Mathematical Justification for the Transition to Negative Pressures of the New Ideal Liquid[*]

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Abstract—Negative pressure also means negative energy and, therefore, "holes," antiparticles. Continuation across infinity to negative energies is accomplished by using a parastatistical correction to the Bose–Einstein distribution.

Difficulties related to the notion of infinity cannot be completely solved within the framework of nonstandard analysis. After the appearance of this theory, the great mathematician Gödel[1] also thought that the issue remains open when he told his colleague Academician S. I. Adian[2] that mathematics would have taken the correct path if it had followed the path designated by Leibniz rather than that paved by Newton. He said that if mathematics had developed along a different path at the same rate, there would not be any need to overcome established methods. They struck root in the traditions and conscience of mathematicians in such a way that it is hard to change the established notions.

The prominent physicist Ya. Frenkel spoke more critically about physics and the notions established in physics: "We easily get used to the monotonous and unchanging, we stop noticing it. What we are used to seems natural to us, things we are not used to seem unnatural and non-understandable. ... Essentially, we are unable to understand, we can only *get used to*."[3]

The simplest problem involving infinity is the arithmetic mean of the natural series. Mathematicians cannot "calculate" it, but physicists can. The spectrum of the Hamiltonian operator for the oscillator coincides, up to a constant, with the natural series. The mean value of the series is its mean energy.

Physicists identify mean energy with temperature. Van der Waals proposed to normalize temperature as follows: $T_{red} = T/T_c$, where T_{red} is the reduced temperature and T_c is the critical temperature.

The critical temperature T_c was called the absolute boiling temperature by D. Mendeleev. What he meant was that the liquid state is impossible above this temperature. What is temperature on the Kelvin scale? It is counted off from absolute zero. However, the absolute zero temperature is unattainable in the sense that however small the absolute (Kelvin) temperature is, it is nevertheless infinitely far-off from absolute zero. In this context, it is more useful to consider the logarithm of the temperature. Then $\log T|_{T=0} = -\infty$, and it is easy to see that the distance between the arbitrarily small absolute temperature and absolute zero is infinite. Hence, in mathematics, we deal with an analog of "nonstandard analysis" and an infinitely large temperature, although it can be equal to, for example, one hundredth of a degree on the Kelvin scale.

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[1]As is well known, Godel's Imcompleteness Theorem in logic for the natural numbers affected the famous mathematician Hilbert, destroying his conception about the "elements" of mathematics, in such a way that he suffered from depression for two weeks. Albert Einstein remarked towards the end of his career that he only went to his office at Princeton "just to have the privilege of walking home with Kurt Gödel" (P. Yourgrau, *The Forgotten Legacy of Gödel and Einstein*).

[2]Oral communication.

[3]B. Ya. Frenkel, *Yakov Il'ich Frenkel* (Nauka Publ., Moscow–Leningrad, 1966) [in Russian], p. 63).

If we consider the relative quantities $T_{red} = T/T_c$ and $P_{red} = P/P_c$, then we can see that, in these coordinates, isotherms, isochores, and isobars for different gases strongly differing by their critical temperature T_c exhibit similar behavior in all coordinate planes (pressure−volume), (temperature−density), etc. (the Van der Waals "law of corresponding states"). This is valid for nitrogen, argon, carbon dioxide, methane, and other gases.

Example 1. Consider the standard Hilbert space of functions H defined on the closed interval $[0, \pi]$ and the basic collection of orthogonal functions

$$\{e^{inx}\}_{n=-\infty}^{n=\infty}, \tag{1}$$

as well as two self-adjoint operators

$$\widehat{A}_1 = -ih\frac{\partial}{\partial x}, \qquad \widehat{A}_2 = -2ih\frac{\partial}{\partial x} \tag{2}$$

depending on the parameter $h = 1/k$, where k is an integer.

The initial reference points in the basis $\{e^{inx}\}$ are $n = 1$.

Consider $n_0^{(1)} = k$ and $n_0^{(2)} = k/2$. We find that

$$\widehat{A}_1 e^{in_0^{(1)}x} \approx 1, \qquad \widehat{A}_2 e^{in_0^{(2)}x} \approx 1.$$

Obviously,

$$\left\{e^{i(n-k)x}\right\}_{n=-\infty}^{n=\infty} \qquad \text{and} \qquad \left\{e^{i(n-k/2)x}\right\}_{n=-\infty}^{n=\infty}$$

are isomorphic to

$$\left\{e^{inx}\right\}_{n=-\infty}^{n=\infty}.$$

Thus, as $h \to 0$, the reference point of the infinite basis shifts from the point $n = 1$ to the point $n = 1/h$ in the first case $\widehat{A}_1$ and to the point $n = 1/(2h)$ in the second case $\widehat{A}_2$. On the closed interval $[0, \pi]$, acting on the functions x expanded in the basis (1), the operators $\widehat{A}_1$ and $\widehat{A}_2$ will behave in exactly the same way as the usual function numbering operator $\widehat{A}$. Shifting the numbering by $n - k$ and $n - k/2$, we obtain rapidly oscillatory functions. As $h \to 0$, these spaces are orthogonal to each other and to the original Hilbert space H in the sense that

$$\int \sin\frac{x}{h}\varphi(x)\,dx \to 0 \quad \text{as} \ \ h \to 0, \qquad \varphi(x) \in H.$$

Denoting $e^{ikx} = \Psi_c$ and $e^{ik/2x} = \varphi_c$, we find that the space of functions corresponding to a basis of the form Ψ_n/Ψ_c, where $\Psi_n = e^{inx}$, is isomorphic to the space of functions Ψ_n/φ_c as $h \to 0$, ($h = 0$ and $k = \infty$ in "nonstandard analysis").

A similar pattern is observed for Dirichlet series and the collection of eigenfunctions of the Schrödinger equation (of the Hamiltonian operator $\widehat{H}$) for an oscillator, etc. (cf. [1]).

This isomorphism of infinitely large quantities of "nonstandard analysis" accounts, to a certain extent, for the empirical law of corresponding states of Van der Waals, who discovered a similar "almost" isomorphism.

For a mathematician, it is difficult to explain ideas of the new thermodynamics to a physicist. But the most difficult thing is to explain that such a small temperature as $T = 1/100K$, is an infinitely large quantity.

In the old (Boltzmann−Maxwell) theory of the ideal gas, the number of degrees of freedom depends on the number of atoms in the molecule and their maximum possible number was calculated. However, in [2], it was shown that, as temperature decreases, certain degrees of freedom can be "frozen out." This means that the number of degrees of freedom of a molecule depends on its energy. Temperature is the mean energy. Along with temperature, it is also natural to consider the mean (noninteger) number of degrees of freedom.

The author pointed out in several papers that an instrument measuring the density inside some volume in the vessel, small compared to the volume of the whole vessel, cannot register the renumbering of particles. Therefore, for such an instrument (and for the experimenter working with it) statistics of Bose–Einstein type, not the Boltzmann statistics, is valid. For classical gas, the degeneracy temperature T_d corresponds to the critical temperature T_c and the mean number of degrees of freedom at the critical point T_c, P_c, ρ_c can be calculated by the formula

$$Z_c = \frac{P_c}{T_c \rho_c} = \frac{\zeta(\gamma + 2)}{\zeta(\gamma + 1)}, \qquad (3)$$

{5y:u444}

where γ is related to the number of degrees of freedom by the relation

$$D = 2(\gamma + 1).$$

For each pure gas, the number Z_c is known from experiments (real ones or numerical simulation). This yields the value of the parameter γ together with the number of degrees of freedom. By the parameter γ we mean the fractional dimension in a distribution of Bose–Einstein type.

For $\gamma = 0$, we have $D = 2$ and $Z_c = \infty$. As was previously shown, $D = 2$ corresponds to the growth of energy levels proportional to the natural series.

As the prominent mathematician Yu. I. Manin puts it in his book [3], "there is an obvious tendency to (at least) admit number theory to the world of ideas of modern theoretical physics." (p. 209 (Russian transl.)).

This relationship (including the relation $N/\log N$ given by Manin for the number of primes not exceeding N) was studied in greater detail in [4].

We are concerned with the extension of the compressibility factor Z to the domain $\gamma < 0$ for classical gas. The question is: How to extend "analytically" to the negative domain? First, note that the main logical difficulties involving infinity occur for $\gamma = 0$, i.e., for the natural series. The natural series corresponds to quantum oscillators, i.e., photons. Only, for $\gamma = 0$, there is no negative number. The only particle whose antiparticle coincides with it is the photon.

The first extension to negative domains was performed by the author in the case of money: negative money is debt. [4]

To the number of degrees of freedom corresponds the number of degrees of unfreedom. The value of the maximal number of degrees of unfreedom must be determined experimentally (for a numerical simulation, see [5]). This value of the minimum pressure allows us to fully determine the missing constants in the Bose–Einstein distribution.

Indeed, the point of minimum of the negative pressure and the temperature corresponding to this point determine two constants: one of them is Λ, while the other is the value of Young's liquid extension modulus. But, as was already stated, this critical point of transition from the liquid to the dispersed phase is very hard to determine experimentally. It is much easier to determine the triple point for a given pure gas. It is assumed that, below this point, the formula for the old Maxwell ideal gas holds. From our point of view, this means that the chemical potential of the new ideal gas is equal to $-\infty$. Therefore, the value of the triple point determines the endpoint of the binodal in the plane $\{\rho, T\}$ and other planes. The experimental value of this point uniquely determines these two constants. Hence, for the new ideal liquid, we can also determine the critical point of transition from the liquid phase to "solidified foam" like pumice.

Let us present the formula for the entropy of a nonequilibrium Bose gas derived in the book [8, 184] (Formula 54.6):

$$S = \sum_j G_j [(1 + \bar{n}_j) \ln(1 + \bar{n}_j) - \bar{n}_j \ln \bar{n}_j], \qquad (4)$$

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where the $\bar{n}_j$, $n_j = N_j/G_j$, are the mean occupation numbers of quantum states.

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[4]The great chemist D. Mendeleev and the specialist in thermodynamics I. A. Vyshnegradskii, who was appointed Minister of Finance of Russia in 1887 and interested Mendeleev in financial matters, paid significant attention to the relationship between thermodynamics and economics; so did the author of the "main law of economics" Irving Fisher, a disciple of Gibbs.

We have

$$\sum_j N_j = \sum_j G_j \bar{n}_j = N, \qquad \sum_j \varepsilon_j N_j = \sum_j \varepsilon_j G_j \bar{n}_j = E. \tag{5}$$

Hence, under conditions (5), using the method of indeterminate Lagrange multipliers for the maximum of the entropy (4), we obtain[5]

$$\bar{n}_j = \frac{1}{e^{a+b\varepsilon_j} - 1}. \tag{6}$$

Here $a = -\mu/T$, $b = 1/T$, T is the temperature, and $\mu \leq 0$ is the chemical potential. Thus, the indeterminate Lagrange multipliers are expressed in terms of the temperature and the chemical potential of the gas.

Since, by the Bose–Einstein formulas, sums of the form

$$\sum_{i=0}^{\infty} \lambda_i n_i = E, \qquad \sum_{i=0}^{\infty} n_i = N,$$

are replaced by integrals, it follows that it is impossible to extract the point $i = 0$ and the condensate phenomenon can occur only in a neighborhood of the point $i = 0$, but not at the point $i = 0$ itself. Note that, for $D \leq 2$, the Euler–Maclaurin formulas do not apply (i.e., it is impossible to pass from sums to integrals) and, therefore, the point $i = 0$ can be isolated.

However, will the superfluous particles for $N > N_c$ accumulate particularly at the point $i = 0$? In fact, this question is: Will the number of zeros in the problem under consideration be much greater than the number of 1's as $E \to \infty$, i.e., will the relation

$$n_1 = o(n_0)$$

be valid even for $N > N_c$?

A computer calculation yielded a negative answer. Specialists in number theory also answer in the negative. Especially for $D > 2$, when the Euler–Maclaurin estimates can already be applied also for $i \gg 1$ and we can pass to the integral, it is impossible to consider one point $i = 0$, as was done in [8].

In fact, the convergence to the integral (for example, in the three-dimensional case) as $N \to \infty$ and $\mu \to 0$ is simultaneously accompanied by an increase of the bell-shaped function of the form

$$\sqrt{N} e^{-Nx^2}, \qquad \text{where} \quad \frac{1}{\ln N} > x \geq 0 \tag{7}$$

(cf. [9]).

In this general conclusion, it is necessary to understand what G_j is in the general case. The $2D$-dimensional phase space can be divided into a lattice, and G_j is determined by the formula

$$G_i = \frac{\Delta p_j \Delta q_j}{(2\pi\hbar)^D}. \tag{8}$$

Here $\hbar$ is the Planck constant.

In the case of the Laplace operator and the Schrödinger equation, there is no sense in considering fractional dimensions. In the statistical case, fractional dimensions are meaningful, but in a different way. They determine the mean number of degrees of freedom, which can be different for molecules with different velocities, and the mean number of degrees of freedom[6], for the whole gas will be fractional. It corresponds to the mean energy, i.e., the temperature. For the unbounded probability theory that we consider, it is the fractional "dimension" (i.e., the mean number of degrees of freedom with respect to the whole "general" population) that is meaningful.

The author proved that both the degeneracy energy for the Bose gas and the number of particles in the state of the degeneracy coincide up to normalization with the critical values of N_c and E_c (i.e., for $\mu = 0$)

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[5]We repeat word for word the derivation given in [8].
[6]The term is taken from sociology and economics.

of the classical noninteracting gas. In the D-dimensional case, the degeneracy energy (degeneration) is of the form

$$E_d = \int_0^\infty \frac{\frac{|p|^2}{2m} \, d\varepsilon}{e^{\frac{|p|^2}{2m}/T_d} - 1}, \tag{9}$$

where

$$d\varepsilon = \frac{|p|^2}{2m} \frac{dp_1 \ldots dp_D \, dV_D}{(2\pi\hbar)^D}. \tag{10}$$

Hence we obtain the coefficients C and Λ^D in the formula

$$E_d = C\Lambda^D T_d^{2+\gamma} \zeta(1 + D/2)\Gamma(1 + D/2). \tag{11}$$

It is convenient to consider the other normalization

$$d\varepsilon' = \frac{1}{\Gamma(D/2 + 1)} \, d\varepsilon \tag{12}$$

for the energy and, respectively,

$$d\varepsilon'' = \frac{1}{\Gamma(D/2)} \, d\varepsilon \tag{13}$$

for the number of particles N.

After that, it is convenient to pass to the polylogarithm $\mathrm{Li}_{2+\gamma}(e^{-\mu/T})$ for the energy and to the polylogarithm $\mathrm{Li}_{1+\gamma}(e^{-\mu/T})$ for the number of particles:

$$E = C\Lambda^D T^{2+\gamma} \, \mathrm{Li}_{2+\gamma}(a), \tag{14}$$

$$N = C\Lambda^D T^{1+\gamma} \, \mathrm{Li}_{1+\gamma}(a), \tag{15}$$

where a is the activity ($a = e^{-\mu/T}$).

Then the compressibility factor is

$$Z = \frac{E}{NT} = \frac{\mathrm{Li}_{2+\gamma}(a)}{\mathrm{Li}_{1+\gamma}(a)}. \tag{16}$$

Here the coefficients C and Λ^D cancel out.

Thus, a and b are the Lagrange multipliers (see (4)–(6)) for determining the maximum of S_γ:

$$dS = a \, dN + b \, d(E), \qquad N = E, \tag{17}$$

$$dE = \frac{1}{b} \, dS - \frac{a}{b} \, dN. \tag{18}$$

Denote the quantity $1/b$ by T and the quantity a/b by μ ($\mu \le 0$). In thermodynamics, we also add the term $P \, dV$, where P is the pressure and V is the volume.

The Hartley entropy S_γ for $\gamma > 0$ is of the form

$$S = N\left[(2 + \gamma) + \frac{\mu}{T}\right]. \tag{19}$$

For $\mu = 0$ and $N > N_c$, the entropy S is extended by continuity by a constant (in the particular case $D = 3$, this phenomenon is similar to the physical notion of "Bose–Einstein condensate").

The relation $N > N_c$ is equivalent to the relation $T < T_c$. Further, *for $N \gg k \gg 1$* (see (7)), the number of "supercritical" values is

$$\sum_{i=k}^\infty n_i = N_c.$$

As $N \to \infty$ and $\mu \to 0$, the other numbers $N - N_c$ accumulate on the relatively decreasing interval $i \leq k \ll N$. Here $k \gg 1$ is the *support of the bell-shaped function.*

The transition to the integral is accompanied by the approximation of the "bell-shaped" function to the δ-function ("Bose condensate"). Therefore, in formula (19) for $N > N_c$ or $T < T_c$, the values of dN and dV can be equated to zero. As a result, the entropy of the liquid phase takes the form

$$dE = T\,dS. \tag{20}$$

In the plane $\{P, Z = PV/NT\}$, in the Van der Waals normalization: $P_r = P/P_c$, $T_r = T/T_c$, the gas critical isotherm takes the form

$$P = \mathrm{Li}_{\gamma_c+2}(e^{-\mu/T_c}), \qquad Z = \frac{\mathrm{Li}_{\gamma_c+2}(e^{-\mu/T_c})}{\mathrm{Li}_{\gamma_c+1}(e^{-\mu/T_c})} \tag{21}$$

and the unknown parameters C и Λ^D in (14)–(15) must be expressed as $\Lambda^{\gamma-\gamma_c}$ (we retain the same symbol).

To each pure gas there corresponds

$$Z_c = \frac{\zeta(\gamma_c + 2)}{\zeta(\gamma_c + 1)}.$$

For argon, $Z_c = 0.29$, for methane, $Z_c = 0.29$, for oxygen and other gases, the value of Z_c is also taken from experiments. The "basic" spinodal (i.e., the set of points for $\mu = 0$) in the plane $\{P, Z\}$ is the closed interval $Z = Z_c$, $P = \{0, 1\}$. The isotherm $T_r \leq 1$ is in good agreement on the closed interval $-\infty < \mu < \mu_l$ up to the phase transition "gas–liquid" when the chemical potentials μ_g of the gas isotherm and μ_l of the gas isotherm are equal [14].

On the liquid isotherm, the value $N = T_r^{\gamma_c+1}\zeta(\gamma_c + 1)$ is constant. Therefore, it is described by a straight line passing through the points $\{P = T_r^{\gamma_c+2}\zeta(\gamma_c + 2),\ Z = Z_c\}$ and $\{P = 0, Z = 0\}$.

The origin, the point $\gamma = 0$, corresponds to the problem "partitio numerorum" (see formulas (11)-(14) of [10]). The main difficulty is to determine the chemical potential of the liquid phase μ_l. To this end, we continue the isotherms of the liquid phase "analytically" to the domain of negative values of Z and P.

For $T < T_c$, the liquid isotherm passes into the domain $\gamma < 0$ on the diagram $\{Z, P\}$ (in thermodynamics, $Z = (PV)/(NT)$ is the compressibility factor and P is the pressure) and reaches the spinodal point (the endpoint of the metastable state) for $\tilde{\mu} = 0$ (see below). This the point of maximum of the entropy on the isotherm of the liquid phase. In the extension to the domain $\gamma < 0$, we attain the domain of negative values of Z, and hence also the domain of negative pressures.

Remark. There are not many theoretical physicists with classical thermodynamic background who know what "negative pressure" really is. In [2], this phenomenon was explained as follows. PV is the energy, while negative energy according to Dirac is represented by holes, i.e., antiparticles. At present, this field is under study by remarkable experimenters, such as S. Balibar [11], V. G. Baidakov [12], K. I. Shmulevich [13], and others. At some temperature T_{cl} (the temperature of transition of the liquid phase to pumice), the minimum of negative energy corresponds to a new critical point.

Assuming that, for $T = T_c$, there is no transition to the liquid phase, we equate the chemical potentials μ and $\tilde{\mu}$ for the liquid and gas phase, respectively, on the isotherm $T = T_c$. Further, we obtain the point μ, the point of transition to the liquid phase for $T < T_c$ by equating the chemical potentials of the liquid and the gas phase. This scheme is given below.

By an energy gap we mean the difference between the solution of problem (22)

$$\sum_{i=0}^{\infty} N_i = N, \qquad \varepsilon \sum_{i=0}^{\infty} (i^\gamma - 1)N_i = E - N\varepsilon \tag{22}$$

and the solution of problem (23)

$$\sum_{i=0}^{\infty} N_i = N, \qquad \varepsilon \sum_{i=0}^{\infty} (i^\gamma - 1)N_i < E - N\varepsilon \tag{23}$$

where, for the Schrödinger equation,

$$\varepsilon = \frac{\hbar^2}{2mV^{2/D}}.$$

By passing to the Bose condensate point, the parameter γ of the liquid phase also has a gap in the domain of negative energies. Let us find the isotherm−isobar point of the liquid as $\varkappa = -\mu/T$ slowly converges to zero.

First, we take into account the fact that although N_c is large, but finite; therefore, we must still use a parastatistical correction. Let us recall the relation for the potential Ω:

$$\Omega = -\Lambda^{\gamma-\gamma_c}T\sum_k \log\left(\frac{1 - e^{N(\mu-\varepsilon_k)/T}}{1 - e^{(\mu-\varepsilon_k)/T}}\right). \tag{24}$$ {6y:u444}

Hence (see [6, Chap. 2]), for the dimension $D = 2(\gamma + 1)$,

$$N = \Lambda^{\gamma-\gamma_c}\sum_{j=1}^n \left(\frac{j^\gamma}{e^{bj+\varkappa} - 1} - \frac{kj^\gamma}{e^{bkj+\varkappa}}\right). \tag{25}$$ {7y:u444}

Lemma. *The following relation holds*:

$$N = \Lambda^{\gamma-\gamma_c}\sum_{j=1}^n \left(\frac{j^\gamma}{e^{bj+\varkappa} - 1} - \frac{kj^\gamma}{e^{bkj+\varkappa}}\right) = \frac{\Lambda^{\gamma-\gamma_c}}{\gamma + 1}\int_0^\infty \left(\frac{1}{e^{bx+\varkappa} - 1} - \frac{k}{e^{bkx+\varkappa} - 1}\right) d(x^{\gamma+1}) + R, \tag{26}$$ {8y:u444}

where $k = N$ and $b = 1/T$, while

$$R \le \frac{Ck^{1-\gamma}}{1 + (k\kappa)^{1-\gamma}},$$

where C is a constant.

Proof. Set $\alpha = \gamma + 1$. Here the remainder R satisfies the estimate

$$|R| \le \frac{\Lambda^{\gamma-\gamma_c}}{\alpha}\int_0^\infty |f'(x)|\, dx^\alpha, \qquad \text{where} \quad f(x) = \frac{1}{e^{bx+\varkappa} - 1} - \frac{k}{e^{k(bx+\varkappa)} - 1}.$$

Calculating the derivative, we obtain

$$f'(x) = \frac{bk^2 e^{k(bx+\varkappa)}}{(e^{k(bx+\varkappa)} - 1)^2} - \frac{be^{bx+\varkappa}}{(e^{bx+\varkappa} - 1)^2}, \qquad |R| \le \frac{\Lambda^{\gamma-\gamma_c}}{\alpha}\int_0^\infty \left|\frac{k^2 e^{k(y+\varkappa)}}{(e^{k(y+\varkappa)} - 1)^2} - \frac{e^{y+\varkappa}}{(e^{y+\varkappa} - 1)^2}\right| dy^\alpha.$$ (27)   {9y:u444}

We also have

$$\frac{e^y}{(e^y - 1)^2} = \frac{1}{y^2} + \psi(y),$$

where $\psi(y)$ is a smooth function and $|\psi(y)| \le C(1 + |y|)^{-2}$. Substituting this formula into (27), we obtain

$$|R| \le \frac{\Lambda^{\gamma-\gamma_c}}{\alpha}\int_0^\infty \left|k^2\psi\big(k(y + \varkappa)\big) - \psi(y + \varkappa)\right| dy^\alpha \le \Lambda^{\gamma-\gamma_c}\left(k^{2-\alpha}\int_{k\varkappa}^\infty |\psi(y)|\, dy^\alpha + \int_\varkappa^\infty |\psi(y)|\, dy\right)$$

$$\le \frac{Ck^{2-\alpha}}{1 + (k\varkappa)^{2-\alpha}},$$

where C is a constant. The lemma is proved. $\square$

For example, if $\varkappa \sim (\log k)^{-1/4}$, then $|R|$ has the estimate

$$|R| \sim O((\ln k)^{(2-\alpha)/4}).$$

We have (see [7])

$$\int_0^\infty \left\{ \frac{1}{e^{b\xi} - 1} - \frac{k}{e^{kb\xi} - 1} \right\} d\xi^\alpha = \frac{1}{b^\alpha} \int_0^\infty \left(\frac{1}{e^\xi - 1} - \frac{1}{\xi} \right) d\xi^\alpha + \frac{1}{b^\alpha} \int_0^\infty \left(\frac{1}{\xi} - \frac{1}{\xi(1 + (k/2)\xi)} \right) d\xi^\alpha$$

$$- \frac{k^{1-\alpha}}{b^\alpha} \int_0^\infty \left\{ \frac{k^\alpha}{e^{k\xi} - 1} - \frac{k^\alpha}{k\xi(1 + (k/2)\xi)} \right\} d\xi^\alpha$$

$$= \frac{c(\gamma)}{b^\alpha} (k^{1-\alpha} - 1).$$

Since $k \gg 1$, for $k = N$ we finally obtain

$$N|_{\tilde{\mu}/T_r = o(1)} = A(\gamma)T_r, \tag{28}$$

where

$$A(\gamma) = (\Lambda^{\gamma - \gamma_c} c(\gamma))^{1/(1+\gamma)}, \qquad c(\gamma) = \int_0^\infty \left(\frac{1}{\xi} - \frac{1}{e^\xi - 1} \right) \xi^\gamma \, d\xi. \tag{29}$$

We have seen that the relation

$$N = \Lambda^{\gamma - \gamma_c} T_r^{\gamma+1} \operatorname{Li}_{\gamma+1}(a), \qquad \text{where} \quad a = e^{-\tilde{\mu}/T_r}, \tag{30}$$

is the activity conjugated to the linear relation $N = A(\gamma)T_r$ for $P_r < 0$.

We can perform the normalization of the activity a at the point $T = T_c$ and find a_0 by conjugating the liquid and gas branches for $T = T_c$ for a pressure such that there is no phase transition for $T = T_c$.

Further, for $T < T_c$, let us normalize the activity by the value a_0 calculated below. Then the chemical potential (in thermodynamics, the Gibbs thermodynamic potentials for the liquid and gas branches) will coincide and, therefore, for $T = T_c$, there will be no phase transition "gas−liquid."

Now, for the isochore−isotherm of an incompressible liquid to exist, we must also construct it for smaller density than $N_c = \zeta(\gamma_c + 1)$, namely, for

$$N(T_r) = \Lambda^{\gamma - \gamma_c} T_r^{\gamma_c + 1} \zeta(\gamma_c + 1). \tag{31}$$

We obtain the value of $\gamma(T_r)$ from the implicit equation

$$A(\gamma) = \Lambda^{\gamma - \gamma_c} T_r^{\gamma_c} \zeta(\gamma_c + 1).$$

Thus, for each $T_r < 1$, we find the "spinodal" curve, (i.e., the points where $\tilde{\mu}/T_r \ll 1$ (i.e., e.g.,for the points where $\tilde{\mu} \sim T_r(\ln N(T_r))^{-1/4}$) in the domain of negative pressures.

$$A(\gamma) = \Lambda^{-(\gamma - \gamma_c)\gamma/(1+\gamma)} c(\gamma)^{1/(1+\gamma)} = T_r^{\gamma_c} \zeta(\gamma_c + 1), \tag{32}$$

$$N = A(\gamma)T_r = T_r^{1+\gamma_c} \zeta(\gamma_c + 1),$$

$$Z = \frac{P_r}{A(\gamma)T_r^2} = -\frac{T_r^{2+\gamma} \zeta(2 + \gamma)}{T_r^{2+\gamma_c} \zeta(\gamma_c + 1)} = -\frac{\zeta(2 + \gamma)}{T_r^{\gamma_c - \gamma} \zeta(\gamma_c + 1)}.$$

$$P_r = ZT_r N = ZT_r^{2+\gamma_c} \zeta(\gamma_c + 1) = -T_r^{2+\gamma} \zeta(2 + \gamma). \tag{33}$$

The values of $P_{\min}$ and $T_{\min}$ are known from experiments.

Using (33), let us find the value of $\gamma = \gamma_{\min}$, where $-1 < \gamma_{\min} < 0$:

$$P_{\min} = -T_{\min}^{2+\gamma_{\min}} \zeta(2 + \gamma_{\min}).$$

From the formula

$$A(\gamma_{\min}) = T_{\min}^{\gamma_c} \zeta(\gamma_c + 1), \tag{34}$$

we find the value of $\Lambda(T_{\min}, \gamma_{\min})$.

Now, substituting $T_r = 1$ into (32), we seek to find the solution for γ.

From the two values of the solution of (32), we choose the largest (the least in absolute value) and denote it by γ_0:

$$\Lambda(T_{\min}, \gamma_{\min})^{-(\gamma_0 - \gamma_c)\gamma_0/(\gamma_0 + 1)} = \zeta(\gamma_c + 1).$$

The second solution corresponds to the dispersive system into which the liquid has passed at the critical point $T_{\min}, P_{\min}, Z_{\min}$. The actual interaction between the particles leads to a significantly greater deviation of the second solution from γ_0 (see [13]).

The jump from $\gamma = 0$ to $\gamma = \gamma_0 < 0$ is the transition gap across the singularity $\gamma = 0$ (compare with the gap for $\mu = 0$ [15],[16]). Such a gap is always formed for media with different moduli (for different Young compression and extension moduli); see [17]. Its physical meaning is that it characterizes Young's liquid extension modulus.

Suppose that $a_g = e^{-\mu/T_r}$ is the activity of the gas and $a_l = e^{-\tilde{\mu}/T_r}$ is the activity of the liquid. Let us present the condition of the equality of E and the activity at the phase transition point:

$$T_r^{\gamma_c} \operatorname{Li}_{2+\gamma_c}(a_g) = T_r^{-|\gamma(T_r)|} \operatorname{Li}_{2-|\gamma(T_r)|}\left(\frac{a_l}{a_0}\right), \qquad (35) \quad \{x:u444\}$$

$$\operatorname{Li}_{2+\gamma_0}(a_0) = 1, \qquad (36) \quad \{xx:u444\}$$

where

$$a_g = \frac{a_l}{a_0}. \qquad (37)$$

These two equalities determine the value of the chemical potential

$$\mu = \tilde{\mu} = T_r \ln a_g$$

at which the phase transition "gas−liquid" occurs.

Suppose that

$$T_{\min} = \min_{-1 < \gamma < 0} A(\gamma).$$

For $\Lambda = 1$, the minimum is attained at the point $\gamma_{\min} = -0.39$. Thus, for each $T_0 < T < T_c$, we find a spinodal point in the domain of negative pressures and, for

$$T_r^{2+\gamma(T_r)} \operatorname{Li}_{2+\gamma(T_r)}(a_l) = T_r^{2+\gamma_c} \zeta(2 + \gamma_c),$$

we obtain the value $a_r = a_l/a_0$ (a_l is the activity of the liquid).

Thus, if the values of $T_{\min}$ and $P_{\min}$ are known from the experiment [11]–[13], then we can find the values of Λ и γ^0 from the above relations. The value $T_{\min}$ (reduced) is a bound for the value of the temperature of the binodal constructed in the way described above. Below this temperature (and the corresponding energy), the Bose condensate effect disappears. Roughly speaking, the decrease in the internal energy must not attain the energy interval which is the support of the bell-shaped function (7). Otherwise, the Bose condensate phenomenon disappears. In that case, the condition $N = \text{const}$ and relation (31) are no longer valid. Here, for $T_{red} < T_{\min}$, we must set $\tilde{\mu} = -\infty$, i.e., assume that the conditions for existence of the Boltzmann ideal gas hold.

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