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Multidimensional Bony Attractors*

Yu. S. Ilyashenko

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To the memory of Israel Moiseevich Gelfand

Abstract. In this paper we study attractors of skew products, for which the following dichotomy is ascertained. These attractors either are not asymptotically stable or possess the following two surprising properties. The intersection of the attractor with some invariant submanifold does not coincide with the attractor of the restriction of the skew product to this submanifold but contains this restriction as a proper subset. Moreover, this intersection is thick on the submanifold, that is, both the intersection and its complement have positive relative measure. Such an intersection is called *a bone*, and the attractor itself is said to be *bony*. These attractors are studied in the space of skew products. They have the important property that, on some open subset of the space of skew products, the set of maps with such attractors is, in a certain sense, prevalent, i.e., "big." It seems plausible that attractors with such properties also form a prevalent subset in an open subset of the space of diffeomorphisms.

Key words: attractor, skew product, invariant set.

1. Introduction

As far as the author is aware, all attractors of dynamical systems on a compact manifold with an invariant submanifold which do not coincide with the manifold and have been studied up to 2010 possess the following two properties:

the intersection of the attractor with the invariant submanifold of the system coincides with the attractor of the restriction of the system to this submanifold;

this intersection either has k-dimensional Lebesgue measure 0 *or coincides with the whole submanifold*; *here k stands for the dimension of the submanifold.*

In the present work we construct attractors that possess neither of these properties, while the dimension k is arbitrary. The purpose of this paper is not only to prove the results formulated in it but also to present the construction used to prove them. We believe that this construction deserves further examination. The major component of this construction is the presence of two fiber maps such that the attractor of one of them intersects the repeller of the other and this intersection is irremovable by a small perturbation.

This construction is motivated by the following example, which is, however, destroyed by a small perturbation.

Let Σ^k denote the set of all doubly infinite sequences over a k-element alphabet $\{0,1,\dots,k-1\}$.

Consider a skew product over Σ^2 with circle fiber, that is, a map of the form

$$F\colon X=\Sigma^2\times S^1\to X,\quad (\omega,x)\mapsto(\sigma\omega,f_{\omega_0}x). \tag{1}$$

The diffeomorphisms f_0 and f_1 in this formula are called the *fiber maps*. Suppose that f_0 and f_1 possess the following properties.

1. The rotation number of both maps is 0. All their fixed points are hyperbolic.
2. The map f_0 is a north-south map with an attractor a_0 and a repeller r_0.

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3. The map f_1 has attractors a_1 and a_2 and repellers r_1 and r_2, and $r_1 = a_0$. Moreover, on the arc $[r_0, r_2]$ (the smallest of the two arcs with endpoints r_0 and r_2), there are no attractors of the maps f_0 and f_1.

4. The multiplicators of the fiber maps at the point $a_0 = r_1$ satisfy the inequality

$$f_0'(a_0)f_1'(a_0) < 1. \tag{2}$$

Theorem 1. *A skew product* (1) *with fiber maps f_0 and f_1 described above has a Lyapunov unstable Milnor attractor.*

This theorem was proved by Shilin [5]. It motivates the consideration of those skew products whose attractor and repeller (which do not necessarily consist of one point) intersect.

The unusual property of the attractor of the map (1) is due to the coincidence of the attractor of the fiber map f_0 and the repeller of the fiber map f_1. This coincidence can be removed by a small perturbation. But if the fibers are multidimensional, the attractor of the map f_0 is a submanifold, and the repeller of f_1 is a submanifold of complementary dimension, then the intersection of these two manifolds cannot be destroyed by a small perturbation. The study of such skew products leads to our main result.

2. Main Result

2.1. A theorem and a conjecture. We say that an invariant set of a map F is *Lyapunov stable* if, for every neighborhood of this set, there is another neighborhood such that the points of the second neighborhood never leave the first neighborhood under positive iterations of F.

An invariant subset of a map F is said to be *asymptotically stable* if it has a neighborhood $\mathscr{W}$ such that, for any other neighborhood $\mathscr{V}$ of this subset, there is a number k_0 with the property

$$F^k(\mathscr{W}) \subset \mathscr{V} \quad \text{for every } k \geqslant k_0. \tag{3}$$

Definition 1. Consider a self-diffeomorphism F of a compact metric space X with a probability measure. The minimal closed set in X that contains ω-limit sets of almost all points of the space X under the iterations of F is called the *Milnor attractor*. The Milnor attractor is denoted by A_M.

Theorem 2. *In the space of step skew products over the Bernoulli shift with fiber the torus of an arbitrary dimension greater than* 3, *there exists an open set in which metrically generic* (*prevalent*) *are maps whose Milnor attractor either is not asymptotically stable or possesses the following two properties.*

1. *There exists an invariant fiber such that its intersection with the Milnor attractor has nonempty interior and nonempty complement.*

2. *The restriction of the skew product to this fiber has an attractor that is a proper subset in the intersection of the Milnor attractor with this fiber.*

A. Okunev has recently modified the construction presented below in such a way that the dichotomy mentioned in the abstract disappears, and there remain only those systems that possess properties 1 and 2 in the statement of the theorem.

When the intersection of the Milnor attractor A_M with the fiber possesses the properties specified in the theorem, this intersection is called a *bone*. An attractor possessing property 1 is said to be *thick on an invariant subset.*

Comment. By the definition of a skew product of the form (1), the invariant fiber lies over a fixed point of the Bernoulli shift. In our construction, this is the sequence consisting solely of ones. The fiber and the attractor are both closed, so their intersection is closed too. Therefore, both the bone and its complement contain an open subset.

In [6], Kudryashov studied attractors of skew products over Σ^2 with fiber the interval for which the theorem stated above is true. These attractors were called bony. A general definition of such attractors is given below. In Kudryashov's thesis [7] it was also proved that the maps with bony attractors form an open set in the space of self-diffeomorphisms of the toric slice (the product of the two-dimensional torus and the interval). In the present paper we propose a construction of a skew product over Σ^k with fiber of *arbitrary dimension* for which the theorem formulated above is true. It seems plausible that the attractors constructed here have a series of other curious properties.

The following conjecture refers to the generalization of Theorem 2 to the set of diffeomorphisms.

Conjecture 1. *In the space of diffeomorphisms of the n-torus, $n > 5$, there is an open set in which the maps whose attractors either are thick on an invariant submanifold of dimension $n - 2$ or are not asymptotically stable form a generic subset.*

2.2. On the definition of an attractor. The initial definition of an attractor (see [8] and the references therein) included requirement (3). Attractors in the sense of this definition (often called *maximal*) are automatically asymptotically stable. However, not every diffeomorphism has an attractor with property (3). The simplest example is a diffeomorphism of the circle with a single fixed point. This point is inevitably of saddle-node type. We must either regard the whole circle as an attractor (which is unnatural, since all trajectories tend to the saddle-node) or change the definition of an attractor. Such a change was made by Milnor [8], who defined the Milnor attractor, and the author [1], who defined minimal and statistical attractors. Unlike maximal attractors, these attractors exist for any diffeomorphism of a compact Riemannian manifold. Although, they are not necessarily Lyapunov stable. The simplest example is the circle diffeomorphism with a single saddle-node point discussed above. Moreover, maximal attractors are often rather excessive in the sense that they contain big subsets which are superfluous from the point of view of numerical experiments, i.e., unobservable. Minimal, statistical, and Milnor attractors are less excessive. In this paper we consider Milnor attractors.

2.3. Bony attractors. A *component of the Milnor attractor* A_M is a subset $K \subset A_M$ that possesses the following properties: (a) some neighborhood of the set K contains no points of A_M that do not belong to K; (b) there are no proper subsets of K that possess property (a).

By definition, a *bony attractor* of a homeomorphism F is the Milnor attractor of the map F or a component K of such an attractor provided that the following condition holds. The restriction of F to the bony attractor is topologically equivalent to a skew product of the form

$$F\colon (b, x) \mapsto (h(b), f_b(x)). \tag{4}$$

Here $h\colon B \to B$ is a homeomorphism of a metric measure space and

$$K = \bigsqcup_{b \in B} K_b, \qquad f_b(K_b) = K_{h(b)}, \tag{5}$$

where the set B is called *a base*, and the K_b are called *fibers*. Note that not all fibers K_b are homeomorphic to each other. Moreover, *there are fibers of different dimensions among them.* Fibers of maximal dimension are called *bones*, and the attractor itself is said to be *bony*.

It follows from (4) and (5) that the map F permutes fibers.

There exist examples of bony attractors for which the sum of the dimensions of the bones and the base equals the dimension of the phase space. However, in all known examples of bony attractors, the measure of these attractors is zero.

Problem 1. *Does there exist an open set in the space of diffeomorphisms such that each map from this set has* a thick attractor, *i.e.*, *a Milnor attractor of positive measure whose complement is nonempty* (*therefore*, *this complement is open and has positive measure too*)?

3. Construction of Bony Attractors

Here we give conditions on skew products which are sufficient for the existence of bony attractors. Examples of such skew products are given in Section 5.

Let M be a closed k-manifold, and let Σ^k be, as before, the set of doubly infinite sequences of k symbols. We set $X = \Sigma^{k+3} \times M$ and

$$F\colon X \to X, \quad (\omega, x) \mapsto (\sigma\omega, f_{\omega_0} x). \tag{6}$$

Here f_{ω_0} is one of the diffeomorphisms $f_0, f_1, \dots, f_{k+2}$. These diffeomorphisms are called *fiber maps*. Suppose that they satisfy the following conditions.

1. The diffeomorphism f_0 has a global attracting surface A_0 normally hyperbolic in the sense of Hirsch–Pugh–Shub with index greater than r. This means that the orbits with initial points close to A_0 are exponentially attracted to A_0. The rate of exponential convergence of orbits to the attracting manifold A_0 is more than r times greater than the similarly defined exponential approach rate of the orbits on A_0. Below we usually take $r = 1$. A more detailed version of these definitions (which is still simplified in comparison with the original) can be found in [1]. What is important for us is that the map f_0 possesses the property formulated below in Remark 1.

Let A_0 be a global attractor of f_0, which means that all points of M, except those of a stratified embedded submanifold, tend to A_0 under the action of f_0.

2. The diffeomorphism f_1 has a hyperbolic attractor and an invariant repelling surface R_1 (it is often homeomorphic to a torus in what follows). The repelling basin of R_1 (which, by definition, coincides with the attraction basin of the surface R_1 for f_1^{-1}) will be denoted by $\mathscr{R}$. This is an open subset of the manifold M. By (1) we denote the sequence that consists solely of ones. We shall show that the basin $\mathscr{R}$ in the fiber over the sequence (1) is a bone in the Milnor attractor of the map F.

Suppose that R_1 is a normally hyperbolic invariant submanifold for f_0 and its index is greater than r.

Remark 1. From the condition on the indexes it follows that, under a small perturbation of the maps f_0 and f_1, the perturbed maps have C^r-smooth invariant attracting and repelling manifolds close to A_0 and R_1, respectively.

3. Suppose that the restrictions $f_0|_{A_0}$ and $f_1|_{R_1}$ are Anosov diffeomorphisms.

Now, we make some assumptions about the "interaction" of the maps f_j. We say that a map f has a *repelling region* W if $\overline{W} \subset f(W)$.

4. All fiber maps $f_0, f_1, \dots, f_{k+2}$ have a common repelling region W.

To formulate property 5, we need the following lemma.

Lemma 1 Hutchinson's lemma [4]**.** *Consider a k-manifold M, an open domain $\Omega \subset M$, a number $q < 1$, and maps $g_1, \dots, g_{k+1}$ such that*

$$g_l(\Omega) \subset \Omega, \tag{7}$$

$$\operatorname{Lip} g_l|_\Omega \leqslant q. \tag{8}$$

Suppose there is also a subdomain Ω_1 contained with its closure in Ω such that

$$\Omega_1 \subset \bigcup_1^{k+1} g_l(\Omega_1). \tag{9}$$

Then, for any domain U that has nonempty intersection with Ω_1, there exists a map g in the semigroup $G^+(g_1, \dots, g_{k+1})$ such that

$$g(\Omega) \subset U. \tag{10}$$

Now, we can formulate the last two conditions on the skew product (6).

5. The fiber maps $f_2, \dots, f_{k+2}$ have a common attracting region Ω, and each point of the manifold M belongs to the basin of attraction of Ω under the iterations of one of the maps $f_2, \dots, f_{k+2}$. On the domain Ω, these maps satisfy the conditions of Hutchinson's lemma that refer to the maps $g_1, \dots, g_{k+1}$, respectively; moreover, $\Omega_1 \cap A_0 \neq \varnothing$.

6. Suppose that

$$\dim A_0 + \dim R_1 = \dim M \tag{11}$$

and the surfaces A_0 and R_1 intersect transversally in a set that contains a point p such that its future orbit under the restriction $f_1|_{R_1}$ is dense in R_1 and its past orbit under $f_0|_{A_0}$ is dense in A_0.

Let P denote an equidistributed Bernoulli measure on Σ^{k+3}. Let m be a Lebesgue measure on a fiber, and let $\mu = P \times m$ be a measure on X. The metric on X is the Cartesian product of the metrics on the base and on the fiber. These definitions allow us to talk about the Milnor attractor of the map (6).

Theorem 3. *Suppose that a skew product satisfies conditions* 1–6 *in this section. Then its Milnor attractor either is not asymptotically stable or possesses properties* 1 *and* 2 *in Theorem* 2.

Proposition 1. *The set of skew products with these properties is metrically dense in some open subset of the space* $(\mathrm{Diff}^1 \mathbb{T}^m)^{k+3}$ *with* $m \geqslant 4$.

This proposition is proved in Section 5.

Together, Theorem 3 and Proposition 1 imply Theorem 2.

4. Hunting for a Bone

In this section we prove Theorem 3. If the Milnor attractor of the map described in this theorem is not asymptotically stable, then there is nothing to prove. It remains to consider the case in which this attractor is asymptotically stable.

4.1. Sketch of the proof. Theorem 3 is proved in four steps, which correspond to the four lemmas below. In Sections 4.1–4.4 we assume the conditions of Section 3 to be satisfied.

Lemma 2. *The surface* $\{(0)\} \times A_0$ *lies in the Milnor attractor* $A_M(F)$ *of the map* F.

Lemma 3. *Let* $\omega^{01} = 0^-|1^+$. *Then*
either the Milnor attractor $A_M(F)$ *of the map* F *is not asymptotically stable,*
or the surface $\{\omega^{01}\} \times A_0$ *is contained in* $A_M(F)$.

Lemma 4. *Let* $p = A_0 \cap R_1$ *be the intersection point mentioned in condition* 6. *Suppose that* $P = ((1), p)$. *Then*

$$\omega(P) \subset A_M(F).$$

From the Hirsch–Pugh–Shub theory it follows that, in some neighborhood of the manifold R_1, the map f_1 has an invariant foliation $\mathscr{W}^u$ expanding under the action of f_1. Each fiber of this foliation intersects the surface R_1 transversally in a single point. This foliation can be extended to the whole repelling basin of the repelling surface R_1 with the help of positive iterations of the map f_1. For the extended foliation we use the same notation $\mathscr{W}^u$, and for its fiber through a point Q, the notation W^u_Q.

Lemma 5. *Let* P *be the same point as in Lemma* 4. *Then, together with every point* $T \in \omega(P)$, *the attractor* $A_M(F)$ *contains the fiber* W^u_T.

Theorem 3 follows from Lemmas 2–5, because the repelling basin $\mathscr{R}$ of the repelling surface R_1 equals the union of expanding unstable fibers:

$$\mathscr{R} = \bigcup_{Q \in R_1} W^u_Q.$$

This basin is an open domain. By Lemma 5, the set $\{(1)\} \times \mathscr{R}$ lies in the Milnor attractor of the map F. On the other hand, the complement to the Milnor attractor in the invariant fiber $\{(1)\} \times M$ contains a repelling region $\{(1)\} \times W$. Therefore, the intersection of the Milnor attractor with the invariant fiber $\{(1)\} \times M$ has nonempty interior, and the (open) complement of this intersection in the fiber is nonempty too. Note that the restriction of the map F to the fiber under consideration is the map f_1. The attractor of this map is a hyperbolic set (see property 2 in Section 3). By the Bowen theorem [3], this set has k-dimensional measure zero and cannot contain an open subset of the fiber, such as the repelling basin of the surface $\mathscr{R}$. Hence the intersection of the attractor with this fiber does not coincide with the attractor of the restriction of the map to the fiber. This proves Theorem 3.

We proceed to the proof of the lemmas.

4.2. An attracting surface in a fiber as a part of the Milnor attractor.

Proof of Lemma 2. Let $p \in \{(0)\} \times (A_0 \cap \Omega_1)$ be an arbitrary point whose past orbit under f_0 is dense in A_0. Consider an arbitrary neighborhood of p of the form

$$\mathscr{V}_p = C_{w^0(m)} \times V_p;$$

here V_p is a neighborhood of the point p in the fiber M, C_w is the cylinder corresponding to a word w, and $w^0(m) = \underbrace{0\dots0}_{m}|\underbrace{0\dots0}_{m+1}$, where the mark | is next to the left of the zero position. Here and below, we denote neighborhoods in the fiber M by romans and neighborhoods in the phase space X by italics. Subscripts indicate sets whose neighborhoods are taken.

Proposition 2. *Under the positive iterations of the map F, almost every points of the set $X \setminus (\Sigma^{k+3} \times W)$ falls in the neighborhood $\mathscr{V}_p$ at least once.*

This proposition easily implies a similar statement in which "at least once" is replaced by "infinitely many times." Indeed, by virtue of Proposition 2, the set of points that are never taken to $\mathscr{V}_p$ under the positive iterations of the map F has measure zero. The image of this set under F^{-k} is the set of points that do not fall into $\mathscr{V}_p$ after k iterations of the map F; thus, it has measure zero as well. The complement to all of these sets consists of the points that hit $\mathscr{V}_p$ infinitely many times, and this complement has full measure.

The arbitrariness of the neighborhood $\mathscr{V}_p$ implies $p \in A_M(F)$.

Since the Milnor attractor is invariant, we obtain

$$\mathrm{orb}_F((0), p) = \{(0)\} \times \mathrm{orb}_{f_0}\, p \subset A_M(F).$$

Recall that the orbit $\mathrm{orb}_{f_0}\, p$ is dense in A_0, and the Minor attractor is closed. These two facts now imply Lemma 2. □

Proof of Proposition 2. We prove that, under conditions 1–6 of Section 3, the following condition holds.

7. Let Ω and Ω_1 be the same as in condition 5 of Section 3. Then, for every domain U such that $U \cap \Omega_1 \cap A_0 \neq \varnothing$ and every point $x \in M$, there is a word $w = w(x)$ such that

$$f_w(x) \in U.$$

Here

$$f_w = f_{\omega_{k-1}} \circ \dots \circ f_{\omega_0},$$

provided that $w = \omega_0 \cdots \omega_{k-1}$. Let us derive Proposition 2 from condition 7.

Let $\mathscr{A}$ denote the set $X \setminus \Sigma^{k+3} \times W$, and let $\mathscr{B}(U)$ be the set of points in $\mathscr{A}$ that hit the domain U at least once under the positive iterations of the map F. Then Proposition 2 acquires the following form.

Proposition 3. *Let $\mathscr{V}_p$ be the same neighborhood as at the beginning of this subsection. Then almost all points of the set $\mathscr{A}$ belong to the set $\mathscr{B}(\mathscr{V}_p)$.*

Proof. Let $C(x) = (\{x\} \times \Sigma^{k+3}) \setminus \mathscr{B}(\mathscr{V}_p)$. We shall prove that

$$P(C(x)) = 0, \tag{12}$$

where P is the Bernoulli measure on Σ^{k+3} mentioned above.

Let m be the same as in the definition of the neighborhood $\mathscr{V}_p$. Choose $m' \geqslant m$ so that $U \cap \Omega_1 \neq \varnothing$, where $U = f_0^{-m'}(V_p)$ and the neighborhood V_p is the same as at the beginning of this subsection. This is possible because the past orbit of the point p under f_0 is dense in A_0.

The set $M \setminus W$ is compact. Therefore, there exists a number N_0 such that, for every $x \in M \setminus W$, the word $w = w(x)$ in condition 7 that corresponds to the point x and the domain U has length at most N_0. For this word w, we have $f_w(x) \in U$. Let $\widetilde{w}(x)$ and $\widehat{w}(x)$ denote the words obtained from $w(x)$ by adding, respectively, $2m'+1$ and m' zeros on the right. We set $|\widehat{w}(x)| = K$ and $|\widetilde{w}(x)| = L$. We have $f_{\widehat{w}(x)}(x) \in V_p$. For any sequence $\omega \in \Sigma^2$, let

$$\omega|_a^b = \omega_a \cdots \omega_b$$

be the subword of the sequence ω that begins at the position a and ends at the position b. Let $N = 2m' + 1 + N_0$, $N \geqslant L$. For every ω such that $\omega|_0^{L-1} = \widetilde{w}(x)$, we obtain

$$F^K(\omega, x) \in \mathscr{V}_p.$$

Thus, the relative measure of the set formed by those points of the horizontal fiber $\{x\} \times \Sigma^{k+3}$ which do not belong to $C(x)$ is less than 2^{-N}, and therefore the relative measure of those points of this fiber which belong to $C(x)$ is at most $\nu = \nu(N) = (2^N - 1)/2^N$.

We shall prove by induction on l that this measure is actually not larger than ν^l for any l, i.e., equals zero. For every word w' of length N that does not contain a subword $\widetilde{w}(x)$ beginning at the zero position, consider $y = f_{w'}(x)$ and the word $\widetilde{w}(y)$. Let $K = |\widetilde{w}(y)| < N$, and let $\omega|_0^{N+K-1} = w'\widetilde{w}(y)$. Then

$$F^{N+K}(\omega, x) \in \mathscr{V}_p.$$

Therefore, the relative measure of the set of those points which belong to $C(x)$ does not exceed ν^2.

This concludes the inductive step from 1 to 2. The inductive step for an arbitrary l is similar. This proves relation (12).

Proposition 3 now follows from the Fubini theorem. □

The proof of Proposition 3 uses a method due to Kudryashov [6].

Proposition 3 directly implies Proposition 2.

Proof of condition 7. By condition 5 of Section 3, for every $x \in M$, there is a word $w^0 = w^0(x)$ such that $f_{w^0}(x) \in \Omega$. By Hutchinson's lemma, for every $y \in \Omega$, there exists a word $w^1 = w^1(y)$ such that $f_{w^1}(y) \in U$. Take such a word for $y = f_{w^0}(x)$. The word $w^0(x)w^1(y)$ is as required. □

Note that in the proof of Proposition 3 we have obtained not only relation (12) but also the inequality $\dim_H C(x) < 2$.

4.3. The unstable manifold of the attracting surface.

Proof of Lemma 3. Suppose, as before, that the attractor $A_M(F)$ is asymptotically stable. As mentioned above, this means that this attractor has a neighborhood $\mathscr{W}$ such that, for any other its neighborhood $\mathscr{V}$, there is a number k_0 such that $F^k(\mathscr{W}) \subset \mathscr{V}$ for every $k \geqslant k_0$.

Suppose now that there exists a point $Q = (\omega^{01}, q) \in \{\omega^{01}\} \times A_0$ that does not belong to $A_M(F)$. Then there exist neighborhoods $\mathscr{V}_{A_M(F)}$ and $\mathscr{V}_Q$ such that $\mathscr{V}_{A_M(F)} \cap \mathscr{V}_Q = \varnothing$. Let $\mathscr{W}$ be the same neighborhood as above. Take k_0 such that $F^k(\mathscr{W}) \subset \mathscr{V}_{A_M(F)}$ for every $k \geqslant k_0$. By Lemma 2, the domain $\mathscr{W}$ is at the same time a neighborhood of the surface $\{(0)\} \times A_0$. Take $m \geqslant k_0$ such that $C_{w^0(m-1)} \times A_0 \subset \mathscr{W}$. We have

$$F^{-m}(Q) = (\sigma^{-m}\omega^{01}, f_0^{-m} q) \in C_{w^0(m-1)} \times A_0 \subset \mathscr{W}.$$

Consequently,

$$F^m(\mathscr{W}) \cap \mathscr{V}_Q \ni Q \neq \varnothing.$$

But $F^m(\mathscr{W}) \subset \mathscr{V}_{A_M(F)}$. This contradiction proves the lemma. □

4.4. The bone.

Proof of Lemma 4. As above, we suppose that the Milnor attractor of the skew product under consideration is asymptotically stable. The lemma follows from the fact that the Milnor attractor is invariant and closed. Let $p = A_0 \cap R_1$ and $P = ((1), p)$ be the same as in the statement of the lemma, and let $Q = (\omega^{01}, p)$. By Lemma 3, we have $Q \in A_M(F)$. By virtue of invariance, the same is true for the orbit of the point Q, and since the attractor is closed, it contains the ω-limit set of this orbit. The orbits of the points P and Q under F asymptotically approach each other, because the left shifts of the sequence ω^{01} tend to (1):

$$F^n(Q) = (\sigma^n\omega^{01}, f_1^n(p)), \quad F^n(P) = ((1), f_1^n(p)), \quad \operatorname{dist}(F^n(P), F^n(Q)) \to 0.$$

Therefore, $\omega(P) = \omega(Q) \subset A_M(F)$. This completes the proof of Lemma 4. □

Proof of Lemma 5. This proof makes use of the fact that not only the point $Q = (\omega^{01}, p)$ but also the whole surface $\omega^{01} \times A_0$ is contained in the Milnor attractor.

Consider the images

$$F^n(\omega^{01} \times A_0) = (\sigma^n\omega^{01} \times f_1^n(A_0)).$$

The surface $f_1^n(A_0)$ contains the point $f_1^n(p) \in R_1$. Points on the repelling surface diverge from each other at a rate slower than the rate of repelling from R_1. Hence, for a sufficiently small neighborhood V_{R_1} of the repelling surface in M, the intersections $f_1^n(A_0) \cap V_{R_1}$ approach the fibers $W^u_{f_1^n(p)}$. This is proved from the same considerations as the well-known Λ-lemma, so we omit the details.

Note that $\omega(P)$ is dense in $\{(1)\} \times R_1$. Therefore,

$$\{(1)\} \times R_1 \subset A_M(F). \tag{13}$$

Inclusion (13) proves Lemma 5 and, thereby, Theorem 2. □

5. Metric Genericity

Here we prove the local metric genericity (prevalence) of the set of skew products which satisfy conditions 1–6 in Section 3. At first, we construct a tuple $f_0^0, \dots, f_{k+2}^0$ satisfying conditions 1–5 in Section 3 but not satisfying condition 6, that is, such that the attractor and the repeller surface of the first two maps have complementary dimensions, but one of them is a subset of the other. Appropriate perturbations of this tuple yield the required tuples.

5.1. Description of the first two fiber maps of the auxiliary tuple. Here we construct maps f_0^0 and f_1^0 that have normally hyperbolic attracting and repelling surfaces and a common repelling region.

Let $k \geqslant 4$ be arbitrary, and let $M = \mathbb{T}^k$. We decompose k into two summands as $k = l + m$, $2 \leqslant l \leqslant m$, and rewrite $\mathbb{T}^k$ in the form $\mathbb{T}^l \times \mathbb{T}^m = \mathbb{T}^l \times \mathbb{T}^{m-l} \times \mathbb{T}^l$.

We begin with the map f_0^0. Let $G^0\colon \mathbb{T}^l \to \mathbb{T}^l$ be a Morse–Smale map such that all of its nonwandering points are fixed. We assume that all Morse–Smale maps that appear in the sequel satisfy this condition. Let a be an attracting fixed point of the map G^0, and let r be a repelling point. Suppose that $A_l\colon \mathbb{T}^l \to \mathbb{T}^l$ is a hyperbolic diffeomorphism of the torus. In this notation, $A_m\colon \mathbb{T}^m \to \mathbb{T}^m$ is a hyperbolic automorphism of the torus $\mathbb{T}^m$. We set

$$f_0^0 = G^0 \times A_m.$$

The map f_0^0 has a globally attracting surface $A_0 = \{a\} \times \mathbb{T}^m$, whose restriction to A_0 is a hyperbolic diffeomorphism. The normal hyperbolicity of the surface A_0 is guaranteed by choosing the map G^0 to be a strong enough contraction in a neighborhood of the point a.

Now, we proceed to the construction of the map f_1^0.

Let $\widetilde{G}^0\colon \mathbb{T}^l \to \mathbb{T}^l$ be a Morse–Smale map for which both points a and r are repelling. Let $\widetilde{G}^1\colon \mathbb{T}^{m-l} \to \mathbb{T}^{m-l}$ be another Morse–Smale map with a repeller $\tilde{r} \in \mathbb{T}^{m-l}$. Then $G^1 = \widetilde{G}^0 \times \widetilde{G}^1\colon \mathbb{T}^m \to \mathbb{T}^m$ is a Morse–Smale map with the repeller $(a, \tilde{r}) \in \mathbb{T}^l \times \mathbb{T}^{m-l}$. We set

$$f_1^0 = G^1 \times A_l.$$

The map f_1^0 has the repelling surface $R_1 = \{(a, \tilde{r})\} \times \mathbb{T}^l \subset A_0$. The restriction of this map to the surface R_1 is a hyperbolic diffeomorphism. The normal hyperbolicity of the surface R_1 is guaranteed by choosing the map G^1 to be strongly expanding in some neighborhoods of its repelling points.

The maps f_0^0 and f_1^0 have a common repelling region W. Indeed, both maps G^0 and $\widetilde{G}^0$ have a common repelling point r. Therefore, they have a common repelling neighborhood $V \subset \mathbb{T}^l$ of this point. The domain $W = V \times \mathbb{T}^m$ is as required.

The repelling basin $\mathscr{R}_1$ of the repelling surface R_1 of the map f_1^0 is an open set, and its complement contains the open set W. The attractor of the map f_1^0 is hyperbolic.

Conditions 1–4 in Section 3 are satisfied, while condition 6 is not; in what follows, we satisfy condition 6 by applying a small perturbation to the map f_0^0.

Condition 5 refers to the maps $f_2, \dots, f_{k+2}$, and we proceed to construct these maps.

5.2. Description of the remaining fiber maps. Let $g_2^0 \colon \mathbb{T}^k \to \mathbb{T}^k$ be a Morse–Smale map with a globally attracting fixed point b. More specifically, all points of the torus $\mathbb{T}^k$, except the points of some stratified $(k-1)$-manifold S, are attracted to b under the action of g_2^0. Suppose that, in some coordinate neighborhood of the point b, the diffeomorphism g_2^0 is a scalar contraction with coefficient close to 1. Consider diffeomorphisms $h_j \colon \mathbb{T}^k \to \mathbb{T}^k$ close to the identity and satisfying the following conditions.

1. In the same coordinate neighborhood of the point b, the maps h_j are translations, and there is no hyperplane containing the points $h_j(b)$.

2. The intersection $\bigcap_2^{k+2} h_j(S)$ is empty.

We set

$$f_j = h_j \circ g_2^0 \circ h_j^{-1}.$$

Note that $\mathbb{T}^k \setminus h_j(S)$ is the basin of attraction of the point $h_j(b)$ under f_j; we denote this basin by $\mathscr{A}_j$. Condition 2 guarantees that any point $x \in \mathbb{T}^k$ belongs to at least one basin $\mathscr{A}_j$.

Let Ω denote the simplex in the same coordinate neighborhood with vertices $h_j(b)$. For the domain Ω_1 we take a simplex with the same mass center as Ω and obtained from Ω by applying a contraction with coefficient close to 1. The maps $f_2^0, \dots, f_{k+2}^0$ thus constructed satisfy condition 5 in Section 3.

Note that conditions 1–5 in Section 3 are C^1-robust. Consider a neighborhood of the tuple $f_0^0, \dots, f_{k+2}^0$ in the space C^1. For a generic tuple from this neighborhood, the attracting surface of the map f_0, which is close to A_0, transversally intersects the repelling surface of the map f_1, which is close to R_1. The set of tuples for which this intersection contains a point whose past orbit under f_0 is dense in A_0 and future orbit under f_1 is dense in R_1 is metrically generic, because almost all points of A_0 and R_1 possess the required property.

This concludes the proof of Theorem 2.

6. Problems

Consider a skew product that satisfies the conditions in Section 3 except condition 6, which is replaced by the following condition.

$6'$. The surfaces A_0 and R_1 intersect transversally in a finite set consisting of periodic points of the map f_1.

Problem 2. *Suppose that the map F (see (6)) satisfies conditions 1–5 in Section 3 and condition $6'$. Let $Q = A_0 \cap R_1 \subset \operatorname{Per} f_1$ be a finite set. Is it true that the intersection of the Milnor attractor $A_M(F)$ and the fiber $\{(1)\} \times M$ has the form*

$$A_M(F) \cap \{(1)\} \times M = \bigcup_{q \in Q} W_q^u, \tag{14}$$

where W_q^u is the strong unstable manifold of the point q, which was defined in Section 4.1?

If relation (14) is true, then the attractor $A_M(F)$ is Lyapunov unstable because of the hyperbolicity of the map $f_1|_{R_1}$. On the other hand, the maps with properties 1–5 and $6'$ are dense in the space of skew products of the form (6). The validity of (14) would give us an example of a set dense in some open subset of the space of skew products over the Bernoulli shift such that any map from this set has Lyapunov unstable Milnor attractor.

Such a phenomenon has never been observed. Note that in [2] a residual set in an open subset of the space of diffeomorphisms was constructed, in which all maps have Lyapunov stable Milnor attractors not being asymptotically stable.

In conclusion, we formulate the following problem.

Problem 3. *Is there an open set in the space of diffeomorphisms of some closed manifold that consists of maps with Lyapunov unstable Milnor attractors?*

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References

[1] V. I. Arnold, V. S. Afraymovich, Yu. S. Ilyashenko, and L. P. Shilnikov, "Theory of bifurcations," in: Dynamical Systems-5, Itogi Nauki i Tekhniki, Current Problems in Mathematics, Fundamental direction, vol. 5 [in Russian], VINITI, Moscow, 1986, 5–218.

[2] C. Bonatti, M. Li, and D. Yang, On the existence of attractors, preprint #568 of Institut Mathématique de Bourgogne, 2009.

[3] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Math., vol. 470, Springer-Verlag, Berlin–New York, 1975.

[4] J. Hutchinson, "Fractals and self-similarity," Indiana Univ. Math. J., **30**:5 (1981), 713–747.

[5] Yu. S. Ilyashenko and I. S. Shilin, "Relatively unstable attractors," Trudy Mat. Inst. Steklov., **277** (2012), 91–100.

[6] Yu. G. Kudryashov, "Bony attractors," Funkts. Anal. Prilozhen., **44**:3 (2010), 73–76; English transl.: Functional Anal. Appl., **44**:3 (2010), 219–222.

[7] Yu. Kudryashov, Des orbites périodiques et des attracteurs des systémes dynamiques, Ph.D. Thesis, École Normale Supérieure de Lyon, Lyons, 2010.

[8] J. Milnor, "On the concept of attractors," Comm. Math. Phys., **99**:2 (1985), 177–196.

Cornell University, USA
Moscow State University
Independent Moscow University
Steklov Mathematical Institute
National Research University "Higher School of Economics"
e-mail: yulijs@gmail.com

Translated by I. S. Shilin