ISSN 0001-4346, Mathematical Notes, 2012, Vol. 92, No. 6, pp. 807–819.

Local Finitely Smooth Equivalence of Real Autonomous Systems with Two Pure Imaginary Eigenvalues

V. S. Samovol*

National Research University Higher School of Economics, Moscow

Received October 6, 2011

Abstract—The paper deals with real autonomous systems of ordinary differential equations in a neighborhood of a nondegenerate singular point such that the matrix of the linearized system has two pure imaginary eigenvalues, all other eigenvalues lying outside the imaginary axis. It is proved that, for such systems having a focus on the center manifold, the problem of finitely smooth equivalence is solved in terms of the finite segments of the Taylor series of their right-hand sides.

DOI: 10.1134/S0001434612110260

Keywords: *autonomous system of ordinary differential equations, finitely smooth equivalence of systems, pseudonormal form, resonance, shearing transformation.*

1. INTRODUCTION

The present paper is a continuation of the studies begun in [1]–[3]. Here we study the problem of local finitely smooth equivalence of real autonomous systems of ordinary differential equations with two pure imaginary eigenvalues of the matrix of the linear part, whose right-hand sides have Taylor series differing by high-order terms (the so-called *problem of finite-definite germs of vector fields*). Most papers concerned with this subject area deal with systems with a nondegenerate singular point (this range of problems is extensively covered in the book [4]). With regard to partially degenerate systems, see [5]. The papers [6], [7], study the problem of infinitely smooth equivalence of formally equivalent systems with one zero root or two pure imaginary eigenvalues. Our approach to the solution of the problem of finitely smooth equivalence is based on the reduction of such systems to a special pseudonormal form (see [3]). Although transformations with singularities are used, the proposed method, nevertheless, allows us to establish a criterion for finitely smooth equivalence of the systems under consideration.

Consider the real autonomous system

$$\dot{\xi} = \frac{d\xi}{dt} = Q(\xi), \tag{1}$$

where ξ, $Q(\xi) \in \mathbb{R}^{n+2}$, $n > 0$, $Q(\xi)$ is a C^∞ function in a neighborhood of the origin, $Q(0) = 0$, and the matrix $\widetilde{A} = Q'(0)$ has n eigenvalues lying outside the imaginary axis and a pair of pure imaginary (conjugate) eigenvalues. Suppose that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix $\widetilde{A}$ with nonzero real part, $\pm i\omega$ constitute the pair of pure imaginary eigenvalues of the given matrix, $\omega > 0$, and i is the imaginary unit.

Using a standard linear transformation, let us reduce system (1) to the following form, where the matrix $\widetilde{A}$ is a Jordan one:

$$\begin{aligned} \dot{x}_1 &= i\omega x_1 + f_1(x, y), \\ \dot{x}_2 &= -i\omega x_2 + f_2(x, y), \\ \dot{y}_j &= \varepsilon_j y_{j+1} + \lambda_j y_j + g_j(x, y), \qquad j = 1, \dots, n. \end{aligned} \tag{2}$$

*E-mail: 555svs@mail.ru

Here x, y are complex coordinates, $x = (x_1, x_2)$, $x_2 = \overline{x}_1$, $y = (y_1, \dots, y_n)$, complex conjugate variables satisfy complex conjugate equations (see, for example, [8, pp. 167–168]), and the Taylor series of the functions f_1, f_2, g_j, $j = 1, \dots, n$ do not contain linear terms. We shall say that the variables x are *degenerate*, while the variables y, respectively, are nondegenerate. It is important that all subsequent transformations of complex systems have the property that complex conjugate variables and the corresponding equations are taken to complex conjugate ones. To every transformation of this kind there corresponds a real transformation of the original system. We refer to this principle (condition) as the *reality principle* (*condition*).

By Theorem 1 from [3] we assume that the Taylor series of the right-hand side of system (2) is the sum of resonant monomials over all variables (both degenerate and nondegenerate).

Let us recall some notions introduced in [3]. An important feature of the systems under consideration are resonances, which arise in the cases where $(\lambda_{j_1} - \lambda_{j_2})/i\omega$ is an integer for any $1 \le j_1 \ne j_2 \le n$. Such eigenvalues, as well as the corresponding variables and equations in the system, will be said to be *equivalent* to each other. If the above-mentioned resonances are absent, then system (1) can always be reduced to polynomial normal form by a transformation of finite smoothness [9, Theorem 3]. Of the above-mentioned resonances, we single out those for which $\lambda_{j_1} = \overline{\lambda_{j_2}}$. This means that $2\omega^{-1}\operatorname{Im}\lambda_{j_1}$ is an integer. The resonances for which this number is odd will be said to be *singular*. The variables corresponding to the eigenvalues λ for which $2\omega^{-1}\operatorname{Im}\lambda$ is odd will be referred to as *singular* variables as well, while if this number is not odd (or is noninteger), then we say that the corresponding variables are *nonsingular*.

By the arguments given in [3], we assume that, in system (2), there is a two-dimensional invariant central manifold defined by the equation $y = 0$; on this manifold, the integral curves in a neighborhood of the singular point are either closed (the case of a center) or are spirals (the case of a focus). The present paper studies systems of the form (1) having a focus on a center manifold. Consider such a system reduced to the form (2). Following [3], [10], we consider the system that, on a given central manifold, is of the form

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})), \\ \dot{x}_2 &= x_2(-i\omega - i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})). \end{aligned} \tag{3}$$

Here and elsewhere,

$$\widetilde{r} = x_1x_2, \qquad \varphi(\widetilde{r}) = \sum_{j=1}^{m} \varphi_j \widetilde{r}^{\,j}, \qquad \widetilde{b}(\widetilde{r}) = b\widetilde{r}^{\,m} + c\widetilde{r}^{\,2m},$$

$m \ge 1$ is an integer, $\varphi_1, \dots, \varphi_m$, b, c are real numbers, and $b \ne 0$. Note that the variable $\widetilde{r}$ satisfies the equation $\dot{\widetilde{r}} = 2\widetilde{r}\widetilde{b}(\widetilde{r})$.

Systems (1) in which $\widetilde{b}(\widetilde{r})$ is a flat function constitute an *exceptional set* (by the terminology used in [7]) of codimension equal to infinity. We consider systems not appearing in this set.

Consider the linear (in nondegenerate coordinates) part of system (2)

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})), \\ \dot{x}_2 &= x_2(-i\omega - i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})), \\ \dot{y} &= A(x)y. \end{aligned} \tag{4}$$

Note that, by a nondegenerate C^∞ transformation and the corresponding numbering of the variables, we can ensure that the matrix $A(x)$ is of block-diagonal form, where the individual blocks will contain only groups of equivalent variables (see [3, Introduction]).

Let us transform system (4) using the so-called shearing and weakly degenerate transformations (see [1]–[3], [11]). Let us present the corresponding definitions.

Definition 1. By *shearing transformations* we mean transformations of the form

$$y = S(\widetilde{r})z, \qquad y = S_1(x_1)z, \qquad y = S_2(x_2)z, \tag{5}$$

where $S(\eta) = \operatorname{diag}(\eta^{\delta_1}, \eta^{\delta_2}, \dots, \eta^{\delta_n})$, the δ_q are rational numbers, $S_j(\eta) = \operatorname{diag}(\eta^{h_{1j}}, \eta^{h_{2j}}, \dots, \eta^{h_{nj}})$, and the h_{qj} are integers.

Definition 2. A *weakly degenerate transformation* is a change of variables of the form

$$\widetilde{r} = du^l, \qquad y = T^* z = BVTz. \tag{6}$$

Here $B = S_1(x)S_2(x)$, $T = T(\widetilde{r})$ is the product of finitely many transformations, some of which are shearing transformations of the form $S(\widetilde{r})$ and the remaining ones are C^∞ transformations close to the identity. The number l is a positive integer, and $d = \mathrm{const} > 0$. The transformation V has the same block-diagonal structure $V = \mathrm{diag}(V_1, \dots, V_K)$ as the matrix $A(x)$. Moreover, the V_j are the identity transformations for the blocks that do not include complex conjugate equations (i.e., the corresponding numbers $2\omega^{-1}\operatorname{Im}\lambda_q$ are noninteger), and the transformations V_j have the form

$$V_j = \mathrm{diag}(e^{i\widetilde{h}\alpha}E_1, e^{-i\widetilde{h}\alpha}E_1, E_2), \qquad \alpha = \arg x_1, \quad e^{i\alpha} = \frac{x_1}{\sqrt{\widetilde{r}}},$$

where $\widetilde{h}$ is the least of the positive numbers $h_q = \omega^{-1}\operatorname{Im}\lambda_q$ belonging to the block considered. Here E_1 and E_2 are identity matrices of sizes corresponding to the groups of complex and real variables in the block in question.

Remark 1. For blocks consisting of singular variables, the number $2\widetilde{h}$ is odd; therefore, the matrix $e^{i\widetilde{h}\alpha}E_1$ from Definition 2 can be expressed as

$$e^{i\widetilde{h}\alpha}E_1 = \sqrt{e^{i\alpha}}\, e^{ih_1\alpha}E_1,$$

where $h_1 = \widetilde{h} - 1/2$ is an integer. In this case, the transformation (6) is discontinuous at the points where x_1 is a real positive number and nondegenerate of class C^∞ at all other points where $x \neq 0$.

The main result concerning system (4) is the following theorem (Theorem 2 from [3]).

Theorem 1. *There exists a weakly degenerate transformation* (6) *reducing system* (4) *to the following pseudonormal form*:

$$\begin{aligned} \dot u_1 &= u_1(i\omega + i\psi(u) + b_1u^p + c_1u^{2p}),\\ \dot u_2 &= u_2(-i\omega - i\psi(u) + b_1u^p + c_1u^{2p}),\\ \dot z &= (A_0 + uA_1 + \dots + u^{p-1}A_{p-1} + u^pA_p)z. \end{aligned} \tag{7}$$

Here $u = u_1u_2$, $\psi(u) = \sum_{h=1}^{p}\psi_h u^h$, $\psi_1, \dots, \psi_p$, b_1, and c_1 are real numbers, $A_0, A_1, \dots, A_{p-1}$ are constant diagonal matrices, A_p is a constant Jordan matrix, and $p = 2ml$. The matrix A_p has the property that if two diagonal entries $\lambda^h_{j_1}$ and $\lambda^h_{j_2}$ of some matrix A_h, $0 \le h \le p-1$, are distinct, then the corresponding diagonal entries of the Jordan matrix A_p belong to distinct Jordan blocks. The variable $u = u_1u_2$ satisfies the equation

$$\dot u = 2(b_1u^{p+1} + c_1u^{2p+1}).$$

By an appropriate choice of d in the transformation (6)*, one can ensure that $|b_1|$ is an arbitrary given positive number.*

Remark 2. Any finite segments of the Taylor series of the C^∞ transformations occurring as factors in T are determined, according to Theorem 1, by finite segments of the Taylor series of the matrix $A(x)$ of system (4). The total number of factors forming T depends as well only on a finite segment of the Taylor series of $A(x)$, the length of this segment depending on m and n. The shearing transformations occurring as factors in T are determined by a finite segment of the Taylor series of $A(x)$, the length of this segment being some number $m_1(m,n)$. The number l depends on n alone. Note also that, for $x \neq 0$, the transformation T in (6) is a nondegenerate C^∞ transformation. The same is true for V, provided that none of the numbers $2\omega^{-1}\operatorname{Im}\lambda_q$ is odd (i.e., there are no singular variables). If some of the integers $2\omega^{-1}\operatorname{Im}\lambda_q$ are odd (i.e., there are singular variables), then the transformation V is discontinuous at the points where x_1 is a real positive number and nondegenerate of class C^∞ at all other points where $x \neq 0$.

Let us now pass to the problem of finitely smooth equivalence of the systems of equations. Consider two systems of the form (1) whose N-jets of the right-hand side coincide.

Remark 3. If two systems of the form (1) differ by terms of order N, where N is greater than the number $m_1 = m_1(m, n)$ (see Remark 2), then system (7) from Theorem 1 will coincide with them. In addition, the shearing transformations occurring as multipliers in the weakly degenerate transformation (6) reducing the linear parts of the systems to the form (7) will also coincide. Further, the multipliers of class C^∞ occurring in (6) will differ for our systems; however, they differ by terms of order higher than some number $N_1 = N_1(N)$, with $\lim_{N\to\infty} N_1(N) = \infty$.

The following theorem solves the problem of the finitely smooth equivalence of systems of the form (1) having a focus on a center manifold.

Theorem 2. *For any integer $k > 0$, there exists an integer N possessing the following property*: *if the Taylor series of the right-hand side of two systems of the form* (1)*, having a focus on a center manifold and not belonging to an exceptional set, differ by terms of order higher than N* (*their N-jets coincide*)*, then these systems are locally C^k-equivalent, i.e., there exists a nondegenerate* (*close to identical*) *C^k transformation reducing one system to the other in a small neighborhood of the origin. The number N depends on the numbers k, m, n, as well as on the coefficients of the segment of the Taylor series of the matrix $A(x)$ whose length is equal to some number $m_1(m, n)$.*

For nondegenerate systems, the assertion of Theorem 2 is due to Sternberg and Chen (see Theorem 12.2 from [4, Chap. IX]). A certain general result for partially degenerate systems was obtained in [5], where the corresponding statement was proved under the condition that the segments of Taylor series in the expansions of the right-hand sides of systems in terms of nondegenerate coordinates coincide. Theorem 2 generalizes the Sternberg–Chen theorem to the case of systems with two pure imaginary eigenvalues. A similar theorem for a system with one zero eigenvalue was proved in [2].

2. EXAMPLES

Here we illustrate the proof of Theorem 2 on finitely smooth equivalence using two examples.

Example 1. Consider the system of equations

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + x_1x_2) + f_1(x, y), \\ \dot{x}_2 &= x_2(-i\omega + x_1x_2) + f_2(x, y), \\ \dot{y}_1 &= \lambda_1 y_1 + g_{11}(x)y_1 + g_{12}(x)y_2 + g_1(x, y), \\ \dot{y}_2 &= \lambda_2 y_2 + g_{21}(x)y_1 + g_{22}(x)y_2 + g_2(x, y). \end{aligned} \tag{8}$$

Here the variables x_1 and x_2, as well as y_1 and y_2, are complex conjugate, the same is true for the pair (λ_1, λ_2), and complex conjugate variables satisfy complex conjugate equations. Moreover,

$$f_l(x, 0) = g_{lj}(0) = 0, \quad |g_l(x, y)| = o(\|y\|), \qquad 1 \le l, j \le 2.$$

Let

$$\lambda_1 = a + \beta i, \quad \lambda_2 = a - \beta i, \qquad a \ne 0, \quad \beta > 0, \qquad \text{and} \qquad \omega > 0.$$

In contrast to the general case (see system (3)), we assume that $\varphi(\widetilde{r}) = 0$ and $\widetilde{b}(\widetilde{r}) = \widetilde{r}$. Note, however, that all the results remain valid in the general case.

We can assume, without loss of generality, that the Taylor series of the functions on the right-hand sides in this system are sums of resonant monomials. However, the resonant monomials in the last two equations of the system can only be linear in the nondegenerate variables, while the first two equations contain no such resonant monomials related to the functions $f_j(x, y)$, $j = 1, 2$, at all. Consequently, all functions $f_j(x, y)$ and $g_j(x, y)$ are flat. Since, according to [7], the formal equivalence of systems of the form (8) implies their C^∞ equivalence, we can assume that these functions are zero. Thus, without loss of generality, we assume that the system in question has the form

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + x_1x_2), \\ \dot{x}_2 &= x_2(-i\omega + x_1x_2), \\ \dot{y}_1 &= \lambda_1 y_1 + g_{11}(x)y_1 + g_{12}(x)y_2, \\ \dot{y}_2 &= \lambda_2 y_2 + g_{21}(x)y_1 + g_{22}(x)y_2. \end{aligned} \tag{9}$$

Also consider the following system similar to (9):

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + x_1x_2), \\ \dot{x}_2 &= x_2(-i\omega + x_1x_2), \\ \dot{\widetilde{y}}_1 &= \lambda_1 \widetilde{y}_1 + \widetilde{g}_{11}(x)\widetilde{y}_1 + \widetilde{g}_{12}(x)\widetilde{y}_2, \\ \dot{\widetilde{y}}_2 &= \lambda_2 \widetilde{y}_2 + \widetilde{g}_{21}(x)\widetilde{y}_1 + \widetilde{g}_{22}(x)\widetilde{y}_2. \end{aligned} \tag{10}$$

Assume that the corresponding functions from the right-hand side of systems (9) and (10) have coinciding N-jets.

The monomials $x_1^p x_2^q$ occurring in the Taylor series of the function $g_{12}(x)$ satisfy the resonance equation $(p-q)i\omega + \lambda_2 = \lambda_1$. A similar equation holds for $g_{21}(x)$. Consequently, if $h = (\lambda_1 - \lambda_2)/(i\omega)$ is not an integer, then the functions $g_{12}(x)$ and $g_{21}(x)$ are flat and can be assumed to be zero (see the above reasoning). In this case, the reduction of system (9) to normal form as a polynomial of third degree can be achieved by a C^∞ transformation (e.g., see Lemma 2 in [9]). All this is also valid for system (10). This implies that if, in the case under consideration, the 3-jets of systems (9) and (10) coincide, then these systems are C^∞-equivalent.

Now consider the case in which

$$h = \frac{\lambda_1 - \lambda_2}{i\omega} = \frac{2b}{\omega}$$

is an integer. The functions $g_{lj}(x)$, $1 \le l, j \le 2$, can be expressed as

$$g_{ll}(x) = d_{ll}(\widetilde{r}), \quad g_{lj}(x) = x_l^h d_{lj}(\widetilde{r}), \qquad l \ne j, \quad \widetilde{r} = x_1x_2,$$

where $d_{ll}(\widetilde{r})$ and $d_{lj}(\widetilde{r})$ are C^∞ functions. In addition, note that

$$d_{22}(\widetilde{r}) = \overline{d_{11}(\widetilde{r})}, \qquad d_{21}(\widetilde{r}) = \overline{d_{12}(\widetilde{r})}.$$

Let us make the change of variables

$$\begin{gathered} y = V(x)z, \qquad V(x) = \begin{pmatrix} e^{i\widetilde{h}\alpha} & 0 \\ 0 & e^{-i\widetilde{h}\alpha} \end{pmatrix}, \\ \widetilde{h} = \frac{h}{2}, \qquad e^{i\alpha} = \frac{x_1}{r}, \qquad \alpha = \arg x_1, \qquad r = \sqrt{\widetilde{r}}. \end{gathered} \tag{11}$$

In this example, we restrict ourselves to the case in which the number h is even and, therefore, there are no singular variables. The equations for the nondegenerate variables in system (9), together with the equation for $\widetilde{r}$, take the form

$$\begin{aligned} \dot{z}_1 &= az_1 + d_{11}(\widetilde{r})z_1 + \widetilde{r}^{\widetilde{h}} d_{12}(\widetilde{r})z_2, \\ \dot{z}_2 &= az_2 + \widetilde{r}^{\widetilde{h}} d_{21}(\widetilde{r})z_1 + d_{22}(\widetilde{r})z_2, \qquad a = \operatorname{Re}\lambda_1. \\ \dot{\widetilde{r}} &= 2\widetilde{r}^2, \end{aligned} \tag{12}$$

The resulting system of equations can be written as

$$\begin{aligned} \dot{z} &= aEz + \widetilde{r}B_1 z + o(\widetilde{r})z, \\ \dot{\widetilde{r}} &= 2\widetilde{r}^2, \end{aligned} \tag{13}$$

where E is the identity matrix.

If the matrix B_1 has no eigenvalues differing by a nonzero integer then (see [1]) system (13) can be reduced by a nondegenerate C^∞ transformation satisfying the reality condition to the form

$$\begin{aligned} \dot{w} &= aEw + \widetilde{r}A_1 w, \\ \dot{\widetilde{r}} &= 2\widetilde{r}^2, \end{aligned} \tag{14}$$

where A_1 is a constant Jordan matrix. Note that the entries of the matrix A_1 in this system are determined by the numbers $d_{ll}(0)$, $d_{lj}(0)$ from system (12). But if the eigenvalues of the matrix B_1 differ by a nonzero integer, then, using certain shearing transformations (see [1]), we can make the eigenvalues of the matrix B_1 equal to one another and, further, using a nondegenerate C^∞ transformation reduce the system to the form (14).

Note that, for an even h, all these changes of variables depend on the jet of the original system of some length P, where P depends on the eigenvalues of the matrix B_1. These replacements are nondegenerate for $x \neq 0$.

Thus, by making the change of variables

$$y = V(x)T(\widetilde{r})w, \tag{15}$$

the original system (9) reduces to the pseudonormal form (14), where $V(x)$ is the transformation (11) and $T(\widetilde{r})$ is the product of finitely many multipliers, some of which are shearing transformations and the remaining ones are C^∞ nondegenerate transformations.

Note that the similar transformation $\widetilde{y} = V(x)\widetilde{T}(\widetilde{r})w$ for system (10) differs from (15) only by the C^∞ multipliers referred to above. Moreover, these multipliers will have coinciding N-jets.

Now consider the transformation

$$y = R(x)\widetilde{y} = VT\widetilde{T}^{-1}V^{-1}\widetilde{y}. \tag{16}$$

Obviously, this transformation reduces system (9) to the form (10). Let us now show that, for a sufficiently large N, this transformation belongs to the class C^k with given $k > 0$. The corresponding multipliers of class C^∞ in the transformations T and $\widetilde{T}$ have coinciding N-jets and all the derivatives of the shearing multipliers up to order k in the transformations T and $\widetilde{T}^{-1}$ do not exceed in the norm the quantity $Dr^{-\widetilde{L}}$, where $D = \text{const}$, and $\widetilde{L} > 0$ is a number depending on k. The number of these multipliers is at most some number L_1 depending on the eigenvalues of the matrices B_1 in system (13), which, in turn, are determined by the numbers $d'_{11}(0)$, $d_{12}(0)$ from system (12). It is readily seen that the derivatives up to order k of the entries of the matrix $P(x) = T\widetilde{T}^{-1} - E$ do not exceed (in absolute value) the quantities

$$Dr^{N_1}, \qquad N_1 = N_1(N, k, \widetilde{L}, L_1) = N - L_2(k, \widetilde{L}, L_1), \qquad \lim_{N\to\infty} N_1 = \infty.$$

In addition, note that, for an even h, the matrix $V(x)$ is nondegenerate for $x \neq 0$. The derivatives up to order k of the entries of the matrices V, V^{-1} (for an even h) do not exceed (in absolute value) the value of $Dr^{-(h+k)}$.

It follows from these estimates that, for a sufficiently large N, the transformation (16) belongs to the class C^k and, therefore, the assertion concerning the finitely smooth equivalence of systems (9) and (10) is proved.

Example 2. Consider the system of equations

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + \widetilde{r}), \\ \dot{x}_2 &= x_2(-i\omega + \widetilde{r}), \\ \dot{y}_1 &= \lambda_1 y_1 + x_1 y_2 + \widetilde{r}^N y_3^2, \\ \dot{y}_2 &= \lambda_2 y_2 + x_2 y_1 + \widetilde{r}^N y_4^2, \\ \dot{y}_3 &= \lambda_3 y_3, \\ \dot{y}_4 &= \lambda_4 y_4. \end{aligned} \tag{17}$$

Here the variables x_1 and x_2, y_1 and y_2, and y_3 and y_4 are pairwise complex conjugate; the same is also true for the numbers λ_1 and λ_2 and λ_3 and λ_4; further, $\lambda_1 = 2\lambda_3$, $\lambda_1 = a + bi$, $a \neq 0$, $b > 0$, $\omega = 2b$, $\widetilde{r} = x_1 x_2$, and $N > 0$ is an integer. Here the singular variables are y_1 and y_2.

At the same time, consider the corresponding linear (in the nondegenerate variables) system of equations

$$\begin{aligned}
\dot{x}_1 &= x_1(i\omega + \widetilde{r}),\\
\dot{x}_2 &= x_2(-i\omega + \widetilde{r}),\\
\dot{\widetilde{y}}_1 &= \lambda_1 \widetilde{y}_1 + x_1 \widetilde{y}_2,\\
\dot{\widetilde{y}}_2 &= \lambda_2 \widetilde{y}_2 + x_2 \widetilde{y}_1,\\
\dot{\widetilde{y}}_3 &= \lambda_3 \widetilde{y}_3,\\
\dot{\widetilde{y}}_4 &= \lambda_4 \widetilde{y}_4.
\end{aligned} \tag{18}$$

Our goal is to prove Theorem 2 for system (17), i.e., to construct, for a given integer $k > 0$, an invertible C^k transformation reducing system (17) to the form (18); note that the number N will depend on k. Let us divide the construction of such a transformation into three stages. At the first stage, we construct the linear transformation T reducing both systems to pseudonormal form. Because of the singular variables, this transformation is discontinuous at the points where x_1 is a real positive number; in addition, it is noninvertible for $x = 0$. At all the other points, it is nondegenerate of class C^∞. Note that, at points of discontinuity, this transformation remains to invertible. At the second stage, we construct the transformation H reducing the pseudonormal form of system (17) resulting from the first stage to the pseudonormal form of system (18). At the third stage, we apply the transformation T^{-1} to the systems thus obtained (at the points where $x \neq 0$). Further, we show that the resulting transformation $R = T^{-1}HT$, extended by the identical one at $x = 0$, is close to the identical finitely smooth C^k transformation (for $2N > k + 3$) reducing system (17) to the form (18). Note that this scheme is realized in the proof of Theorem 2 in the general case.

In the following transformations, the variables x_1, x_2, y_3, y_4, $\widetilde{y}_3$, $\widetilde{y}_4$ and the corresponding equations will not vary; therefore, for simplicity, these equations will be omitted. Let us begin to carry out the first stage of our scheme.

Let us make the following transformations:

$$Y = V(\alpha)Z, \quad \widetilde{Y} = V(\alpha)\widetilde{Z}, \qquad Y^T = (y_1, y_2), \quad \widetilde{Y}^T = (\widetilde{y}_1, \widetilde{y}_2),$$
$$Z^T = (z_1, z_2), \quad \widetilde{Z}^T = (\widetilde{z}_1, \widetilde{z}_2),$$
$$V(\alpha) = \operatorname{diag}(e^{i\beta}, e^{-i\beta}), \qquad \beta = \frac{\alpha}{2}, \qquad \alpha = \arg x_1.$$

In this example, T denotes transposition.

The corresponding equations of systems (17) and (18) take the form

$$\dot{Z} = A_1 Z + Q, \qquad A_1 = \begin{pmatrix} a & \sqrt{\widetilde{r}} \\ \sqrt{\widetilde{r}} & a \end{pmatrix}, \qquad Q = \widetilde{r}^N F, \tag{19}$$

$$\dot{\widetilde{Z}} = A_1 \widetilde{Z}, \tag{20}$$

$$a = \operatorname{Re} \lambda_1, \qquad F^T = (F_1, F_2), \quad F_1 = e^{-i\beta} y_3^2, \quad F_2 = e^{i\beta} y_4^2.$$

The transformations

$$Z = BW, \quad \widetilde{Z} = B\widetilde{W}, \qquad B = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \qquad \widetilde{r} = r^2$$

reduce these systems to the pseudonormal form

$$\dot{W} = A_2 W + G, \tag{21}$$

$$\dot{\widetilde{W}} = A_2\widetilde{W}. \tag{22}$$

where

$$A_2 = \begin{pmatrix} a+r & 0 \\ 0 & a-r \end{pmatrix}, \qquad G = r^{2N}B^{-1}F,$$

The first stage is accomplished. We now pass to the second stage.

First, note that since the variable β satisfies the equation

$$\dot{\beta} = \frac{\omega}{2} = b, \qquad b = \operatorname{Im}\lambda_1,$$

in view of the system

$$\dot{\beta} = \frac{\omega}{2}, \qquad \dot{\widetilde{y}}_j = \lambda_j\widetilde{y}_j, \quad j = 3, 4,$$

we see that the derivative $\dot{U}$ of the function

$$U = (U_1, U_2)^T = B^{-1}F(\beta, \widetilde{y}_3, \widetilde{y}_4)$$

satisfies the equation $\dot{U} = aU$.

The purpose of this stage is to construct a transformation reducing system (21) to the form (22).

A direct verification shows that there exist functions of class C^∞

$$c_j(r) = o(r^{2N-2}), \qquad j = 1, 2,$$

such that, as the result of the transformation

$$W = W^1 + C(r)U, \qquad C(r) = \operatorname{diag}(c_1(r), c_2(r))$$

system (21) will take the form

$$\dot{W}^1 = A_2W^1 + h(r)U,$$

where $h(r) = \operatorname{diag}(h_1(r), h_2(r))$ is a flat matrix. To make this matrix equal to zero, we must perform the additional transformation

$$W^1 = W^2 + g(r)U, \qquad g(r) = \operatorname{diag}(g_1(r), g_2(r)),$$

where the diagonal entries of the matrix $g(r)$ are the flat solutions of the equations

$$g_j'(r)r^3 = (-1)^{j+1}rg_j(r) + h_j(r), \qquad j = 1, 2.$$

These equations have the required flat solutions (see, for example, [6, pp. 49–50]).

Thus, we have obtained the transformation

$$W = \widetilde{W} + H^*, \qquad H^* = C_1(r)U, \quad C_1(r) = C(r) + g(r) = o(r^{2N-2})$$

reducing system (21) to the form (22).

As the result of the two stages, we have constructed the following transformations:

$$Y = V(\alpha)BW, \qquad \widetilde{Y} = V(\alpha)B\widetilde{W}, \qquad W = \widetilde{W} + H^*.$$

Let us proceed to the third stage. It follows from the foregoing that the transformation

$$Y = \widetilde{Y} + V(\alpha)BH^* \tag{23}$$

reduces system (17) to the form (18). Let us show that this transformation is invertible and, for $2N > k+3$, belongs to the class C^k. The following equalities are valid:

$$V(\alpha) = \xi V_1(\alpha), \quad \xi = e^{i\beta}, \qquad V_1 = \operatorname{diag}(1, e^{-i\alpha}), \qquad H^* = C_1(r)U,$$
$$U = B^{-1}F, \qquad F^T = (e^{-i\beta}\widetilde{y}_3^2, e^{i\beta}\widetilde{y}_4^2) = e^{-i\beta}(\widetilde{y}_3^2, e^{i\alpha}\widetilde{y}_4^2),$$

$$U = \xi^{-1} B^{-1} F^*, \qquad F^* = (\widetilde{y}_3^2, e^{i\alpha} \widetilde{y}_4^2)^T.$$

This implies that

$$H^* = \xi^{-1} C_1(r) B^{-1} F^* \qquad \text{and} \qquad V(\alpha) B H^* = V_1(\alpha) B C_1(r) B^{-1} F^*,$$

which shows that the transformation (23) is invertible and belongs to the class C^k for $2N > k+3$. The proof is complete.

3. FINITELY SMOOTH EQUIVALENCE

The following definition is similar to Definition 3 from [2].

Definition 3. Let $L \geq 0$ be an integer. A function $Q = Q(x, y)$ is said to be *L-small* if

$$Q = Q(x, y) = \|x\|^L h(x, y), \qquad h \in C^L,$$

A matrix is said to be *L-small* if all of its entries are L-small functions. If the number L depends on an integer parameter N, where $\lim_{N\to\infty} L(N) = \infty$, then the set of $L(N)$-small functions (matrices) will be denoted by the symbol Ω. A set of mappings Ψ for which the difference $\Psi - E$ belongs to Ω (where E is the identity mapping) will be denoted the symbol Ω_1.

Following the terminology used in [7], we introduce another notion.

Definition 4. The mapping $T(w) = T(x, y)$ belonging to C^∞ for $x \neq 0$ is said to be *admissible* if there exist constants $d_I > 0$ and α_I such that, for $x \neq 0$,

$$\left\| \frac{\partial^{|I|} T(w)}{\partial w^I} \right\| \leq d_I \|x\|^{\alpha_I}, \qquad \left\| \frac{\partial^{|I|} T^{-1}(w)}{\partial w^I} \right\| \leq d_I \|x\|^{\alpha_I}.$$

The numbers α_I can also be negative. Here I are nonnegative integer collections.

The functions $\|x\|^N$, $\|x\|^{\sqrt{N}}$, etc., are examples of functions belonging to Ω. At the same time, the functions $\|x\|^2$, $\|x\|^{100}$, etc., do not belong to Ω. Shearing transformations are examples of admissible mappings.

Obviously, if the mapping Φ belongs to Ω, and T is an admissible mapping, then ΦT and $T\Phi$ belong to Ω. The superposition of mappings belonging to Ω (or to Ω_1) also belongs to Ω (Ω_1).

Proof of Theorem 2. Consider two systems of the form (1) with coinciding N-jets of the right-hand side where the number N will be specified later, but now we assume $N > m_1(m, n)$. By [5], there exists a number $\widetilde{N} = \widetilde{N}(k, \Lambda)$ such that if the expansion series of the right-hand sides of two systems of the form (1) in the nondegenerate variables differ by terms of order higher than $\widetilde{N}$, then such systems are C^k-equivalent. Here Λ is the nondegenerate part of the spectrum of the matrix $\widetilde{A}$ of the linear part of system (1). By [5], we can assume that the right-hand sides of the systems are finite sums of monomials in nondegenerate variables (the degree of these monomials not exceeding $\widetilde{N}$), with coefficients that are functions of a degenerate variable of class C^∞ whose N-jets coincide. If $N > m_1(m, n)$, then, by Remark 2, weakly degenerate transformations T_1^* and T_2^* of the form (6) reducing the linear parts of both systems to one system of the form (7) differ only by multipliers of class C^∞ and these multipliers have coinciding N_1-jets where $N_1 = N_1(N)$, $\lim_{N\to\infty} N_1(N) = \infty$.

Consider the systems obtained by applying transformations T_1^* and T_2^* to the original systems. Since the resulting systems will have a similar form, their subsequent transformations can be constructed in a similar way; we restrict ourselves to the study of one of them.

Let us isolate two groups of variables. The first group will comprise singular variables, while the second, nonsingular variables. The vector composed of singular variables will be denoted by Z_1, while that of nonsingular variables, by Z_2. Applying the transformations (6) to system (2), we obtain the nonlinear system

$$\begin{aligned} \dot r &= b_1 r^{2p+1} + c_1 r^{4p+1} + f_1, \\ \dot\alpha &= \omega + \psi(r^2) + f_2, \\ \dot Z_l &= A_l(r) Z_l + H_l, \qquad l = 1, 2. \end{aligned} \tag{24}$$

Here and elsewhere, we use the polar coordinates

$$r = \sqrt{u_1 u_2}, \qquad \alpha = \frac{1}{2i} \ln\left(\frac{u_1}{u_2}\right).$$

The functions H_1 are sums of monomials of the form

$$\xi^{|S_1|-1} Z_1^{S_1} Z_2^{S_2} \mu^K c_{S_1 S_2 K}(r), \tag{25}$$

and the functions f_1, f_2, while H_2 are sums of monomials of the form

$$\xi^{|S_1|} Z_1^{S_1} Z_2^{S_2} \mu^K c_{S_1 S_2 K}(r), \tag{26}$$

where $c_{S_1S_2K}(r)$ is a C^∞ functions. This follows from the form of the transformation (6) and Remark 1. Here S_1, S_2 are collections of nonnegative integers, the $Z_l^{S_l}$ are the corresponding monomials, K is an integer, $\xi = \sqrt{e^{i\alpha}}$, $\mu = e^{i\alpha}$, and $|S_1|$ is the weight of the collection. We use the standard notation from [1]–[3]: $Z^S = z_1^{s_1} \cdots z_p^{s_p}$, $S = (s_1, \dots, s_p)$, $s_1, \dots, s_p$ are nonnegative integers, and $|S|$ is the weight of the collection S, $|S| = s_1 + \dots + s_p$.

Consider the Taylor series

$$\widehat{c}_{S_1S_2K}(r) = \sum_L c_{LS_1S_2K} r^L$$

of the functions $c_{S_1S_2K}(r)$. Since the N-jets of the original systems coincide, we can assume that the corresponding coefficients of both systems coincide for

$$L \le N_2, \qquad N_2 = N_2(N), \quad \lim_{N\to\infty} N_2(N) = \infty.$$

In what follows, it will be convenient to replace the variable α in system (24) by the variable $\mu = e^{i\alpha}$; as a result, this system will take the form

$$\begin{aligned} \dot r &= b_1 r^{2p+1} + c_1 r^{4p+1} + f_1, \\ \dot\mu &= i\mu(\omega + \psi(r^2) + f_2), \\ \dot Z_l &= A_l(r) Z_l + H_l, \qquad l = 1, 2. \end{aligned} \tag{27}$$

Let us try to transform the Taylor series of the functions $c_{S_1S_2K}(r)$ into polynomials of degree N_2. To do this, we arrange the collections (S_1, S_2) in increasing order (see [2, p. 286]). Let us describe our principle of ordering the collections (and the corresponding monomials) as follows. If the weights of two different collections $\widetilde S = (\widetilde S_1, \widetilde S_2)$ and $S = (S_1, S_2)$ are identical, then we assume that $\widetilde S > S$ if $\widetilde S_1 > S_1$ in the sense of a definition from [2].

To achieve our goal, we use the changes of variables

$$\begin{aligned} z_j &= w_j + \sum d_{jSK} \xi_1^{|S_1|-1} W_1^{S_1} W_2^{S_2} \mu_1^K r_1^{L-2(q+1)}, && 1 \le j \le n_1, \\ z_j &= w_j + \sum d_{jSK} \xi_1^{|S_1|} W_1^{S_1} W_2^{S_2} \mu_1^K r_1^{L-2(q+1)}, && n_1 + 1 \le j \le n, \\ \mu &= \mu_1 + \sum a_{1SK} \xi_1^{|S_1|} W_1^{S_1} W_2^{S_2} \mu_1^K r_1^{L-2(q+1)}, \\ r &= r_1 + \sum a_{2SK} \xi_1^{|S_1|} W_1^{S_1} W_2^{S_2} \mu_1^K r_1^{L-2(q+1)}, \end{aligned} \tag{28}$$

where $L > N_2$, $n_1, n_2 = n - n_1$ is the number of singular and nonsingular variables, respectively, q is the level of the corresponding monomial, and $\xi_1 = \sqrt{\mu_1}$. The monomials $\xi^S Z_1^{S_1} Z_2^{S_2} \mu^K r^L$ for $L > N_2$, $s = |S_1| - 1$, and $s = |S_1|$, are, obviously, removable (see the definition in [3]); therefore, the coefficients d_{jSK}, a_{1SK}, a_{2SK} are given by the corresponding formulas (see Sec. 3 from [3]), $S = (S_1, S_2)$. The sums contain collections for which $|S_1|$ and $|S_2|$ are fixed.

Carrying out induction on L, we establish the existence of transformations of the form (28), where d_{jSK}, a_{1SK}, a_{2SK} are functions of r of class C^∞; as the result of these transformations, the coefficients of the monomials

$$\xi^{|S_1|-1} W_1^{S_1} W_2^{S_2} \mu_1^K, \qquad \xi^{|S_1|} W_1^{S_1} W_2^{S_2} \mu_1^K$$

in all the equations of the system become sums of polynomials of degree N_2 and of flat functions. To make these flat coefficients equal to zero, we perform an additional transformation of the form (28), where the coefficients d_{jSK}, a_{1SK}, a_{2SK} are the required functions of r. To achieve this goal, these functions must satisfy equations of the form

$$g'(r)(br^l + cr^{l+h}) = g(r)(a_0 + a_1 r + \cdots + a_{l-1} r^{l-1}) + a(r).$$

Here $g(r)$ is the required function, $l \geq 1$, $h > 0$ are integers, b, c, a_j are constants of the quantity, $b \neq 0$, and $a(r)$ is a flat function. This equation has a flat solution (see, for example, [6, pp. 49–50]). Thus, we have constructed the required transformation. It can be written as follows:

$$\begin{aligned} Z_1 &= W_1 + \sum g_{1SK}(r_1) \xi^{|S_1|-1} W_1^{S_1} W_2^{S_2} \mu_1^K, \\ Z_2 &= W_2 + \sum g_{2SK}(r_1) \xi^{|S_1|} W_1^{S_1} W_2^{S_2} \mu_1^K, \\ \mu &= \mu_1 + \sum f_{1SK}(r_1) \xi_1^{|S_1|} W_1^{S_1} W_2^{S_2} \mu_1^K, \\ r &= r_1 + \sum f_{2SK}(r_1) \xi_1^{|S_1|} W_1^{S_1} W_2^{S_2} \mu_1^K. \end{aligned} \tag{29}$$

Here all the sums are finite and $S = (S_1, S_2)$. The coefficients $g_{lSK}(r)$, $f_{lSK}(r)$, $l = 1, 2$ are C^∞ functions depending on r of the order of $o(r^{N_3})$, $N_3 = N_3(N)$, $\lim_{N\to\infty} N_3(N) = \infty$.

Denote the transformations constructed for our systems by

$$H_{11} = H_{11}(\xi, Z, \mu, r) \qquad \text{and} \qquad H_{21} = H_{21}(\xi, Z, \mu, r).$$

Let us apply the same procedure again to the systems obtained by these transformations. As a result, we obtain a finite sequence of transformations

$$H_{q1}, H_{q2}, \dots, H_{qJ}, \qquad g = 1, 2,$$

whose application takes the right-hand sides of both systems to sums of two terms. The first summand is the same for both systems and is a sum of monomials of the form (25) and (26) in which $|S| \leq \widetilde{N}$, $\widetilde{N} = \widetilde{N}(k, \Lambda)$ is the number introduced at the beginning of the proofs of our theorem, and the functions $c_{S_1 S_2 K}(r)$ are identical for both systems and are polynomials of degree N_2. The second summand is composed of convergent series of monomials of the same form, but with $|S| > \widetilde{N}$ (here the functions $c_{S_1 S_2 K}(r)$ are different for the different systems and, in general, are not polynomials, but belong to C^∞). Denote the final transformations of our systems by

$$H_q = \prod_{0 \leq l \leq J-1} H_{qJ-l}, \qquad q = 1, 2.$$

Let us now apply the transformation $(T_1^*)^{-1}$ to the systems obtained by these transformations and consider the following final transformations of the two original systems:

$$R_1 = (T_1^*)^{-1} H_1 T_1^*, \qquad R_2 = (T_1^*)^{-1} H_2 T_2^*.$$

These transformations can be expressed as the following products:

$$R_1 = \widetilde{H}_{1J} \cdots \widetilde{H}_{11}, \qquad \widetilde{H}_{1l} = (T_1^*)^{-1} H_{1l} T_1^*, \quad 1 \leq l \leq J,$$

$$R_2 = \widetilde{H}_{2J} \cdots \widetilde{H}_{21}, \qquad \widetilde{H}_{2l} = (T_2^*)^{-1} H_{2l} T_2^*, \quad 1 \le l \le J-1, \quad \widetilde{H}_{2J} = (T_1^*)^{-1} H_{2J} T_2^*.$$

The next statement is that the mappings $R_{1,2}$ belong to Ω_1. To prove it, it suffices to show that $\widetilde{H}_{1l}, \widetilde{H}_{2l} \in \Omega_1$, $1 \le l \le J$. We restrict ourselves to proving that $\widetilde{H}_{11} \in \Omega_1$. For the other mappings, the proof is similar. This justification will be given after the conclusion of the proof of the theorem.

Since the transformation $(T_1^*)^{-1}$ is linear in the nondegenerate variables, it follows that the right-hand sides of the systems obtained are sums of two summands, the first of which, identical for both systems, is a polynomial in the nondegenerate variables of degree $\widetilde{N}$, while the second summands are C^∞ functions of the nondegenerate variables whose Taylor series begin with terms of degree higher than $\widetilde{N}$. Since the final transformations belong to the class C^{N_4}, where $N_4 = N_4(N)$, $\lim_{N\to\infty} N_4(N) = \infty$, it follows that, for a sufficiently large N, we can assume that the first and second summands belong to the class C^M, where $M \ge \widetilde{N}$. Furthermore, the right-hand sides of the systems differ only by terms of order higher than $\widetilde{N}$ in the nondegenerate variables. But, by [5], in that case, the systems in question will be C^k-equivalent. Thus, the proof of the theorem will be complete.

Let us now consider the transformation $\widetilde{H}_{11}$. Recall that $T_1^* = BVT_1$, where B, V, and $T_1 = T_1(\sqrt{r})$ are the transformations described in Definition 2. Note that the transformations B, V, and T_1 are block-diagonal:

$$B = \operatorname{diag}(B_1, B_2), \qquad V = \operatorname{diag}(D_1, D_2), \qquad T_1 = \operatorname{diag}(T_{11}, T_{12}),$$

the square matrices B_1, D_1, and T_{11} have dimension n_1, and the matrices B_2, D_2, and T_{12} have dimension n_2. The matrices B_q, D_q, $q = 1, 2$, can be written as

$$B_q = B_q(r,\mu) = B_{q1}(r) B_{q2}(\mu), \quad q = 1,2, \qquad D_2 = D_2(\mu), \qquad D_1 = \xi C(\mu),$$

where B_{ql}, C, and D_2 are diagonal matrices whose diagonals contain their arguments in integer powers.

Denote

$$G_q(r_1,\mu_1) = B_q(r_1,\mu_1) D_q(\mu_1) T_{1q}(r_1), \qquad \widetilde{Y}_q = G_q(r_1,\mu_1) W_q, \quad q = 1,2,$$
$$\widetilde{G}_1(r,\mu) = B_1(r,\mu) C(\mu) T_{11}(r).$$

Taking into account the fact that $Y_q = G_q(r,\mu) Z_q$, $q = 1, 2$, and using (29), we obtain

$$\begin{aligned}
Y_1 &= G_1(r,\mu) G_1^{-1}(r_1,\mu_1)\widetilde{Y}_1 + G_1(r,\mu) \sum g_{1SK}(r_1) \xi_1^{|S_1|-1} (G_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K,\\
Y_2 &= G_2(r,\mu) G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2 + G_2(r,\mu) \sum g_{2SK}(r_1) \xi_1^{|S_1|} (G_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K,\\
\mu &= \mu_1 + \sum f_{1SK}(r_1) \xi_1^{|S_1|} (G_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K,\\
r &= r_1 + \sum f_{2SK}(r_1) \xi_1^{|S_1|} (G_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K.
\end{aligned}$$

Taking into account the fact that $G_1(r,\mu) = \xi \widetilde{G}_1(r,\mu)$, we find that

$$\begin{aligned}
Y_1 &= \xi\xi_1^{-1} \widetilde{G}_1(r,\mu) \widetilde{G}_1^{-1}(r_1,\mu_1)\widetilde{Y}_1\\
&\qquad + \xi\xi_1^{-1} \widetilde{G}_1(r,\mu) \sum g_{1SK}(r_1) (\widetilde{G}_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K,\\
Y_2 &= G_2(r,\mu) G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2 + G_2(r,\mu) \sum g_{2SK}(r_1) (\widetilde{G}_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K,\\
\mu &= \mu_1 + \sum f_{1SK}(r_1) (\widetilde{G}_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K,\\
r &= r_1 + \sum f_{2SK}(r_1) (\widetilde{G}_1^{-1}(r_1,\mu_1)\widetilde{Y}_1)^{S_1} (G_2^{-1}(r_1,\mu_1)\widetilde{Y}_2)^{S_2} \mu_1^K.
\end{aligned}$$

Now note that these formulas imply that $\mu\mu_1^{-1} \in \Omega_1$ and $\xi\xi_1^{-1} \in \Omega_1$.

In addition, if the admissible transformation Q is of the form $(x, y) \to (x, P(x)y)$ and the transformation H belongs to Ω_1, then $QHQ^{-1} \in \Omega_1$. This implies the following assertions:

1) if $S(\eta) = \operatorname{diag}(\eta^{\delta_1}, \eta^{\delta_2}, \dots, \eta^{\delta_n})$, δ_j are rational numbers, and $H \in \Omega_1$, then $S(r) H S^{-1}(r_1) \in \Omega_1$;

2) if $S(r) = r + B(r)$, $B(r) \in C^\infty$, $\|B(r)\| = o(r)$, and $H \in \Omega_1$, then $S(r)HS^{-1}(r_1) \in \Omega_1$;

3) if $H \in \Omega_1$, then $C(\mu)HC^{-1}(\mu_1) \in \Omega_1$.

In addition, if the admissible transformation Q is of the form (in polar coordinates) $(\alpha, r, y) \to (\alpha, \sqrt[l]{r}, y)$, where $l > 0$ is an integer and the transformation H belongs to Ω_1, then $QHQ^{-1} \in \Omega_1$.

Using these assertions and taking into account the fact that the product of an admissible transformation and a transformation from the set Ω also belongs to Ω, we find that the transformation constructed by us is of the form

$$x = \widetilde{x} + H_1(\widetilde{x}, \widetilde{Y}), \quad Y = \widetilde{Y} + H_2(\widetilde{x}, \widetilde{Y}), \qquad H_{1,2} \in \Omega.$$

Here

$$Y = (Y_1, Y_2), \quad \widetilde{Y} = (\widetilde{Y}_1, \widetilde{Y}_2), \qquad x = (r\mu, r\mu^{-1}), \quad \widetilde{x} = (r_1\mu_1, r_1\mu_1^{-1}).$$

Thus, we have proved that the mapping $\widetilde{H}_{11}$ belongs to Ω_1. Theorem 2 is proved. □

ACKNOWLEDGMENTS

The author wishes to express deep gratitude A. D. Bryuno for useful discussions of the present paper and a number of valuable remarks that led to improvements in the exposition.

REFERENCES

1. V. S. Samovol, "On new resonances and normal forms of autonomous systems with one zero eigenvalue," Mat. Zametki **88** (1), 63–77 (2010) [Math. Notes **88** (1), 67–78 (2010)].
2. V. S. Samovol, "Finitely smooth local equivalence of autonomous systems with one zero root," Mat. Zametki **88** (2), 275–287 (2010) [Math. Notes **88** (2), 251–261 (2010)].
3. V. S. Samovol, "On the pseudonormal form of real autonomous systems with two pure imaginary eigenvalues," Mat. Zametki **92** (5), 731–746 (2012) [Math. Notes **92** (5), 673–685 (2012)].
4. Ph. Hartman, *Ordinary Differential Equations* (John Wiley, New York–London–Sydney, 1964; Mir, Moscow, 1970).
5. V. S. Samovol, "Equivalence of systems of differential equations in a neighborhood of a singular point," Trudy Moskov. Mat. Obshch. **44**, pp. 213–234 (1982) [Trans. Mosc. Math. Soc. (2), 217–237 (1983).
6. A. N. Kuznetsov, "Differentiable solutions to degenerate systems of ordinary equations," Funktsional. Anal. Prilozhen. **6** (2), 41–51 (1972) [Functional Anal. Appl. **6** (2), 119–127 (1972)].
7. G. R. Belitskii, "Smooth equivalence of germs of vector fields with a single zero eigenvalue or a pair of purely imaginary eigenvalues," Funktsional. Anal. Prilozhen. **20** (4), 1–8 (1986) [Functional Anal. Appl. **20** (4), 253–259 (1986)].
8. A. D. Bryuno, *A Local Method for the Nonlinear Analysis of Differential Equations* (Nauka, Moscow, 1979) [in Russian].
9. V. S. Samovol, "Finitely smooth normal form of an autonomous system with two pure imaginary roots," Mat. Zametki **80** (2), 270–281 (2006) [Math. Notes **80** (2), 261–271 (2006)].
10. A. D. Bryuno and V. Yu. Petrovich, *Normal Forms of a System of ODE*, Preprint no. 18 (Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, 2000) [in Russian].
11. W. Wazow, *Asymptotic Expansions for Ordinary Differential Equations* (Interscience Publishers–John Wiley, New York–London–Sydney, 1965; Mir, Moscow, 1968).