu = \varkappa}} \frac{1}{|\mathrm{Aut}(\Gamma, \mu)|} \prod_{v \in V(\Gamma)} \bar{a}_{\nu(v)}. \quad (3.23)$$

Note that all the values of $\bar{a}_{\nu(v)}$ are ± 1 and according to (3.22) negative factors may arise only from vertices of $\eta(\mu)$:

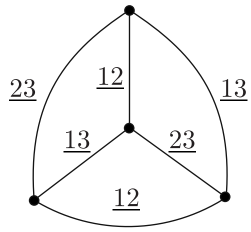
$$\prod_{v \in V(\Gamma)} \bar{a}_{\nu(v)} = \prod_{\text{Vertices of } \eta(\mu)} \bar{a}_{\nu(v)}. \quad (3.24)$$

Since any cycle $\eta \in H_1(\Delta(\varkappa), \mathbb{Z}_2)$ can be represented as a union of disjoint simple cycles,

$$\prod_{v \in V(\Gamma)} \bar{a}_{\nu(v)} = \prod_{\substack{\text{Simple cycles} \\ c \subset \eta(\mu)}} \prod_{\text{Vertices of } c} \bar{a}_{\nu(v)}. \quad (3.25)$$

We will see that for a simple cycle c the sign of the product

$$\prod_{\text{Vertices of } c} \bar{a}_{\nu(v)} \quad (3.26)$$



A nonorientable coloring corresponding to the embedding of the graph in the real projective plane. The borders of the three 2-cells are the three 4-cycles of this graph; the three 2-cells are colored by 1, 2, and 3.

can be expressed in topological terms arising from the consideration of embeddings of the graph Γ in a compact surface S. By definition (see [5]) this means that there is a cell decomposition of the surface S and an isomorphism of the graph (considered as a one-dimensional cell complex) and the one-dimensional skeleton of the cell decomposition of the surface. Denote by $F(S)$ the set of two-dimensional cells of the cell decomposition; the elements of $F(S)$ will be called *faces*.

Consider some coloring of faces

$$\varphi\colon\ F(S)\to\{1,\dots,n\}. \tag{3.27}$$

Obviously the coloring φ defines a reduced coloring $\varkappa(\varphi)$ of the set of edges of Γ (with the color set $\Omega_0\cup\Omega_+$) satisfying (3.16): for an edge e incident to faces i and j we put

$$\varkappa(\varphi)(e)=\underline{ij}. \tag{3.28}$$

In the next section we prove that any reduced coloring $\varkappa$ is generated by some embedding; but in general such an embedding is not unique.

Proposition 3.4. 1. *Any reduced coloring* $\varkappa\colon E(\Gamma)\to\Omega_0\cup\Omega_+$ *satisfying* (3.16) *is generated by some embedding of the graph* Γ *in a surface* S *and by some coloring* φ *of faces* (3.27).

2. *The embedding and the coloring* φ (3.27) *are determined by* $\varkappa$ *uniquely if and only if* $\Gamma=\Delta(\varkappa)$.

Fix some coloring of edges μ satisfying (3.16) and (3.17) and an embedding of the graph Γ in a compact surface S generating the coloring $\varkappa_\mu$. Consider a simple cycle $c\in H_1(\Delta(\varkappa_\mu),\mathbb{Z}_2)$. Then the neighborhood of the cycle c in S is homeomorphic either to a cylinder (orientable case) or to a Möbius strip (nonorientable case). In the next section we prove that this determines the sign of the product in (3.26) and therefore does not depend on the embedding.

Proposition 3.5. *For a simple cycle* $c\subset\eta$

$$\prod_{\text{Vertices of } c}\bar a_{\nu(v)}=\begin{cases}1 & \text{if the neighborhood of } c \text{ is orientable},\\ -1 & \text{if the neighborhood of } c \text{ is nonorientable}.\end{cases} \tag{3.29}$$

Corollary 3.2. *Fix a reduced coloring* $\varkappa$. *For each cycle* $\eta\in H_1(\Delta(\varkappa),\mathbb{Z}_2)$ *consider the coloring* $\mu(\varkappa,\eta)$ *defined in Proposition* 3.3 *and the corresponding values of* $\bar a_N$. *Then the mapping*

$$\varepsilon\colon\ H_1(\Delta(\varkappa),\mathbb{Z}_2)\to\{\pm1\} \tag{3.30}$$

defined by

$$\varepsilon(\eta)=\prod_{\text{Vertices of } \eta}\bar a_{\nu(v)} \tag{3.31}$$

is a homomorphism.

Let us call a reduced coloring $\varkappa$ *orientable* if the neighborhoods of all cycles $c \in H_1(\Delta(\varkappa), \mathbb{Z}_2)$ are orientable; otherwise the coloring will be called *nonorientable*. Thus a coloring is orientable if and only if the homomorphism (3.30) is trivial. (A simple example of a nonorientable reduced coloring for $n = 3$ is presented in the figure.)

Now let us fix a reduced coloring $\varkappa$. Consider the action of $\mathrm{Aut}(\Gamma, \varkappa)$ on $H_1(\Delta(\varkappa), \mathbb{Z}_2)$. The orbits of this action are in one-to-one correspondence with the colorings μ such that $\varkappa_\mu = \varkappa$. The sign of $\varepsilon(c)$ is constant on the orbits of the action, and therefore

$$\frac{1}{|\mathrm{Aut}(\Gamma,\varkappa)|} \sum_{c \in H_1(\Delta,\mathbb{Z}_2)} \varepsilon(c) = \sum_{\substack{\text{Orbits } \mathrm{Aut}(\Gamma,\varkappa)\cdot c \\ \text{of the action of } \mathrm{Aut}(\Gamma,\varkappa) \\ \text{on } H_1(\Delta(\varkappa),\mathbb{Z}_2)}} \varepsilon(c) \frac{|\mathrm{Aut}(\Gamma,\varkappa)\cdot c|}{|\mathrm{Aut}(\Gamma,\varkappa)|}$$

$$= \sum_{\substack{\text{Orbits } \mathrm{Aut}(\Gamma,\varkappa)\cdot c \\ \text{of the action of } \mathrm{Aut}(\Gamma,\varkappa) \\ \text{on } H_1(\Delta(\varkappa),\mathbb{Z}_2)}} \varepsilon(c) \frac{1}{|\mathrm{Aut}(\Gamma,\varkappa,c)|} = \sum_{\substack{\text{All colorings } \mu \\ \text{of } E(\Gamma) \text{ such} \\ \text{that } \varkappa_\mu = \varkappa}} \frac{1}{|\mathrm{Aut}(\Gamma,\mu)|} \prod_{\text{Vertices of } \eta(\mu)} \bar{a}_{\nu(v)}. \quad (3.32)$$

Corollary 3.3.

$$\sum_{\substack{\text{All colorings } \mu \\ \text{of } E(\Gamma) \text{ such} \\ \text{that } \varkappa_\mu = \varkappa}} \frac{1}{|\mathrm{Aut}(\Gamma,\mu)|} \prod_{\substack{\text{Vertices } v \\ \text{of } \eta(\mu)}} \bar{a}_{\nu(v)} = \begin{cases} \dfrac{2^{b_1(\Delta(\varkappa))}}{|\mathrm{Aut}(\Gamma,\varkappa)|} & \text{for an orientable coloring } \varkappa, \\ 0 & \text{for a nonorientable coloring } \varkappa. \end{cases} \quad (3.33)$$

Here $b_1(\Delta(\varkappa))$ is the first Betti number of the graph $\Delta(\varkappa)$.

This provides the following asymptotic expansions of the matrix integral.

Theorem 3.1.

$$\log \int \exp\left(\frac{\sqrt{-1}}{6\hbar} \operatorname{tr} X^3\right) d\widetilde{\mu}_\Lambda(X) \sim \sum_{\substack{\text{All trivalent} \\ \text{graphs } \Gamma}} \sum_{\substack{\text{All orientable} \\ \text{colorings } \mu \\ \text{of } E(\Gamma)}} \frac{(-\hbar)^{g(\Gamma)-1}}{|\mathrm{Aut}(\Gamma,\mu)|} \prod_{e \in E(\Gamma)} \sigma_{\mu(e)}$$

$$= \sum_{\substack{\text{All trivalent} \\ \text{graphs } \Gamma}} \sum_{\substack{\text{All orientable} \\ \text{reduced} \\ \text{colorings } \varkappa \\ \text{of } E(\Gamma)}} \frac{(-\hbar)^{g(\Gamma)-1} \cdot 2^{b_1(\Delta(\varkappa))}}{|\mathrm{Aut}(\Gamma,\varkappa)|} \prod_{e \in E(\Gamma)} \sigma_{\varkappa(e)}. \quad (3.34)$$

Here $b_1(\Delta(\varkappa))$ means the first Betti number of the graph $\Delta(\varkappa)$.

Of course, the terms in (3.34) are homogeneous in Λ and $\hbar$: the terms corresponding to genus g graphs have degree $-3g + 3$ in Λ.

For convenience we rewrite these expansions once more after the substitution inverse to (3.5):

$$\log \int \exp\left(\frac{\sqrt{-1}}{6} \operatorname{tr} X^3\right) d\mu_\Lambda(X) \sim \sum_{\substack{\text{All trivalent} \\ \text{graphs } \Gamma}} (-1)^{g(\Gamma)-1} \sum_{\substack{\text{All orientable} \\ \text{colorings } \mu \\ \text{of } E(\Gamma)}} \frac{1}{|\mathrm{Aut}(\Gamma,\mu)|} \prod_{e \in E(\Gamma)} \sigma_{\mu(e)}$$

$$= \sum_{\substack{\text{All trivalent} \\ \text{graphs } \Gamma}} (-1)^{g(\Gamma)-1} \sum_{\substack{\text{All reduced} \\ \text{orientable} \\ \text{colorings } \varkappa \\ \text{of } E(\Gamma)}} \frac{2^{b_1(\Delta(\varkappa))}}{|\mathrm{Aut}(\Gamma,\varkappa)|} \prod_{e \in E(\Gamma)} \sigma_{\varkappa(e)}. \quad (3.35)$$

In [4] the same asymptotic expansion is described in terms of ribbon graphs. By definition a ribbon graph is a graph Γ with fixed cyclic order of half-edges at each vertex; let us denote this additional data by $\mathcal{D}$. The pair $(\Gamma,\mathcal{D})$ defines a unique embedding of the graph in an orientable surface and a choice of one side of this surface. A coloring of the faces φ (3.27) defines a coloring $\varkappa(\varphi)$ (3.28) of $E(\Gamma)$, and therefore the factors $\sigma_{\varkappa(\varphi)}(e)$ are defined for the edges of Γ. The description of [4] in these terms looks as follows.

Theorem 3.2 [4].

$$\log\int\exp\left(\frac{\sqrt{-1}}{6}\operatorname{tr}X^3\right)d\mu_\Lambda(X) \sim \sum_{\substack{\text{All trivalent}\\ \text{ribbon graphs }(\Gamma,\mathcal{D})}}(\sqrt{-1})^{|V(\Gamma)|}\sum_{\substack{\text{All colorings}\\ \text{of faces }\varphi}}\frac{2^{|E(\Delta(\varkappa(\varphi)))|-|V(\Gamma)|}}{|\mathrm{Aut}(\Gamma,\mathcal{D},\varphi)|}\prod_{e\in E(\Gamma)}\sigma_{\varkappa(\varphi)}(e). \tag{3.36}$$

Now it is not hard to recognize in (3.35) a refinement of (3.36). The connection is given by the following class formula involving automorphisms of colored ribbon graphs and colored ordinary graphs.

Proposition 3.6. *Let Γ be a trivalent graph and*

$$\varkappa\colon\ E(\Gamma)\to\Omega_0\cup\Omega_+ \tag{3.37}$$

be an orientable reduced coloring satisfying (3.16). *Then*

$$\frac{2^{b_0(\Delta(\varkappa))}}{\mathrm{Aut}(\Gamma,\varkappa)}=\sum_{\substack{\text{All ribbon graph}\\ \text{structures }\mathcal{D}\text{ on }\Gamma}}\ \sum_{\substack{\text{All colorings}\\ \text{of faces }\varphi\text{ such}\\ \text{that }\varkappa(\varphi)=\varkappa}}\frac{1}{\mathrm{Aut}(\Gamma,\mathcal{D},\varphi)}, \tag{3.38}$$

where $b_0(\Delta(\varkappa))$ is the zero Betti number (*the number of connected components*) *of the graph $\Delta(\varkappa)$.*

We prove this formula in the next section; since

$$|E(\Delta(\varkappa))|-|V(\Gamma)|=b_1(\Delta(\varkappa))-b_0(\Delta(\varkappa)), \tag{3.39}$$

formula (3.38) provides a proof of Theorem 3.2 that is independent of [4].

Remark. Let us indicate two extremal special cases of formula (3.38).

In the case of a *proper* coloring $\varkappa$ each edge separates two faces colored by different colors; this means that $\Delta(\varkappa)=\Gamma$ and therefore $b_0(\Delta(\varkappa))=1$. According to Proposition 3.4 the embedding and the coloring in this case are determined uniquely and so the right-hand side of (3.38) has at most two summands, which correspond to two possible choices of a side of the orientable surface S. If the two corresponding ribbon graphs are different, each of them has the same automorphism group as $(\Gamma,\varkappa)$. If these two ribbon graphs are isomorphic, then the right-hand side of (3.38) has only one summand.

In the opposite case $n=1$ we in fact have no colorings at all. So on the right-hand side we have the sum over all possible ribbon graph structures $\mathcal{D}$ on Γ. For this case (3.38) yields

$$\frac{2^{|V(\Gamma)|}}{\mathrm{Aut}(\Gamma)}=\sum_{\substack{\text{All ribbon graph}\\ \text{structures }\mathcal{D}\text{ on }\Gamma}}\frac{1}{\mathrm{Aut}(\Gamma,\mathcal{D})}. \tag{3.40}$$

4. RIBBON GRAPHS AND COLORED GRAPHS

In this section we prove all the topological statements from the previous section that concern colored graphs and ribbon graphs. To this end recall the notion of (discrete) *connection* on a graph (see [6]). Let us fix the following notations. Directed edges of the graph will be denoted by $\overrightarrow{e}$; the same edge with the opposite orientation will be denoted by $\overleftarrow{e}$. The two vertices incident to $\overrightarrow{e}$ are denoted by $s(\overrightarrow{e})$ and $t(\overrightarrow{e})$ so that $\overrightarrow{e}$ is directed from $s(\overrightarrow{e})$ to $t(\overrightarrow{e})$. A *path* $\overrightarrow{l}$ is a sequence of directed edges $\overrightarrow{l} = [\overrightarrow{e_1}, \dots, \overrightarrow{e_m}]$ such that $t(\overrightarrow{e_i}) = s(\overrightarrow{e_{i+1}})$ and $\overrightarrow{e_i} \neq \overleftarrow{e_{i+1}}$; for a closed path we also need to claim $\overrightarrow{e}_1 \neq \overleftarrow{e}_m$ and $t(\overrightarrow{e_m}) = s(\overrightarrow{e_1})$. For a path $\overrightarrow{l}$ we define $s(\overrightarrow{l}) = s(\overrightarrow{e_1})$ and $t(\overrightarrow{l}) = t(\overrightarrow{e_m})$. A path is called *simple* if all the edges $e_1, \dots, e_m$ are different; a simple path in a trivalent graph is evidently a simple directed cycle.

For a graph without tails it is natural to identify the set of half-edges incident to a vertex v with the set of oriented edges directed from v. We will denote this set by $\mathrm{St}(v)$. For a trivalent graph all the sets $\mathrm{St}(v)$ have three elements.

By definition a *connection* ∇ on a graph Γ is a collection of identifications

$$\nabla_{\overrightarrow{e}}\colon\ \mathrm{St}(s(\overrightarrow{e})) \to \mathrm{St}(t(\overrightarrow{e})) \tag{4.1}$$

such that

$$\nabla_{\overrightarrow{e}}(\overrightarrow{e}) = \overleftarrow{e} \tag{4.2}$$

and

$$\nabla_{\overrightarrow{e}}^{-1} = \nabla_{\overleftarrow{e}}. \tag{4.3}$$

Then for any path $\overrightarrow{l} = [\overrightarrow{e_1}, \dots, \overrightarrow{e_m}]$ we can define

$$\nabla_{\overrightarrow{l}} = \nabla_{\overrightarrow{e_m}} \circ \dots \circ \nabla_{\overrightarrow{e_1}}\colon\ \mathrm{St}(s(\overrightarrow{l})) \to \mathrm{St}(t(\overrightarrow{l})); \tag{4.4}$$

for a closed path $\overrightarrow{l}$ with $s(\overrightarrow{l}) = t(\overrightarrow{l}) = v$ there is a monodromy map

$$\nabla_{\overrightarrow{l}}\colon\ \mathrm{St}(v) \to \mathrm{St}(v). \tag{4.5}$$

A path $\overrightarrow{l} = [\overrightarrow{e_1}, \dots, \overrightarrow{e_m}]$ is called *geodesic* if

$$\nabla_{\overrightarrow{e_{i+1}}}(\overleftarrow{e_i}) = \overrightarrow{e_{i+2}}; \tag{4.6}$$

evidently the inverse path $\overleftarrow{l} = [\overleftarrow{e_m}, \dots, \overleftarrow{e_1}]$ is then geodesic too.

It is not hard to prove (see [6]) that all maximal geodesic paths are closed; any two half-edges incident to one vertex define a geodesic uniquely (up to orientation); each edge appears in exactly two different geodesic paths or twice in the same geodesic path. Therefore, gluing the border of a 2-cell to each maximal geodesic, we obtain a compact surface S_∇ with an embedding of the graph Γ in S_∇. Evidently the collection of closed geodesics defines the connection uniquely, and therefore an embedding of the graph in a compact surface defines the connection uniquely.

Now we start with a graph Γ with a reduced coloring of edges

$$\varkappa\colon\ E(\Gamma) \to \Omega_0 \cup \Omega_+. \tag{4.7}$$

Our goal is to construct a connection ∇ on Γ satisfying the following property:

For any closed geodesic there exists i such that all the edges of this geodesic are colored with $\underline{ij}$ for some j (j may be different for different edges of this geodesic). (4.8)

For any edge $e \in E$ colored with $\underline{ij} \in \Omega_+$ (i.e., $i \neq j$), condition (4.8) defines $\nabla_{\overrightarrow{e}}$ uniquely: the remaining two edges incident to $s(\overrightarrow{e})$ are colored with $\underline{ik}$ and $\underline{jk}$ for some k, and the edges incident to $t(\overrightarrow{e})$ are colored with $\underline{il}$ and $\underline{jl}$ for some l. Therefore, $\nabla_{\overrightarrow{e}}$ must map the $\underline{ik}$-edge to the $\underline{il}$-edge and the $\underline{jk}$-edge to the $\underline{jl}$-edge. On the contrary, for $i = j$ both ways of defining $\nabla_{\overrightarrow{e}}$ satisfy (4.8). Thus we obtain the following.

Proposition 4.1. *Let* Γ *be a trivalent graph with reduced coloring of edges* $\varkappa$ (4.7) *satisfying* (3.16).

1. *There is a one-to-one correspondence between embeddings of* Γ *in a compact surface* S *together with colorings of faces*

$$\varphi\colon\ F(S) \to \{1, \dots, n\} \tag{4.9}$$

generating $\varkappa$ *and connections on* Γ *satisfying* (4.8).

2. *A connection* ∇ *on* Γ *satisfying* (4.8) *is defined uniquely on* $\Delta(\varkappa)$ *and may be defined in an arbitrary way on the edges colored with* Ω_0. *Such a connection is unique if and only if* $\Gamma = \Delta(\varkappa)$.

As a consequence we obtain Proposition 3.4.

Next let us prove Proposition 3.5. Consider the coloring

$$\mu\colon\ E(\Gamma) \to \Omega \tag{4.10}$$

defined by the coloring $\varkappa$ (4.7) and by a cycle $\eta \in H_1(\Delta(\varkappa), \mathbb{Z}_2)$. Recall (see Proposition 3.3) that μ is obtained from $\varkappa$ by changing the colors for edges of η from $\underline{ij}$ to $\overline{ij}$. In turn, the coloring μ defines the values of $a_{\nu(v)}$ for vertices of the graph according to formula (3.22). Consider an embedding of the graph Γ in a compact surface S together with a coloring φ (4.9) satisfying (3.28). The cycle η is a sum of simple cycles; pick one of these cycles c and fix an orientation on it. Consider a small neighborhood S_c of the cycle c in the surface S; for a face $f \in F(S)$ denote its intersection with S_c by $\bar{f} = f \cap S_c$. For each edge $\overrightarrow{e}$ of the cycle $\overrightarrow{c}$ consider the two faces $f', f'' \in F(S)$ incident to e; denote $W_e = \bar{f}' \cup \bar{f}''$. Since $e \in \eta$, $\mu(e) \in \Omega_-$ and therefore $\varphi(f') \neq \varphi(f'')$, say, $\varphi(f') > \varphi(f'')$. Then let us choose an orientation on W_e in such a way that the orientation of $\overrightarrow{e}$ is induced by the orientation of f'.

Take any vertex v of c and consider the three edges e_1, e_2, and e_3 incident to v and the three faces f_1, f_2, and f_3 meeting at v, so that e_m is not incident to $\bar{f}_m$ for $m = 1, 2, 3$. Let $e_1, e_2 \in c$ and $e_3 \notin c$; then $W_{e_1} \cap W_{e_2} = \bar{f}_3$. Evidently the defined orientations on W_{e_1} and W_{e_2} coincide on $\bar{f}_3$ if and only if $\bar{a}_{\{\mu(e_1)\mu(e_2)\mu(e_3)\}} = 1$, and the orientations are opposite if and only if $\bar{a}_{\{\mu(e_1)\mu(e_2)\mu(e_3)\}} = -1$. Therefore, the neighborhood S_c is orientable if and only if c contains an even number of negative vertices, which proves Proposition 3.5.

Thus we have proved that if at least for one choice of $c \in H_1(\Delta(\varkappa), \mathbb{Z}_2)$ the product (3.26) is negative, then the surface S is nonorientable. In the previous section we called the corresponding colorings nonorientable. Unfortunately, an orientable coloring may also sometimes be generated by an embedding of the graph in a nonorientable surface. Nevertheless, it is not hard to enumerate the corresponding embeddings in orientable surfaces.

Proposition 4.2. *Let* Γ *be a trivalent graph with a reduced orientable coloring of edges* $\varkappa$ (3.18). *Then there are exactly*

$$2^{b_0(\Delta(\varkappa))-1} \tag{4.11}$$

connections ∇ *on* Γ *satisfying* (4.8) *such that the surface* S_∇ *is orientable.*

Corollary 4.1. *There are exactly*

$$2^{b_0(\Delta)} \tag{4.12}$$

ribbon graph structures (*together with a coloring of faces*) *on* Γ *generating the coloring of edges* $\varkappa$.

Recall that a ribbon graph structure $\mathcal{D}$ on a graph Γ is a collection of cyclic permutations δ_v of half-edges of $\mathrm{St}(v)$ for all the vertices of the graph:

$$\mathcal{D} = \bigl\{\delta_v\colon\ \mathrm{St}(v) \to \mathrm{St}(v),\ \delta_v \text{ is cyclic},\ v \in V(\Gamma)\bigr\}. \tag{4.13}$$

A ribbon graph structure $\mathcal{D}$ defines an embedding of the graph in an orientable surface S together with a choice of one side of this surface and is defined uniquely by these data. Therefore, $\mathcal{D}$ defines a connection $\nabla_{\mathcal{D}}$ on Γ; the opposite ribbon graph structure $\mathcal{D}^{-1} = \{\delta_v^{-1}, v \in V(\Gamma)\}$ corresponding to the choice of another side of the same surface S defines the same embedding and therefore the same connection: $\nabla_{\mathcal{D}} = \nabla_{\mathcal{D}^{-1}}$. The cycles δ_v for adjacent vertices are anti-conjugate under the connection: for a directed edge $\overrightarrow{e}$

$$\delta_{t(\overrightarrow{e})} = \nabla_{\overrightarrow{e}}\, \delta^{-1}_{s(\overrightarrow{e})} \nabla^{-1}_{\overrightarrow{e}}. \tag{4.14}$$

Evidently the converse is also true: any ribbon structure $\mathcal{D}$ satisfying (4.14) for all edges defines exactly the embedding S_∇.

For each vertex $v \in V(\Gamma)$ let us denote by $\mathcal{J}_v$ the set of all cyclic orders on $\mathrm{St}(v)$; for a trivalent graph $\mathcal{J}_v$ has exactly two elements. Define the action ∇_* of ∇ on $\mathcal{J}$ according to (4.14):

$$\nabla_{\overrightarrow{e}*}\colon\ \mathcal{J}_{s(\overrightarrow{e})} \to \mathcal{J}_{t(\overrightarrow{e})}, \tag{4.15}$$

$$\nabla_{\overrightarrow{e}*}(\delta) = \nabla_{\overrightarrow{e}}\, \delta^{-1} \nabla^{-1}_{\overrightarrow{e}}. \tag{4.16}$$

Then for any edge $e \in E(\Gamma)$ the composition $\nabla_{\overleftarrow{e}*} \circ \nabla_{\overrightarrow{e}*}$ is the identity mapping on $\mathcal{J}_{s(\overrightarrow{e})}$. Now for any path $\overrightarrow{l} = [\overrightarrow{e}_1, \dots, \overrightarrow{e}_m]$ we can define

$$\nabla_{\overrightarrow{l}*} = \nabla_{\overrightarrow{e}_m*} \circ \dots \circ \nabla_{\overrightarrow{e}_1*}\colon\ \mathcal{J}_{s(\overrightarrow{l})} \to \mathcal{J}_{t(\overrightarrow{l})}; \tag{4.17}$$

for a closed path $\overrightarrow{l}$ with $s(\overrightarrow{l}) = t(\overrightarrow{l}) = v$ there is a monodromy map

$$\nabla_{\overrightarrow{l}*}\colon\ \mathcal{J}_v \to \mathcal{J}_v. \tag{4.18}$$

A connection ∇ is called *flat*[1] if for any closed path $\overrightarrow{l}$ the monodromy is identical:

$$\nabla_{\overrightarrow{l}*} = \mathrm{Id}_{\mathcal{J}_{s(\overrightarrow{l})}}. \tag{4.19}$$

Evidently a connection is flat if (4.19) holds for any simple cycle $\overrightarrow{l}$.

Proposition 4.3. 1. *A connection ∇ is flat if and only if the surface S_∇ is orientable.*

2. *Let ∇ be a flat connection on Γ. The two possible ribbon structures on Γ arising from the embedding in S_∇ are defined by a choice of a cyclic order $\delta_0 \in \mathcal{J}_{v_0}$ at one fixed vertex $v_0 \in V(\Gamma)$; the corresponding cyclic order at any other vertex $v \in V(\Gamma)$ is defined by $\nabla_{\overrightarrow{l}*}(\delta_0)$, where $\overrightarrow{l}$ is an arbitrary path connecting v_0 and v (i.e., $v_0 = s(\overrightarrow{l})$ and $v = t(\overrightarrow{l})$).*

This proposition is a trivial consequence of our definitions; the following trivial lemma describes the extension of a flat connection defined on a subgraph to a larger subgraph.

Lemma 4.1. 1. *Let Γ' be a connected subgraph of Γ. Consider an edge $e \in E(\Gamma)$, $e \notin E(\Gamma')$, that connects two vertices of Γ' and is colored with $\underline{ii} \in \Omega_0$. Let ∇ be a flat connection on Γ' satisfying* (4.8). *Then ∇ can be uniquely extended to a flat connection on the graph $\Gamma' \cup \{e\}$ satisfying the color property* (4.8).

[1] Note that our definition of a flat connection is different from the one given in [6].

2. *Let* Γ' *and* Γ'' *be two connected subgraphs of* Γ, $\Gamma' \cap \Gamma'' = \varnothing$. *Consider an edge* $e \in E(\Gamma)$, $e \notin E(\Gamma')$, $e \notin E(\Gamma'')$, *that connects a vertex from* Γ' *with a vertex from* Γ'' *and is colored with* $\underline{ii} \in \Omega_0$. *Let* ∇ *be a flat connection on* $\Gamma' \cup \Gamma''$ *satisfying* (4.8). *Then each of the two possible ways of extending* ∇ *to* e *provides a flat connection on* $\Gamma' \cup \Gamma'' \cup \{e\}$ *satisfying the color property* (4.8).

We have seen that for an edge e colored with Ω_0 each of the two possible ways to define ∇_e satisfies (4.8), and thus we only need to ensure the flatness of the extension. For assertion 2 of the lemma, e cannot be included in any simple cycle on $\Gamma' \cup \Gamma'' \cup \{e\}$ and therefore each of the two possible ways of extension is automatically flat. For assertion 1 there are two possibilities: e is either a loop or not a loop. In the former case the flatness condition means that $\nabla_{\vec{e}*}$ is the identity mapping; therefore $\nabla_{\vec{e}}$ is a transposition of the two half-edges of the loop. Evidently this provides a flat connection on $\Gamma' \cup \{e\}$. In the latter case there is a simple path $\vec{l}$ in Γ' connecting the ends of the edge e. Direct $\vec{e}$ in such a way that $s(\vec{e}) = s(\vec{l})$ and $t(\vec{e}) = t(\vec{l})$; then the flatness condition requires

$$\nabla_{\vec{e}*} = \nabla_{\vec{l}*}. \tag{4.20}$$

Since ∇ is flat on Γ', the right-hand side is independent of the choice of the path $\vec{l}$ and therefore (4.20) defines the required flat extension; it is not hard to see that $\nabla_{\vec{e}}$ is defined by (4.20) uniquely.

Lemma 4.1 provides an explicit description of all possible flat connections on Γ satisfying (4.8). First, recall that we have seen that the connection is uniquely determined on $\Delta(\varkappa)$. ($\Delta(\varkappa)$ was defined as a subgraph having the same set of vertices $V(\Gamma)$ but containing only those edges that are colored with Ω_+.) Consider one connected component Γ' of $\Delta(\varkappa)$; we have seen that ∇ is defined on Γ' uniquely. Pick another connected component Γ'' of $\Delta(\varkappa)$ such that there is an edge connecting Γ' and Γ''; part 2 of Lemma 4.1 states that there are exactly two ways of extending ∇ to $\Gamma' \cup \Gamma'' \cup \{e\}$. Now take the latter graph as Γ' and find another connected component of $\Delta(\varkappa)$ that is connected with it by an edge. We will repeat the above procedure until we join all the components of $\Delta(\varkappa)$. Clearly the number of possible flat connections is

$$2^{b_0(\Delta(\varkappa))-1}. \tag{4.21}$$

It remains to extend ∇ to all the remaining edges colored with Ω_0; according to assertion 1 of the lemma this extension exists and is unique. Thus we have proved the following.

Proposition 4.4. *A trivalent graph* Γ *with a reduced orientable coloring of its edges*

$$\varkappa\colon\ E(\Gamma) \to \Omega_0 \cup \Omega_+ \tag{4.22}$$

satisfying (3.16) *admits exactly*

$$2^{b_0(\Delta(\varkappa))-1} \tag{4.23}$$

flat connections satisfying the color property (4.8). *Each such flat connection defines an embedding of* Γ *in a compact orientable surface* S_∇ *together with a coloring of the faces*

$$\varphi\colon\ F(S_\nabla) \to \{1, \dots, n\} \tag{4.24}$$

generating the coloring of edges $\varkappa$ (*i.e., satisfying* (3.28)).

Since each embedding corresponds to exactly two ribbon structures, we obtain the following corollary.

Corollary 4.2. *There are exactly*

$$2^{b_0(\Delta(\varkappa))} \tag{4.25}$$

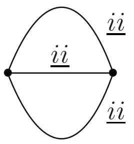
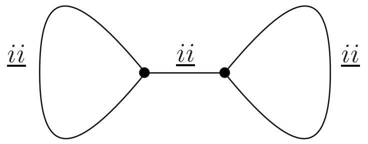
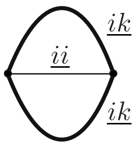
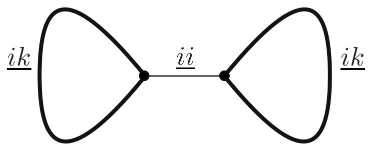
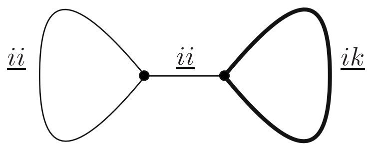
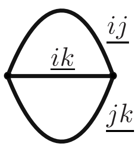
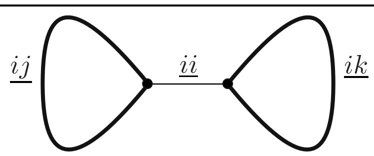
ribbon structures $\mathcal{D}$ *on* Γ (*together with colorings of the faces* φ (4.24)) *generating the reduced orientable coloring of edges* $\varkappa$.

Now the proof of Proposition 3.6 becomes trivial. The group $\mathrm{Aut}(\Gamma, \varkappa)$ acts on the set of all ribbon structures on Γ, and the orbits of this action correspond to all ribbon graphs with a coloring of faces (4.24) compatible with the coloring $\varkappa$.

5. THE FIRST TERM

In the table we list all genus 2 colored graphs corresponding to the first term of the expansion (3.34). There are only two stable genus 2 graphs, and it is not hard to see that all reduced colorings of these graphs are orientable.

Table

Graph Γ	$\lvert\mathrm{Aut}(\Gamma,\varkappa)\rvert$	$b_1(\Delta(\varkappa))$	$\dfrac{2^{b_1(\Delta(\varkappa))}}{\lvert\mathrm{Aut}(\Gamma,\varkappa)\rvert}\prod_{e\in E(\Gamma)}\sigma_{\varkappa(e)}$
	12	0	$\frac{1}{12}\frac{1}{\lambda_i^3}$
	8	0	$\frac{1}{8}\frac{1}{\lambda_i^3}$
 $i \neq k$	4	1	$\frac{1}{2}\frac{1}{\lambda_i}\frac{1}{(\lambda_i+\lambda_k)^2}$
 $i \neq k$	8	2	$\frac{1}{2}\frac{1}{\lambda_i}\frac{1}{(\lambda_i+\lambda_k)^2}$
 $i \neq k$	4	1	$\frac{1}{2}\frac{1}{\lambda_i^2}\frac{1}{\lambda_i+\lambda_k}$
 $i \neq j,\ i \neq k,\ j \neq k$	2	2	$\frac{2}{(\lambda_i+\lambda_j)(\lambda_i+\lambda_k)(\lambda_j+\lambda_k)}$
 $i \neq j,\ i \neq k,\ j \neq k$	4	2	$\frac{1}{\lambda_i(\lambda_i+\lambda_j)(\lambda_i+\lambda_k)}$

The edges of the graph $\Delta(\varkappa)$ are drawn by thick lines. Of course, it is not hard to see that the sum of the entries of the last column in accordance with [4] is

$$\frac{1}{6}\left(\sum\frac{1}{\lambda_i}\right)^3+\frac{1}{24}\sum\frac{1}{\lambda_i^3}. \tag{5.1}$$

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