J. Math. Anal. Appl. 402 (2013) 648–659

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

A pointwise selection principle for maps of several variables via the total joint variation

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ARTICLE INFO

Article history:
Received 1 March 2012
Available online 8 February 2013
Submitted by B.S. Thomson

Keywords:
Maps of several variables
Total joint variation
Selection principle
Metric semigroup
Pointwise convergence

ABSTRACT

Given a rectangle in the real Euclidean n-dimensional space and two maps f and g defined on it and taking values in a metric semigroup, we introduce the notion of the total joint variation $\mathrm{TV}(f, g)$ of these maps. This extends similar notions considered by Hildebrandt (1963) [17], Leonov (1998) [18], Chistyakov (2003, 2005) [5,8] and the authors (2010). We prove the following irregular pointwise selection principle in terms of the total joint variation: *if a sequence of maps* $\{f_j\}_{j=1}^{\infty}$ *from the rectangle into a metric semigroup is pointwise precompact and* $\limsup_{j,k\to\infty} \mathrm{TV}(f_j, f_k)$ *is finite, then it admits a pointwise convergent subsequence* (whose limit may be a highly irregular, e.g., everywhere discontinuous, map). This result generalizes some recent pointwise selection principles for real functions and maps of several real variables.

1. Introduction

Pointwise selection principles (PSP) are assertions which state that under certain specified conditions on a sequence (or a family) of functions, their domain and range, the sequence contains a pointwise convergent subsequence. The known PSP can be classified as regular and irregular. Regular PSP usually apply to sequences of regulated functions (i.e., those having finite one-sided limits at each point of the domain) and additionally assert that analytical properties of the pointwise limit of the extracted subsequence are at least as good as those of the members of the sequence (e.g., it belongs to the same functional class of regulated functions). If this is not the case or no information is available about properties of the pointwise limit, the PSP under consideration is termed irregular. Let us illustrate this by examples.

The classical Helly theorem is a regular PSP: *a pointwise bounded sequence of real functions on a closed interval* $[a, b] \subset \mathbb{R}$ *of uniformly bounded variation admits a pointwise convergent subsequence whose pointwise limit is a function of bounded variation*. This theorem, having numerous applications in Analysis [2–4,7,16,17,19,23], has been generalized for functions and maps of one real variable [2,7,10,12,19] and several real variables [1,4,6,13,17,18,20]; see also references in these papers. The above Helly theorem and all enlisted generalizations are based on the Helly theorem for monotone functions (or its counterpart for monotone functions of several variables [4,18]): *a uniformly bounded sequence of real monotone functions on* $[a, b]$ *contains a pointwise convergent subsequence whose pointwise limit is a bounded monotone function*. Thus, the PSP, alluded to above, are regular.

A different kind of a PSP has been presented in [24]. Given a real function f on $[a, b]$, we denote by $T(f)$ the supremum of sums of the form $\sum_{i=1}^{n} |f(t_i)|$ taken over all $n \in \mathbb{N}$ and all finite collections of points $\{t_1, t_2, \ldots, t_n\} \subset [a, b]$ such that either $(-1)^i f(t_i) > 0$ for all $i = 1, 2, \ldots, n$, or $(-1)^i f(t_i) < 0$ for all $i = 1, 2, \ldots, n$, or $(-1)^i f(t_i) = 0$ for all $i = 1, 2, \ldots, n$ (if f is

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doi:10.1016/j.jmaa.2013.02.005

nonnegative on $[a, b]$ or nonpositive on $[a, b]$, we set $T(f) = \sup_{t\in[a,b]} |f(t)|$). The quantity $T(f)$ is called the *oscillation of f* on $[a, b]$. Schrader's generalization of the Helly theorem is as follows: *if a sequence of real functions* $\{f_j\}_{j=1}^{\infty}$ *on* $[a, b]$ *is such that* $\sup_{j,k\in\mathbb{N}} T(f_j - f_k)$ *is finite, then it contains a pointwise convergent subsequence*. In contrast to regular PSP, this result applies to the sequence of non-regulated functions $f_j(t) = (-1)^j \mathcal{D}(t)$, $j \in \mathbb{N}$, $t \in [a, b]$, where $\mathcal{D}$ is the Dirichlet function (which is equal to 1 at rational points and 0 otherwise). Thus, we have an example of an irregular PSP; it is worth noting that it is based on Ramsey's theorem from formal logic (see Theorem A in Section 3). At present even for functions and maps of one real variable only a few irregular PSP are known in the literature [11,12,15], which are, however, more general than PSP based on the Helly theorem for monotone functions.

The purpose of this paper is to present a PSP in the context of maps of several real variables taking values in metric semigroups (i.e., metric spaces equipped with the operation of addition), which, in particular, gives an appropriate framework for treating multifunctions of several variables (cf. [5,7,8,14,22]). In this context a regular PSP has been recently presented in [13] for maps of finite total variation in the sense of Vitali, Hardy and Krause. This paper addresses an irregular PSP, which is expressed in terms of the finite total joint variation and, due to the chosen context, it is of different nature as compared to [15,24] and more close to [11–13].

The paper is organized as follows. In Section 2 we present necessary definitions and our main result (Theorem 1). In order to get to its proof as quickly as possible, in Section 3 we collect all main ingredients and auxiliary facts. Section 4 is devoted to the proof of Theorem 1 and Section 5 contains proofs of the auxiliary results exposed in Section 3.

2. Definitions and the main result

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of positive and nonnegative integers, respectively, and $n \in \mathbb{N}$. Given $x, y \in \mathbb{R}^n$, we write $x = (x_1, \dots, x_n) = (x_i : i \in \{1, \dots, n\})$ for the coordinate representation of x, and set $x + y = (x_1 + y_1, \dots, x_n + y_n)$, and $x - y$ is defined similarly. The inequality $x < y$ is understood componentwise, i.e., $x_i < y_i$ for all $i \in \{1, \dots, n\}$, and similar meanings apply to $x = y$, $x \le y$, $y \ge x$ and $y > x$. If $x < y$ or $x \le y$, we denote by I_x^y the rectangle $\prod_{i=1}^n [x_i, y_i] = [x_1, y_1] \times \cdots \times [x_n, y_n]$. Elements of the set $\mathbb{N}_0^n$ are as usual said to be *multiindices* and denoted by Greek letters and, given $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we set $|\theta| = \theta_1 + \cdots + \theta_n$ (the order of θ) and $\theta x = (\theta_1 x_1, \dots, \theta_n x_n)$. The n-dimensional zero $0_n = (0, \dots, 0)$ and unit $1_n = (1, \dots, 1)$ will be denoted by 0 and 1, respectively (the dimension of 0 and 1 will be clear from the context). We also put $\mathcal{E}(n) = \{\theta \in \mathbb{N}_0^n : \theta \le 1 \text{ and } |\theta| \text{ is even}\}$ (the set of 'even' multiindices) and $\mathcal{O}(n) = \{\theta \in \mathbb{N}_0^n : \theta \le 1 \text{ and } |\theta| \text{ is odd}\}$ (the set of 'odd' multiindices). For elements from the set $\mathcal{A}(n) = \{\alpha \in \mathbb{N}_0^n : 0 \ne \alpha \le 1\}$ we simply write $0 \ne \alpha \le 1$.

The domain of (almost) all maps under consideration is a rectangle I_a^b with fixed $a, b \in \mathbb{R}^n$, $a < b$, called the *basic rectangle*. The range of maps is a *metric semigroup* $(M, d, +)$, i.e., (M, d) is a metric space, $(M, +)$ is an Abelian semigroup with the operation of addition $+$, and d is translation invariant: $d(u, v) = d(u + w, v + w)$ for all $u, v, w \in M$. A nontrivial example of a metric semigroup is as follows [14,22]. Let $(X, \|\cdot\|)$ be a real normed space and M be the family of all nonempty closed bounded convex subsets of X equipped with the Hausdorff metric d given by $d(U, V) = \max\{\mathrm{e}(U, V), \mathrm{e}(V, U)\}$, where $U, V \in M$ and $\mathrm{e}(U, V) = \sup_{u\in U} \inf_{v\in V} \|u - v\|$. Given $U, V \in M$, defining $U \oplus V$ as the closure in X of the Minkowski sum $\{u + v : u \in U, v \in V\}$, we find that the triple $(M, d, \oplus)$ is a metric semigroup.

Note at once that if $(M, d, +)$ is a metric semigroup, then, by virtue of the triangle inequality for d and the translation invariance of d, we have:

$$d(u + u', v + v') \le d(u, v) + d(u', v'), \tag{2.1}$$

$$d(u, v) \le d(u + u', v + v') + d(u', v'), \tag{2.2}$$

for all $u, v, u', v' \in M$. Inequality (2.1) implies the continuity of the addition operation $(u, v) \mapsto u + v$ as a map from $M \times M$ into M.

Given two maps $f, g : I_a^b \to (M, d, +)$ and $x, y \in I_a^b$ with $x \le y$, we define the *Vitali-type n-th joint mixed 'difference' of f and g* on $I_x^y \subset I_a^b$ by

$$\mathrm{md}_n(f, g, I_x^y) = d\Bigg(\sum_{\theta\in\mathcal{E}(n)} f\big(x + \theta(y - x)\big) + \sum_{\eta\in\mathcal{O}(n)} g\big(x + \eta(y - x)\big), \\ \sum_{\eta\in\mathcal{O}(n)} f\big(x + \eta(y - x)\big) + \sum_{\theta\in\mathcal{E}(n)} g\big(x + \theta(y - x)\big)\Bigg). \tag{2.3}$$

As an example, let us exhibit the form of $\mathrm{md}_n(f, g, I_x^y)$ for the first three dimensions $n = 1, 2, 3$. Since $x = (x_1, \dots, x_n) \in I_a^b$, and likewise for $y \in I_a^b$ and $\theta \in \mathbb{N}_0^n$, we note that the i-th coordinate $x_i + \theta_i(y_i - x_i)$ of $x + \theta(y - x)$ is equal to x_i if $\theta_i = 0$ and it is equal to y_i if $\theta_i = 1$. Thus, for $n = 1$ we have $\mathcal{E}(1) = \{0\}$ and $\mathcal{O}(1) = \{1\}$, and so, $\mathrm{md}_1(f, g, I_x^y) = d(f(x) + g(y), f(y) + g(x))$. If $n = 2$, then $\mathcal{E}(2) = \{(0, 0), (1, 1)\}$ and $\mathcal{O}(2) = \{(0, 1), (1, 0)\}$, and so,

$$\mathrm{md}_2(f, g, I_{x_1,x_2}^{y_1,y_2}) = d\big(f(x_1, x_2) + f(y_1, y_2) + g(x_1, y_2) + g(y_1, x_2), f(x_1, y_2) + f(y_1, x_2) + g(x_1, x_2) + g(y_1, y_2)\big).$$

Now, if $n = 3$, we find $\mathcal{E}(3) = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ as well as $\mathcal{O}(3) = \{(1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, which imply that the expression for $\mathrm{md}_3(f, g, I_{x_1,x_2,x_3}^{y_1,y_2,y_3})$ is of the form

$$d\Big(f(x_1, x_2, x_3) + f(y_1, y_2, x_3) + f(y_1, x_2, y_3) + f(x_1, y_2, y_3)$$
$$+ g(y_1, y_2, y_3) + g(y_1, x_2, x_3) + g(x_1, y_2, x_3) + g(x_1, x_2, y_3),$$
$$f(y_1, y_2, y_3) + f(y_1, x_2, x_3) + f(x_1, y_2, x_3) + f(x_1, x_2, y_3)$$
$$+ g(x_1, x_2, x_3) + g(y_1, y_2, x_3) + g(y_1, x_2, y_3) + g(x_1, y_2, y_3)\Big).$$

If g is a constant map, quantity (2.3) is, by the translation invariance of d, independent of g and reduces to the *Vitali-type* n-th *mixed difference of* f on I_x^y, denoted by $\mathrm{md}_n(f, I_x^y)$; cf. [1,5,8,17] if $n = 2$ and [6,9,13,18] if $n \in \mathbb{N}$. Note also that if $x \not< y$ in (2.3), then $\mathrm{md}_n(f, g, I_x^y) = 0$ for all maps f and g (see Remark 2.1 in [13, Part I]).

The *Vitali-type* n-th *joint variation* of $f,\ g : I_a^b \to M$ is defined by

$$V_n(f, g, I_a^b) = \sup_{\mathcal{P}} \sum_{1\le\sigma\le\kappa} \mathrm{md}_n\big(f, g, I_{x[\sigma-1]}^{x[\sigma]}\big), \tag{2.4}$$

the supremum being taken over all multiindices $\kappa \in \mathbb{N}^n$ and all *net partitions* of I_a^b of the form $\mathcal{P} = \{x[\sigma]\}_{\sigma=0}^{\kappa}$, where points $x[\sigma] = (x_1(\sigma_1), \ldots, x_n(\sigma_n))$ from I_a^b are indexed by $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$ with $\sigma \le \kappa$ and satisfy the conditions: $x[0] = a$, $x[\kappa] = b$ and $x[\sigma - 1] < x[\sigma]$ for all $1 \le \sigma \le \kappa$ (in other words, a net partition $\mathcal{P}$ is the Cartesian product of ordinary partitions of closed intervals $[a_i, b_i]$, $i = 1, \ldots, n$). Note that all rectangles $I_{x[\sigma-1]}^{x[\sigma]}$ of a net partition are non-degenerate, non-overlapping and their union is I_a^b.

In order to define the notion of the total joint variation of maps $f,\ g : I_a^b \to M$, we need the notion of a joint variation of f and g of order less than n. Following [9], we define the *truncation of a point* $x \in \mathbb{R}^n$ *by a multiindex* $0 \ne \alpha \le 1$ by $x\lfloor\alpha = (x_i : i \in \{1, \ldots, n\},\ \alpha_i = 1)$, and set $I_a^b\lfloor\alpha = I_{a\lfloor\alpha}^{b\lfloor\alpha}$. Clearly, $x\lfloor 1 = x$ and $I_a^b\lfloor 1 = I_a^b$, and if $x \in I_a^b$, then $x\lfloor\alpha \in I_a^b\lfloor\alpha$. For example, if $x = (x_1, x_2, x_3, x_4)$ and $\alpha = (0, 1, 0, 1)$, we have $x\lfloor\alpha = (x_2, x_4)$ and $I_a^b\lfloor\alpha = [a_2, b_2] \times [a_4, b_4]$. Given $f : I_a^b \to M$ and $z \in I_a^b$, we define the *truncated map* $f_\alpha^z : I_a^b\lfloor\alpha \to M$ *with the base at* z by $f_\alpha^z(x\lfloor\alpha) = f\big(z + \alpha\,(x - z)\big)$ for all $x \in I_a^b$. It follows that f_α^z depends only on $|\alpha|$ variables $x_i \in [a_i, b_i]$, for which $\alpha_i = 1$, and the other variables remain fixed and equal to z_j when $\alpha_j = 0$. In the above example we get $f_\alpha^z(x_2, x_4) = f_\alpha^z(x\lfloor\alpha) = f(z_1, x_2, z_3, x_4)$ for all $(x_2, x_4) \in [a_2, b_2] \times [a_4, b_4]$.

Now, if $f,\ g : I_a^b \to M$ and $0 \ne \alpha \le 1$, the truncated maps $f_\alpha^a,\ g_\alpha^a : I_a^b\lfloor\alpha \to M$ with the base at $z = a$ depend only on $|\alpha|$ variables, and so, making use of the definition (2.4) and (2.3) with n replaced by $|\alpha|$, f—by f_α^a, g—by g_α^a and I_a^b—by $I_a^b\lfloor\alpha$, we get the notion of the (*Hardy–Krause-type* [1,9,13,18]) $|\alpha|$-th *joint variation* of f and g, denoted by $V_{|\alpha|}(f_\alpha^a, g_\alpha^a, I_a^b\lfloor\alpha)$.

We define the *total joint variation of* $f : I_a^b \to M$ *and* $g : I_a^b \to M$ by

$$\mathrm{TV}(f, g, I_a^b) = \sum_{0\ne\alpha\le 1} V_{|\alpha|}(f_\alpha^a, g_\alpha^a, I_a^b\lfloor\alpha), \tag{2.5}$$

the summations here and throughout the paper being taken over *n-dimensional* multiindices in the ranges specified under the summation sign.

The quantities (2.3)–(2.5) are symmetric with respect to f and g and are equal to zero if $f = g$. For a constant map g they reduce to the (already mentioned Vitali-type) n-th mixed difference $\mathrm{md}_n(f, I_x^y)$, the n-th variation $V_n(f, I_a^b)$ and the total variation $\mathrm{TV}(f, I_a^b)$ of the map f, respectively [6,9,13,18]. Denoting by $\mathrm{BV}(I_a^b; M)$ the set of all maps $f : I_a^b \to M$ with $\mathrm{TV}(f, I_a^b) < \infty$, it follows from (2.3)–(2.5) that if $f,\ g \in \mathrm{BV}(I_a^b; M)$, then

$$\mathrm{TV}(f, g, I_a^b) \le \mathrm{TV}(f, I_a^b) + \mathrm{TV}(g, I_a^b), \tag{2.6}$$

which is a consequence of similar inequalities for md_n and V_n in place of TV.

A sequence $\{f_j\} \equiv \{f_j\}_{j\in\mathbb{N}}$ of maps from I_a^b into M is said to be: (a) *pointwise convergent* on I_a^b to a map $f : I_a^b \to M$ if $d(f_j(x), f(x)) \to 0$ as $j \to \infty$ for all $x \in I_a^b$; (b) *pointwise precompact* on I_a^b provided the closure in M of the set $\{f_j(x)\}_{j\in\mathbb{N}}$ is compact for all $x \in I_a^b$.

Let $\alpha_{j,k} \in \mathbb{R}$ for $j, k \in \mathbb{N}$ be a double sequence such that $\alpha_{j,j} = 0$ for all $j \in \mathbb{N}$. It is said to *converge* to a number $l \in \mathbb{R}$, in symbols, $\lim_{j,k\to\infty} \alpha_{j,k} = l$, if for each $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that $\alpha_{j,k} \in [l - \varepsilon, l + \varepsilon]$ for all $j \ge N$ and $k \ge N$ with $j \ne k$ (cf. [11]). Also, we set

$$\limsup_{j,k\to\infty} \alpha_{j,k} = \lim_{N\to\infty} \sup\{\alpha_{j,k} : j \ge N,\ k \ge N,\ j \ne k\}.$$

Our main result, to be proved in Section 4, is the following irregular PSP:

Theorem 1. *A pointwise precompact sequence* $\{f_j\}$ *of maps from the basic rectangle* I_a^b *into a metric semigroup* $(M, d, +)$ *such that*

$$\limsup_{j,k\to\infty} TV(f_j, f_k, I_a^b) \quad \text{is finite} \tag{2.7}$$

contains a subsequence which converges pointwise on I_a^b.

In the context of metric semigroups, as ranges of maps, this result implies a regular PSP from [13, Part II, Theorem 1] (containing as particular cases PSP from [1,17,18]), which asserts that *under the assumptions of* Theorem 1, *if instead of* (2.7) *we have* $C \equiv \sup_{j\in\mathbb{N}} \mathrm{TV}(f_j, I_a^b)$ *is finite, then* $\{f_j\}$ *contains a subsequence which converges pointwise on* I_a^b *to a map* $f \in \mathrm{BV}(I_a^b; M)$ *such that* $\mathrm{TV}(f, I_a^b) \le C$. In fact, it follows from (2.6) that $\mathrm{TV}(f_j, f_k, I_a^b) \le 2C$ for all $j, k \in \mathbb{N}$, and so, (2.7) is satisfied. However, the converse implication is not true in general as the following example shows. Let $n = 2$ and the sequence $f_j : I_0^1 = [0, 1] \times [0, 1] \to \mathbb{R}, j \in \mathbb{N}$, be given by

$$f_j(x, y) = \mathcal{D}(x) \cdot \mathcal{D}(y) + \begin{cases} 1 & \text{if } (x, y) \in [0, 1] \times [1/j, 1], \\ 0 & \text{if } (x, y) \in [0, 1] \times [0, 1/j), \end{cases}$$

where $\mathcal{D}$ is the Dirichlet function on $[0, 1]$. Then we have: $\mathrm{TV}(f_j, I_0^1) = +\infty$ for all $j \in \mathbb{N}$, $\mathrm{TV}(f_j, f_k, I_0^1) = 2$ for all $j, k \in \mathbb{N}$ with $j \ne k$, and $\{f_j\}$ converges pointwise on I_0^1 as $j \to \infty$. Thus, Theorem 1 extends the class of sequences of maps having pointwise convergent subsequences, but we no longer can infer that the pointwise limits of these subsequences are 'regulated' maps.

3. Joint mixed differences and the total joint variation

In this section we collect the main ingredients of the proof of Theorem 1: Theorems 2–4, which establish relations between joint mixed differences of all orders and properties of the total joint variation, and Ramsey's Theorem.

Throughout this section the triple $(M, d, +)$ is a metric semigroup.

Theorem 2. *If* $f, g : I_a^b \to M$, $x, y \in I_a^b$, $x \le y$, *and* $TV(f, g, I_x^y) < \infty$, *then*

$$d(f(x) + g(y), f(y) + g(x)) \le \sum_{0\ne\alpha\le 1} md_{|\alpha|}(f_\alpha^x, g_\alpha^x, I_x^y \lfloor \alpha) \le TV(f, g, I_x^y).$$

This theorem, to be proved in Section 5, is a counterpart of Leonov's (in)equalities [18, Theorem 2 and Corollary 5] (see also [9, Part I, Lemma 6 and (3.5)]) for $n \in \mathbb{N}$ and $M = \mathbb{R}$, which have been generalized to the case of a metric semigroup M in [1, Theorem 1(a)] and [8, Part I, Lemma 2(a) and its proof] (for $n = 2$) and [13, Part I, Theorem 2] (for $n \in \mathbb{N}$).

Theorem 3. *Suppose* $f, g : I_a^b \to M$, $x, y \in I_a^b$, $x \le y$, $0 \ne \gamma \le 1$, *and* $TV(f, g, I_a^b)$ *is finite. Then*

$$\sum_{0\ne\alpha\le\gamma} V_{|\alpha|}(f_\alpha^x, g_\alpha^x, I_x^y \lfloor \alpha) = TV\big(f, g, I_x^{x+\gamma(y-x)}\big) \le TV(f, g, I_a^{x+\gamma(y-x)}) - TV(f, g, I_a^x). \tag{3.1}$$

This theorem is an extension of [9, Part II, Lemma 8 and (2.8)] (for $M = \mathbb{R}$) and [13, Part II, Theorem B] (for a metric space M). The proof of Theorem 3 is similar to the proof of Lemma 8 from [9, Part II], and so, it is omitted (see, however, comments following Theorem 4 below).

Recall that a function $\varphi : I_a^b \to \mathbb{R}$ is said to be *totally monotone* (cf. [9, Part II, Section 3] and [18]) if, given $0 \ne \alpha \le 1$ and $x, y \in I_a^b$ with $x \le y$, we have: $(-1)^{|\alpha|} \sum_{0\le\theta\le\alpha} (-1)^{|\theta|} \varphi\big(x + \theta(y - x)\big) \ge 0$. Denote by $\mathrm{Mon}(I_a^b; \mathbb{R})$ the set of all totally monotone functions. It is known from the references above that if $\varphi \in \mathrm{Mon}(I_a^b; \mathbb{R})$, then $\varphi \in \mathrm{BV}(I_a^b; \mathbb{R})$ and $\varphi(x) \le \varphi(y)$ and $\mathrm{TV}(\varphi, I_x^y) = \varphi(y) - \varphi(x)$ for all $x, y \in I_a^b$ with $x \le y$.

Theorem 4. *Suppose that* $f, g : I_a^b \to M$ *are such that* $TV(f, g, I_a^b) < \infty$. *Setting* $\nu_{f,g}(x) = TV(f, g, I_a^x)$ *for all* $x \in I_a^b$, *we have:* $\nu_{f,g} \in Mon(I_a^b; \mathbb{R})$ *and* $TV(\nu_{f,g}, I_a^b) = TV(f, g, I_a^b)$.

The proof of Theorem 4 is similar to the proofs of Lemma 9 and Corollaries 10 and 11 from [9, Part II] when $M = \mathbb{R}$, and so, they are omitted. It is to be noted that the proofs of Lemmas 8 and 9 and Corollaries 10 and 11 from [9, Part II] rely on

(i) Equality (3.2) from [9, Part I, Lemma 5],
(ii) Lemma 7 from [9, Part I], and
(iii) The property of *additivity* of the $|\alpha|$-th variation $V_{|\alpha|}$ for each $0 \ne \alpha \le 1$ for real valued functions of n variables.

In our case (in the metric semigroup context) assertions (i), (ii) and (iii) need proper interpretations and different proofs. The respective counterparts of (i), (ii) and (iii) are presented in Section 5 as Lemmas 1–3.

The final ingredient in the proof of Theorem 1 is Ramsey's theorem from formal logic [21, Theorem A], which we recall below as Theorem A.

Given a nonempty set Γ, $k \in \mathbb{N}$ and an injective map $\gamma : \{1, \ldots, k\} \to \Gamma$, the set $\{\gamma(1), \ldots, \gamma(k)\}$ is called a k*-combination* of elements of Γ (note that a k-combination may be generated by $k!$ different injective functions). Let $\Gamma[k]$ denote the family of all k-combinations of elements of Γ.

Theorem A. *Given an infinite set* Γ *and* $k, m \in \mathbb{N}$, *let* $\Gamma[k] = \bigcup_{i=1}^m C_i$ *be a disjoint union of its* m *nonempty subsets* C_i. *Then, under the Axiom of Choice, there exists an infinite subset* $\Delta \subset \Gamma$ *and an* $i_0 \in \{1, \ldots, m\}$ *such that* $\Delta[k] \subset C_{i_0}$.

4. Proof of Theorem 1

Proof of Theorem 1. We apply the induction argument on the dimension n of the basic rectangle $I_a^b \subset \mathbb{R}^n$. For $n = 1$ Theorem 1 has been proved in [11, Corollary 2]. Now, let $n \geq 2$ and assume that Theorem 1 is already established for the domain rectangles of dimension $\leq n-1$.

If there are only finitely many distinct functions in $\{f_j\}$, we may choose a constant subsequence of $\{f_j\}$, and we are done. Otherwise, picking a subsequence of $\{f_j\}$, we may assume that all functions in $\{f_j\}$ are distinct.

Also, note that condition (2.7) implies the existence of an $N_0 \in \mathbb{N}$ and a constant $C > 0$ such that $\sup_{j,k\geq N_0} \mathrm{TV}(f_j, f_k, I_a^b) \leq C$, and so, denoting the subsequence $\{f_{j+N_0-1}\}_{j=1}^\infty$ of $\{f_j\}_{j=1}^\infty$ again by $\{f_j\}$, we find

$$\mathrm{TV}(f_j, f_k, I_a^b) \leq C \quad \text{for all } j, k \in \mathbb{N}. \tag{4.1}$$

The rest of the proof is divided into six steps for clarity. In the first step Theorem A will be applied several times with Γ a subsequence of the sequence $\{f_j\}$ and $k = m = 2$.

Step 1. Let us show that given $x \in I_a^b$, there exists a subsequence $\{f_j^{(x)}\}_{j=1}^\infty$ of $\{f_j\}$, depending on x, such that the double limit

$$\lim_{j,k\to\infty} \mathrm{TV}(f_j^{(x)}, f_k^{(x)}, I_a^x) \quad \text{exists in } [0, C]. \tag{4.2}$$

Let c_0 be the middle point of the interval $[0, C]$ and let C_1^1 be the set of those pairs $\{f_j, f_k\}$ with $j, k \in \mathbb{N}, j \neq k$, for which

$$\mathrm{TV}(f_j, f_k, I_a^x) \in [0, c_0), \tag{4.3}$$

and C_2^1—the set of those $\{f_j, f_k\}$ with $j, k \in \mathbb{N}, j \neq k$, for which the quantity on the left in the inclusion (4.3) belongs to the interval $[c_0, C]$. If C_1^1 and C_2^1 are nonempty, they are disjoint, and so, by Theorem A, there exists a subsequence $\{f_j^1\}_{j=1}^\infty$ of $\{f_j\}$ such that either

(i$_1$) $\{f_j^1, f_k^1\} \in C_1^1$ for all $j, k \in \mathbb{N}, j \neq k$,

or

(ii$_1$) $\{f_j^1, f_k^1\} \in C_2^1$ for all $j, k \in \mathbb{N}, j \neq k$.

If $C_1^1 \neq \varnothing$ and (i$_1$) holds, or if $C_2^1 = \varnothing$, we set $[a_1, b_1] = [0, c_0]$, while if $C_2^1 \neq \varnothing$ and (ii$_1$) holds, or if $C_1^1 = \varnothing$, we set $[a_1, b_1] = [c_0, C]$.

Inductively, if $p \in \mathbb{N}, p \geq 2$, and a subsequence $\{f_j^{p-1}\}_{j=1}^\infty$ of $\{f_j\}$ and an interval $[a_{p-1}, b_{p-1}] \subset [0, C]$ are already chosen, we let c_{p-1} be the middle point of the interval $[a_{p-1}, b_{p-1}]$, C_1^p be the set of those pairs $\{f_j^{p-1}, f_k^{p-1}\}$ with $j, k \in \mathbb{N}, j \neq k$, for which

$$\mathrm{TV}(f_j^{p-1}, f_k^{p-1}, I_a^x) \in [a_{p-1}, c_{p-1}), \tag{4.4}$$

and C_2^p—the set of those $\{f_j^{p-1}, f_k^{p-1}\}$ with $j, k \in \mathbb{N}, j \neq k$, for which the quantity on the left in the inclusion (4.4) belongs to $[c_{p-1}, b_{p-1}]$. If the sets C_1^p and C_2^p are nonempty, they are disjoint, and so, applying Theorem A, we obtain a subsequence $\{f_j^p\}_{j=1}^\infty$ of $\{f_j^{p-1}\}$ such that either

(i$_p$) $\{f_j^p, f_k^p\} \in C_1^p$ for all $j, k \in \mathbb{N}, j \neq k$,

or

(ii$_p$) $\{f_j^p, f_k^p\} \in C_2^p$ for all $j, k \in \mathbb{N}, j \neq k$.

If $C_1^p \neq \varnothing$ and (i$_p$) holds, or if $C_2^p = \varnothing$, we set $[a_p, b_p] = [a_{p-1}, c_{p-1}]$, while if $C_2^p \neq \varnothing$ and (ii$_p$) holds, or if $C_1^p = \varnothing$, we set $[a_p, b_p] = [c_{p-1}, b_{p-1}]$.

In this way for each $p \in \mathbb{N}$ we have nested intervals $[a_{p+1}, b_{p+1}] \subset [a_p, b_p]$ in $[0, C]$ with $b_p - a_p = C/2^p$ and a subsequence $\{f_j^p\}_{j=1}^\infty$ of $\{f_j^{p-1}\}_{j=1}^\infty$ (where $\{f_j^0\}_{j=1}^\infty = \{f_j\}$) such that $\mathrm{TV}(f_j^p, f_k^p, I_a^x) \in [a_p, b_p]$ for all $j, k \in \mathbb{N}, j \neq k$. Let $l \in [0, C]$ be the common limit of a_p and b_p as $p \to \infty$. Denoting the diagonal sequence $\{f_j^j\}_{j=1}^\infty$ by $\{f_j^{(x)}\}$ we infer that the limit in (4.2) is equal to l: in fact, given $\varepsilon > 0$, there exists $p(\varepsilon) \in \mathbb{N}$ such that $a_{p(\varepsilon)}, b_{p(\varepsilon)} \in [l-\varepsilon, l+\varepsilon]$ and, since $\{f_j^{(x)}\}_{j=p(\varepsilon)}^\infty$ is a subsequence of $\{f_j^{p(\varepsilon)}\}_{j=1}^\infty$, we find, for all $j, k \geq p(\varepsilon), j \neq k$,

$$\mathrm{TV}(f_j^{(x)}, f_k^{(x)}, I_a^x) \in [a_{p(\varepsilon)}, b_{p(\varepsilon)}] \subset [l-\varepsilon, l+\varepsilon].$$

Step 2. Given $i \in \{1, \dots, n\}$, denote by Q_i the union of the set of all rational points of the interval $[a_i, b_i]$ and the two-point set $\{a_i, b_i\}$. Then the set $Q = Q_1 \times \cdots \times Q_n$ is an at most countable dense subset of I_a^b, and so, we may assume that $Q = \{y^p\}_{p=1}^\infty$.

We assert that there exists a subsequence of $\{f_j\}$, denoted as the whole sequence $\{f_j\}$, and a totally monotone function $\varphi : Q \to [0, C]$ such that

$$\lim_{j,k\to\infty} \mathrm{TV}(f_j, f_k, I_a^y) = \varphi(y) \quad \text{for all } y \in Q. \tag{4.5}$$

By Step 1, there exists a subsequence $\{f_j^{(y^1)}\}_{j=1}^{\infty}$ of $\{f_j\}$, denoted by $\{f_j^{(1)}\}$, and a number from $[0, C]$, denoted by $\varphi(y^1)$, such that

$$\lim_{j,k\to\infty} \mathrm{TV}(f_j^{(1)}, f_k^{(1)}, I_a^{y^1}) = \varphi(y^1).$$

Inductively, if $p \in \mathbb{N}$, $p \ge 2$, and a subsequence $\{f_j^{(p-1)}\}_{j=1}^{\infty}$ of $\{f_j\}$ is already chosen, we apply Step 1 to pick a subsequence $\{f_j^{(p)}\}_{j=1}^{\infty}$ of $\{f_j^{(p-1)}\}$ such that

$$\lim_{j,k\to\infty} \mathrm{TV}(f_j^{(p)}, f_k^{(p)}, I_a^{y^p}) = \varphi(y^p)$$

for some number $\varphi(y^p) \in [0, C]$. Then (4.5) is satisfied for the diagonal sequence $\{f_j^{(j)}\}_{j=1}^{\infty}$, again denoted by $\{f_j\}$.

Given $j, k \in \mathbb{N}$, we set $\nu_{j,k}(x) = \mathrm{TV}(f_j, f_k, I_a^x)$ for all $x \in I_a^b$. Let us prove that the function φ, defined by the left-hand side of (4.5), is totally monotone on Q, i.e., $(-1)^{|\alpha|}\sum_{0\le\theta\le\alpha}(-1)^{|\theta|}A_\theta \ge 0$ for all $0 \ne \alpha \le 1$ and $x, y \in Q$ with $x \le y$, where $A_\theta = \varphi(x+\theta(y-x))$. By the definition of Q, $x+\theta(y-x) \in Q$ for all $0 \le \theta \le 1$, and so, by (4.5), $A_\theta = \lim_{j,k\to\infty}\nu_{j,k}\big(x+\theta(y-x)\big)$. It follows that for each $\varepsilon > 0$ there exists $n_\theta(\varepsilon) \in \mathbb{N}$, depending on ε and θ, such that for all $j \ge n_\theta(\varepsilon)$ and $k \ge n_\theta(\varepsilon)$ with $j \ne k$, we have $A_\theta - \varepsilon \le \nu_{j,k}\big(x + \theta(y - x)\big) \le A_\theta + \varepsilon$, and so, for any $0 \le \theta \le \alpha$, we get

$$(-1)^{|\theta|}A_\theta - \varepsilon \le (-1)^{|\theta|}\nu_{j,k}\big(x + \theta(y - x)\big) \le (-1)^{|\theta|}A_\theta + \varepsilon.$$

Summing over $0 \le \theta \le \alpha$ and noting that $n(\varepsilon) = \max\{n_\theta(\varepsilon) : 0 \le \theta \le \alpha\}$ depends only on ε, for all $j \ge n(\varepsilon)$ and $k \ge n(\varepsilon)$ with $j \ne k$, we have:

$$\sum_{0\le\theta\le\alpha}(-1)^{|\theta|}A_\theta - 2^{|\alpha|}\varepsilon \le \sum_{0\le\theta\le\alpha}(-1)^{|\theta|}\nu_{j,k}\big(x + \theta(y - x)\big) \le \sum_{0\le\theta\le\alpha}(-1)^{|\theta|}A_\theta + 2^{|\alpha|}\varepsilon.$$

Applying Theorem 4 to $\nu_{j,k}$, the last two inequalities imply

$$0 \le (-1)^{|\alpha|}\sum_{0\le\theta\le\alpha}(-1)^{|\theta|}A_\theta + 2^{|\alpha|}\varepsilon \quad \text{for all } 0 \ne \alpha \le 1 \text{ and } \varepsilon > 0,$$

from which the total monotonicity of φ on Q follows.

We extend the function φ, given by (4.5), from the set Q to the whole rectangle I_a^b as follows (Saks' idea [23] for $n = 1$):

$$\nu(x) = \sup\{\varphi(y) : y \in Q,\ y \le x\} \quad \text{for all } x \in I_a^b. \tag{4.6}$$

Then $\nu : I_a^b \to [0, C]$ is totally monotone (see the beginning of Section 5).

Step 3. It is known [4], [17, III.5.4], [18] that the set of discontinuity points of any totally monotone function on $I_a^b \subset \mathbb{R}^n$ lies on an at most countable set of hyperplanes of dimension $n - 1$ parallel to the coordinate axes. Given $i \in \{1, \ldots, n\}$, denote by Z_i the union of the set of all rational points of the interval $[a_i, b_i]$, the two-point set $\{a_i, b_i\}$ and the set of those points $z_i \in [a_i, b_i]$, for which the hyperplane

$$H_i(z_i) = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{z_i\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n] \tag{4.7}$$

contains points of discontinuity of ν from (4.6). Clearly, Z_i is a countable and dense subset of $[a_i, b_i]$, and so, we may assume that $Z_i = \{z_i(k)\}_{k=1}^{\infty}$.

Setting $H(Z) = \bigcup_{i=1}^{n}\bigcup_{k=1}^{\infty} H_i(z_i(k))$, let us show that

$$\lim_{j,k\to\infty} \mathrm{TV}(f_j, f_k, I_a^x) = \nu(x) \quad \text{for all } x \in I_a^b \setminus H(Z), \tag{4.8}$$

where $\{f_j\}$ is the sequence from Step 2, for which condition (4.5) holds.

Given $\varepsilon > 0$, since ν is continuous at $x \in I_a^b \setminus H(Z)$, there exists a $\delta = \delta(\varepsilon, x) > 0$ such that

$$\nu(y) \in [\nu(x) - \varepsilon, \nu(x) + \varepsilon] \quad \text{for all } y \in I_a^b \cap U_\delta(x), \tag{4.9}$$

where $U_\delta(x) = \{y \in \mathbb{R}^n : \|x - y\| \le \delta\}$ and $\|\cdot\|$ designates the Euclidean norm in $\mathbb{R}^n$. Since the set Q, defined in Step 2, is a dense subset of I_a^b, we find points $\bar{y} = \bar{y}(\varepsilon, x) \in Q \cap U_\delta(x)$ and $\hat{y} = \hat{y}(\varepsilon, x) \in Q \cap U_\delta(x)$ such that $\bar{y} \le x \le \hat{y}$. By (4.5), there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that for all $j \ge N$ and $k \ge N$ with $j \ne k$, we have

$$\mathrm{TV}(f_j, f_k, I_a^{\bar{y}}) \in [\varphi(\bar{y}) - \varepsilon, \varphi(\bar{y}) + \varepsilon] \quad \text{and} \quad \mathrm{TV}(f_j, f_k, I_a^{\hat{y}}) \in [\varphi(\hat{y}) - \varepsilon, \varphi(\hat{y}) + \varepsilon].$$

By Theorem 3, $\mathrm{TV}(f_j, f_k, I_a^{\bar{y}}) \le \mathrm{TV}(f_j, f_k, I_a^x) \le \mathrm{TV}(f_j, f_k, I_a^{\hat{y}})$, and so, (4.9) together with equalities $\nu(\bar{y}) = \varphi(\bar{y})$ and $\nu(\hat{y}) = \varphi(\hat{y})$ yield

$$\mathrm{TV}(f_j, f_k, I_a^x) \in [\varphi(\bar{y}) - \varepsilon, \varphi(\hat{y}) + \varepsilon] \subset [\nu(x) - 2\varepsilon, \nu(x) + 2\varepsilon]$$

for all $j, k \ge N$, $j \ne k$, which establishes (4.8).

Step 4. In order to apply the induction hypothesis, we need an estimate on the $(n-1)$-dimensional total joint variation of functions $f = f_j$ and $g = f_k$ with $j \neq k$ from the sequence $\{f_j\}$ 'over the hyperplane' (4.7) in the sense to be made precise below (cf. also Step 2 in the proof of [13, Part II, Theorem 1]).

Let us fix $i \in \{1, \dots, n\}$ and set $1^i = (1, \dots, 1, 0, 1, \dots, 1)$, where 0 is the i-th coordinate of 1^i and the other coordinates of 1^i are equal to 1. Note that $|1^i| = n-1$. Given $z_i \in Z_i$, we put

$$\overline{a} \equiv \overline{a}(z_i) = (a_1, \dots, a_{i-1}, z_i, a_{i+1}, \dots, a_n). \tag{4.10}$$

The map $f_{1^i}^{\overline{a}} : I_a^b \lfloor 1^i \to M$ with the base at $\overline{a}$, truncated by 1^i, is defined on the $(n-1)$-dimensional rectangle $I_a^b \lfloor 1^i \subset \mathbb{R}^{n-1}$ and given by: if $x \in I_a^b$, then $x \lfloor 1^i \in I_a^b \lfloor 1^i$ and

$$f_{1^i}^{\overline{a}}(x \lfloor 1^i) = f(\overline{a} + 1^i(x - \overline{a})) = f(x_1, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_n). \tag{4.11}$$

Note that the same arguments hold for the map $g_{1^i}^{\overline{a}} : I_a^b \lfloor 1^i \to M$. The $(n-1)$-dimensional total joint variation of $f_{1^i}^{\overline{a}}$ and $g_{1^i}^{\overline{a}}$ on $I_a^b \lfloor 1^i$ is equal to

$$\mathrm{TV}_{n-1}(f_{1^i}^{\overline{a}}, g_{1^i}^{\overline{a}}, I_a^b \lfloor 1^i) = \sum_{0 \neq \alpha \leq 1} V_{|\alpha|}\big((f_{1^i}^{\overline{a}})_\alpha^{a\lfloor 1^i}, (g_{1^i}^{\overline{a}})_\alpha^{a\lfloor 1^i}, (I_a^b \lfloor 1^i)\lfloor \alpha\big), \tag{4.12}$$

the summation being taken over $\alpha \in \mathcal{A}(n-1)$, the set of all $(n-1)$-dimensional multiindices $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ such that $0_{n-1} \neq \alpha \leq 1_{n-1}$. Given $\alpha \in \mathcal{A}(n-1)$, we set $\overline{\alpha} = (\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_i, \dots, \alpha_{n-1})$, where 0 occupies the i-th place, and note that $\alpha = \overline{\alpha} \lfloor 1^i$. We have (cf. [13, Part II, p. 89])

$$(f_{1^i}^{\overline{a}})_\alpha^{a\lfloor 1^i} = f_{\overline{\alpha}}^{\overline{a}} \quad \text{and} \quad (g_{1^i}^{\overline{a}})_\alpha^{a\lfloor 1^i} = g_{\overline{\alpha}}^{\overline{a}} \quad \text{on } (I_a^b \lfloor 1^i)\lfloor \alpha = I_a^b \lfloor \overline{\alpha} = I_{\overline{a}}^b \lfloor \overline{\alpha}.$$

It follows that the $|\alpha|$-th joint variation at the right-hand side of (4.12) is equal to $V_{|\overline{\alpha}|}(f_{\overline{\alpha}}^{\overline{a}}, g_{\overline{\alpha}}^{\overline{a}}, I_{\overline{a}}^b \lfloor \overline{\alpha})$. Noting that the set $\mathcal{A}(n-1)$ is bijective to the set of those $\overline{\alpha} \in \mathcal{A}(n)$, for which $0 \neq \overline{\alpha} \leq 1^i$, and applying Theorem 3 with $x = \overline{a}$, $y = b$ and $\gamma = 1^i$, we get:

$$\begin{aligned}\mathrm{TV}_{n-1}(f_{1^i}^{\overline{a}}, g_{1^i}^{\overline{a}}, I_a^b \lfloor 1^i) &= \sum_{0 \neq \overline{\alpha} \leq 1^i} V_{|\overline{\alpha}|}(f_{\overline{\alpha}}^{\overline{a}}, g_{\overline{\alpha}}^{\overline{a}}, I_{\overline{a}}^b \lfloor \overline{\alpha}) = \mathrm{TV}(f, g, I_{\overline{a}}^{\overline{a}+1^i(b-\overline{a})}) \\ &\leq \mathrm{TV}(f, g, I_a^{\overline{a}+1^i(b-\overline{a})}) - \mathrm{TV}(f, g, I_a^{\overline{a}}) \leq \mathrm{TV}(f, g, I_a^b).\end{aligned} \tag{4.13}$$

Thus, given $j, k \in \mathbb{N}$ with $j \neq k$ and $i \in \{1, \dots, n\}$, setting back $f = f_j$ and $g = f_k$, by virtue of (4.10), (4.13) and (4.1), we find, for all $z_i \in Z_i$ and $\overline{a} = \overline{a}(z_i)$:

$$\mathrm{TV}_{n-1}\big((f_j)_{1^i}^{\overline{a}(z_i)}, (f_k)_{1^i}^{\overline{a}(z_i)}, I_a^b \lfloor 1^i\big) \leq C < \infty. \tag{4.14}$$

Step 5. Now, we make use of the diagonal processes. For $i = 1$ and $z_1 = z_i(1) = z_1(1) \in Z_1$ the sequence $\{(f_j)_{1^i}^{\overline{a}(z_i(1))}\}_{j=1}^\infty = \{(f_j)_{1^1}^{\overline{a}(z_1(1))}\}_{j=1}^\infty$ satisfies the uniform estimate (4.14) on the rectangle $I_a^b \lfloor 1^1$ of dimension $n-1$ and, since each map from this sequence is of the form (4.11) with $z_i = z_1 = z_1(1)$, then it follows from the assumptions of Theorem 1 that the sequence under consideration is pointwise precompact on $I_a^b \lfloor 1^1$. By the induction hypothesis, the sequence $\{f_j\}$ contains a subsequence, denoted by $\{f_j^1\}$, such that $(f_j^1)_{1^1}^{\overline{a}(z_1(1))}$ converges pointwise on $I_a^b \lfloor 1^1$ to a map from $I_a^b \lfloor 1^1$ into M. Since, by (4.11),

$$(f_j^1)_{1^1}^{\overline{a}(z_1(1))}(x_2, \dots, x_n) = (f_j^1)_{1^1}^{\overline{a}(z_1(1))}(x \lfloor 1^1) = f_j^1(z_1(1), x_2, \dots, x_n)$$

with $x = (x_1, \dots, x_n) \in I_a^b$ and $x_i \in [a_i, b_i]$ for $i \in \{2, \dots, n\}$, then the pointwise convergence above means, actually, that the sequence $\{f_j^1\}$ converges pointwise on the hyperplane $H_1(z_1(1)) = \{z_1(1)\} \times [a_2, b_2] \times \dots \times [a_n, b_n]$.

Inductively, if $p \geq 2$ and a subsequence $\{f_j^{p-1}\}_{j=1}^\infty$ of $\{f_j\}$, which is pointwise convergent on $\bigcup_{l=1}^{p-1} H_1(z_1(l))$, is already chosen, then the sequence $\{(f_j^{p-1})_{1^1}^{\overline{a}(z_1(p))}\}_{j=1}^\infty$ satisfies the uniform estimate (4.14) on the rectangle $I_a^b \lfloor 1^1$, where f_j is replaced by f_j^{p-1} and $\overline{a}(z_i)$—by $\overline{a}(z_1(p))$. Moreover, since, as above, the sequence is pointwise precompact on $I_a^b \lfloor 1^1$, then, by the induction hypothesis, there exists a subsequence $\{f_j^p\}_{j=1}^\infty$ of $\{f_j^{p-1}\}_{j=1}^\infty$ such that $(f_j^p)_{1^1}^{\overline{a}(z_1(p))}$ converges pointwise on $I_a^b \lfloor 1^1$ as $j \to \infty$ to a map from $I_a^b \lfloor 1^1$ into M. Again, as above, this pointwise convergence means that the sequence $\{f_j^p\}_{j=1}^\infty$ converges pointwise on the hyperplane $H_1(z_1(p))$ and, as a consequence, on the set $\bigcup_{l=1}^p H_1(z_1(l))$ as well. We infer that the diagonal sequence $\{f_j^j\}_{j=1}^\infty$, which is a subsequence of the original sequence $\{f_j\}$, converges pointwise on the set $H_1(Z_1) = \bigcup_{z_1 \in Z_1} H_1(z_1) = \bigcup_{l=1}^\infty H_1(z_1(l))$; in fact, given $(z_1, x_2, \dots, x_n) \in H_1(Z_1)$, we have $z_1 = z_1(p) \in Z_1$ for some $p \in \mathbb{N}$ and $(x_2, \dots, x_n) \in I_a^b \lfloor 1^1$, and so, noting that $\{f_j^j\}_{j=p}^\infty$ is a subsequence of $\{f_j^p\}_{j=1}^\infty$, we find that

$$f_j^j(z_1, x_2, \dots, x_n) = (f_j^j)_{1^1}^{\overline{a}(z_1(p))}(x_2, \dots, x_n)$$

converges in M as $j \to \infty$.

Let us denote the diagonal sequence $\{f_j^j\}_{j=1}^\infty$ extracted in the last paragraph again by $\{f_j\}$. Then we let $i=2$, $z_2=z_i(1)=z_2(1)\in Z_2$ and, beginning with the sequence $\{(f_j)_{1^i}^{\bar{a}(z_i(1))}\}_{j=1}^\infty=\{(f_j)_{1^2}^{\bar{a}(z_2(1))}\}_{j=1}^\infty$, apply the above arguments of this step. Doing this, we will end up with a diagonal sequence, a subsequence of the original sequence $\{f_j\}$, again denoted by $\{f_j\}$, which converges pointwise on $H_1(Z_1)\cup H_2(Z_2)$. Now suppose that for some $i\in\{2,\ldots,n-1\}$ we have already extracted a (diagonal) subsequence of $\{f_j\}$, again denoted by $\{f_j\}$, which converges pointwise on the set $H_1(Z_1)\cup\ldots\cup H_{i-1}(Z_{i-1})$. Then we let $z_i=z_i(1)\in Z_i$ and apply the above arguments of this step to the sequence $\{(f_j)_{1^i}^{\bar{a}(z_i(1))}\}_{j=1}^\infty$: a subsequence of the original sequence $\{f_j\}$ converges pointwise on the set $H_1(Z_1)\cup\cdots\cup H_i(Z_i)$. In this way after finitely many steps we obtain a subsequence of the original sequence $\{f_j\}$, again denoted by $\{f_j\}$, which converges pointwise on the set $H(Z)=\bigcup_{i=1}^n H_i(Z_i)$.

Step 6. Note that, by virtue of Step 3,

$$\text{the function } \nu \text{ is continuous on } I_a^b\setminus H(Z). \tag{4.15}$$

Finally, let us show that the sequence $\{f_j\}$ converges at each point y from $I_a^b\setminus H(Z)$. For this we show that, given $y\in I_a^b\setminus H(Z)$, the sequence $\{f_j(y)\}$ is Cauchy. If this is already done, the precompactness of $\{f_j(y)\}$ would imply that it is convergent in M as $j\to\infty$ to a point of M denoted by $f(y)$. This observation, the argument at the end of the previous paragraph and equality $I_a^b=H(Z)\cup(I_a^b\setminus H(Z))$ will complete the proof of the theorem.

Let us fix $\varepsilon>0$ arbitrarily. By virtue of (4.15), y is a point of continuity of ν such that its coordinates $a_i<y_i<b_i$ are irrational for all $i\in\{1,\ldots,n\}$, and so, the density of $H(Z)$ in I_a^b yields the existence of a rational point $x=x(\varepsilon)\in H(Z)$ such that $x<y$ and (by properties of totally monotone functions) $0\le\nu(y)-\nu(x)\le\varepsilon$. Applying (4.8), we find a number $N_1=N_1(\varepsilon)\in\mathbb{N}$, depending on ε, x and y, such that if $j\ge N_1(\varepsilon)$, $k\ge N_1(\varepsilon)$ and $j\ne k$, then

$$|\mathrm{TV}(f_j,f_k,I_a^x)-\nu(x)|\le\varepsilon \quad\text{and}\quad |\mathrm{TV}(f_j,f_k,I_a^y)-\nu(y)|\le\varepsilon.$$

Being convergent, the sequence $\{f_j(x)\}$ is Cauchy, and so, there exists a number $N_2=N_2(\varepsilon)\in\mathbb{N}$, depending on ε and x, such that $d(f_j(x),f_k(x))\le\varepsilon$ for all $j\ge N_2(\varepsilon)$ and $k\ge N_2(\varepsilon)$. Applying (2.2), Theorems 2 and 3 with $\gamma=1$ and noting that the number $N=\max\{N_1,N_2\}$ depends only on ε, we get, for all $j\ge N$ and $k\ge N$ with $j\ne k$,

$$\begin{aligned} d(f_j(y),f_k(y)) &\le d(f_j(y)+f_k(x),f_k(y)+f_j(x))+d(f_k(x),f_j(x))\\ &\le \mathrm{TV}(f_j,f_k,I_x^y)+\varepsilon\le \mathrm{TV}(f_j,f_k,I_a^y)-\mathrm{TV}(f_j,f_k,I_a^x)+\varepsilon\\ &\le |\mathrm{TV}(f_j,f_k,I_a^y)-\nu(y)|+(\nu(y)-\nu(x))+|\mathrm{TV}(f_j,f_k,I_a^x)-\nu(x)|+\varepsilon\le 4\varepsilon, \end{aligned}$$

and so, the Cauchy property of $\{f_j(y)\}$ follows. □

5. Proofs of the auxiliary results

In this section we prove Theorem 2 and formulate and prove auxiliary Lemmas 1–3 alluded to on p. 8.

It what follows, given $0\ne\alpha\le 1$, the abbreviation 'ev $\theta\le\alpha$' means '$\theta\in\mathcal{E}(n)$ and $\theta\le\alpha$', and 'od $\theta\le\alpha$' stands for '$\theta\in\mathcal{O}(n)$ and $\theta\le\alpha$'.

We begin by proving that the function ν from (4.6) is totally monotone.

Proof. Given $0\ne\alpha\le 1$ and $x,y\in I_a^b$ with $x\le y$, we have to show that $(-1)^{|\alpha|}\sum_{0\le\theta\le\alpha}(-1)^{|\theta|}B_\theta\ge 0$, where $B_\theta=\nu\big(x+\theta(y-x)\big)$. If $x_i=y_i$ for some $i\in\{1,\ldots,n\}$, for which $\alpha_i=1$, then $\sum_{0\le\theta\le\alpha}(-1)^{|\theta|}B_\theta=0$, and so, we may assume that $x_i<y_i$ for all $i\in\{1,\ldots,n\}$ with $\alpha_i=1$, i.e., $x\lfloor\alpha<y\lfloor\alpha$.

Suppose α is even. Let us show that

$$\sum_{\mathrm{ev}\,\theta\le\alpha}B_\theta-\sum_{\mathrm{od}\,\theta\le\alpha}B_\theta=\sum_{0\le\theta\le\alpha}(-1)^{|\theta|}B_\theta\ge 0. \tag{5.1}$$

On the contrary, assume that (5.1) does not hold. Then the quantity

$$\varepsilon=\frac{1}{2^{|\alpha|-1}}\Bigg(\sum_{\mathrm{od}\,\theta\le\alpha}B_\theta-\sum_{\mathrm{ev}\,\theta\le\alpha}B_\theta\Bigg)\quad\text{is positive.}$$

Given an odd multiindex θ with $0\le\theta\le\alpha$, by virtue of (4.6), we have

$$\nu\big(x+\theta(y-x)\big)=\sup\{\varphi(z):z\in Q,\,z\le x+\theta(y-x)\}, \tag{5.2}$$

and so, there exists a point $z^\theta=\big(z_1^\theta,\ldots,z_n^\theta\big)\in Q$, depending also on ε, such that $z^\theta\le x+\theta(y-x)$ and

$$\varphi(z^\theta)>\nu\big(x+\theta(y-x)\big)-\varepsilon=B_\theta-\varepsilon. \tag{5.3}$$

Note that, given $i\in\{1,\ldots,n\}$, we have $a_i\le z_i^\theta\le x_i$ if $\theta_i=0$, and $a_i\le z_i^\theta\le y_i$ if $\theta_i=1$. However, for those i, for which $\theta_i=1$, we may always assume that $x_i<z_i^\theta\le y_i$. In fact, for each i such that $\theta_i=1$ and $a_i\le z_i^\theta\le x_i$ we choose a rational point r_i such that $x_i<r_i\le y_i$ and replace the coordinate z_i^θ by r_i, the remaining coordinates of z^θ being unchanged. If we

denote the resulting point by $\bar{z}^{\theta}$, then we get $\bar{z}^{\theta} \in Q$ and $z^{\theta} \le \bar{z}^{\theta}$, and so, by the total monotonicity of φ on Q, we obtain (5.3) with z^{θ} replaced by $\bar{z}^{\theta}$. Thus, we assume in (5.3) that the coordinates of z^{θ} satisfy the conditions: $a_i \le z_i^{\theta} \le x_i$ if $\theta_i = 0$, and $x_i < z_i^{\theta} \le y_i$ if $\theta_i = 1$.

Now we pick two points $x' = (x'_1, \ldots, x'_n)$ and $y' = (y'_1, \ldots, y'_n)$, whose coordinates $x'_i, y'_i \in Q_i$ satisfy the following conditions for $i \in \{1, \ldots, n\}$:

$$z_i^{\theta} \le x'_i \le x_i \quad \text{for all odd } \theta \text{ such that } 0 \le \theta \le \alpha \text{ and } \theta_i = 0$$

and

$$z_i^{\theta} \le y'_i \le y_i \quad \text{for all odd } \theta \text{ such that } 0 \le \theta \le \alpha \text{ and } \theta_i = 1.$$

Thus, $x', y' \in Q$ and $x'\lfloor\alpha \le x\lfloor\alpha \le y'\lfloor\alpha \le y\lfloor\alpha$. By the total monotonicity of φ, the inequality (5.3) holds if we replace the point $z^{\theta} \in Q$ by the point $\widetilde{z}^{\theta} = x' + \theta(y' - x') \in Q$ (note that $z^{\theta} \le \widetilde{z}^{\theta}$), i.e.,

$$\varphi(\widetilde{z}^{\theta}) = \varphi\big(x' + \theta(y' - x')\big) > B_{\theta} - \varepsilon.$$

Summing over all odd multiindices θ with $0 \le \theta \le \alpha$, we get:

$$\sum_{\text{od } \theta \le \alpha} \varphi\big(x' + \theta(y' - x')\big) > \sum_{\text{od } \theta \le \alpha} B_{\theta} - 2^{|\alpha|-1}\varepsilon$$

and so, the definition of ε implies

$$\sum_{\text{od } \theta \le \alpha} \varphi\big(x' + \theta(y' - x')\big) > \sum_{\text{ev } \theta \le \alpha} B_{\theta}.$$

Again, by virtue of the total monotonicity of φ on Q and the assumption that α is even, we have (similar to (5.1)):

$$\sum_{\text{od } \theta \le \alpha} \varphi\big(x' + \theta(y' - x')\big) \le \sum_{\text{ev } \theta \le \alpha} \varphi\big(x' + \theta(y' - x')\big).$$

The last two inequalities yield:

$$\sum_{\text{ev } \theta \le \alpha} \nu\big(x + \theta(y - x)\big) = \sum_{\text{ev } \theta \le \alpha} B_{\theta} < \sum_{\text{ev } \theta \le \alpha} \varphi\big(x' + \theta(y' - x')\big),$$

which contradicts the equality for ν from (5.2).

Now suppose α is odd. In order to show that $-\sum_{0 \le \theta \le \alpha} (-1)^{|\theta|} B_{\theta} \ge 0$, we repeat the arguments above replacing even (odd) multiindices θ by odd (even, respectively) multiindices θ. $\square$

Lemma 1. *If $f, g : I_a^b \to M$, $x, y \in I_a^b$, $x \le y$, $z \in I_a^b$ and $0 \ne \alpha \le 1$, then the joint mixed difference $md_{|\alpha|}(f_{\alpha}^z, g_{\alpha}^z, I_x^y\lfloor\alpha)$ is equal to*

$$d\Bigg(\sum_{ev\, \theta \le \alpha} f\big(z + \alpha(x - z) + \theta(y - x)\big) + \sum_{od\, \theta \le \alpha} g\big(z + \alpha(x - z) + \theta(y - x)\big), \\ \sum_{od\, \theta \le \alpha} f\big(z + \alpha(x - z) + \theta(y - x)\big) + \sum_{ev\, \theta \le \alpha} g\big(z + \alpha(x - z) + \theta(y - x)\big)\Bigg). \tag{5.4}$$

The proof of Lemma 1 is the same as that of [9, Part I, Lemma 5] (details are omitted): note only that $\theta' \in \mathbb{N}_0^{|\alpha|}$ and $|\theta'|$ is even (odd) if and only if there exists a unique $\theta \in \mathbb{N}_0^n$ such that $\theta \le \alpha$, $|\theta|$ is even (odd, respectively) and $\theta' = \theta\lfloor\alpha$, and apply definition (2.3) where n is replaced by $|\alpha|$.

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Making suitable modifications, we follow the lines of the proof of Theorem 2 from [13, Part I] and apply auxiliary facts established in that proof. It suffices to prove only the left-hand side inequality, which, by virtue of (2.4) and (2.5), implies the right-hand side inequality. Set $u = f(x) + g(y)$ and $v = f(y) + g(x)$. Taking into account (5.4) with $z = x$, the desired inequality in Theorem 2 can be rewritten equivalently as

$$d(u, v) \le \sum_{0 \ne \alpha \le 1} d\big(u_f(\alpha) + v_g(\alpha), v_f(\alpha) + u_g(\alpha)\big) \tag{5.5}$$

where, given $0 \le \alpha, \theta \le 1$, we set $h_f(\theta) = f(x + \theta(y - x))$, $u_f(\alpha) = \sum_{\text{ev } \theta \le \alpha} h_f(\theta)$ and $v_f(\alpha) = \sum_{\text{od } \theta \le \alpha} h_f(\theta)$, and likewise for $h_g(\theta)$, $u_g(\alpha)$ and $v_g(\alpha)$ (the sum over the empty set is omitted in any context). In order to establish (5.5), given

$j \in \{1, \ldots, n\}$, we also set $u_{f,j} = \sum_{|\alpha|=j} u_f(\alpha)$, $v_{f,j} = \sum_{|\alpha|=j} v_f(\alpha)$ ($u_{g,j}$ and $v_{g,j}$ are defined similarly), $u_j = u_{f,j} + v_{g,j}$ and $v_j = v_{f,j} + u_{g,j}$. By virtue of (2.1), we find

$$d(u_j, v_j) = d\Big(\sum_{|\alpha|=j}\big(u_f(\alpha) + v_g(\alpha)\big), \sum_{|\alpha|=j}\big(v_f(\alpha) + u_g(\alpha)\big)\Big)$$
$$\leq \sum_{|\alpha|=j} d\big(u_f(\alpha) + v_g(\alpha), v_f(\alpha) + u_g(\alpha)\big). \tag{5.6}$$

After summing over $j = 1, \ldots, n$, (5.5) is a consequence of the inequality

$$d(u, v) \leq \sum_{j=1}^{m} d(u_j, v_j), \tag{5.7}$$

which is known to hold [13, Part I, Lemma 7] for odd $m = n$ when sequences $\{u_j\}_{j=1}^m, \{v_j\}_{j=1}^m \subset M$ satisfy the equality

$$u + \sum_{i=1}^{(m-1)/2} u_{2i} + \sum_{i=1}^{(m+1)/2} v_{2i-1} = v + \sum_{i=1}^{(m-1)/2} v_{2i} + \sum_{i=1}^{(m+1)/2} u_{2i-1}, \tag{5.8}$$

and for even $m = n$ when the sequences satisfy the equality

$$u + \sum_{i=1}^{m/2} u_{2i} + \sum_{i=1}^{m/2} v_{2i-1} = v + \sum_{i=1}^{m/2} v_{2i} + \sum_{i=1}^{m/2} u_{2i-1}. \tag{5.9}$$

First, let us verify (5.8). It was shown in [13, p. 684] for odd $m = n$ that

$$f(x) + \sum_{i=1}^{(m-1)/2} u_{f,2i} = \sum_{i=1}^{(m+1)/2} u_{f,2i-1} \quad \text{and} \quad g(y) + \sum_{i=1}^{(m-1)/2} v_{g,2i} = \sum_{i=1}^{(m+1)/2} v_{g,2i-1}, \tag{5.10}$$

and similar equalities hold if we interchange f and g. Summing the two equalities in (5.10) and then summing the two equalities corresponding to the interchanged f and g, we get, respectively,

$$u + \sum_{i=1}^{(m-1)/2} u_{2i} = \sum_{i=1}^{(m+1)/2} u_{2i-1} \quad \text{and} \quad v + \sum_{i=1}^{(m-1)/2} v_{2i} = \sum_{i=1}^{(m+1)/2} v_{2i-1}, \tag{5.11}$$

from which equality (5.8) follows.

Now, let us verify (5.9). It was proved in [13, p. 684] for even $m = n$ that

$$f(x) + \sum_{i=1}^{m/2} u_{f,2i} = f(y) + \sum_{i=1}^{m/2} u_{f,2i-1} \quad \text{and} \quad \sum_{i=1}^{m/2} v_{g,2i} = \sum_{i=1}^{m/2} v_{g,2i-1}, \tag{5.12}$$

and similar equalities hold with interchanged f and g. Summing the two equalities in (5.12) and adding $g(y)$ to the result and then summing the two equalities corresponding to the interchanged f and g and adding $f(y)$ to the result, we find, respectively,

$$u + \sum_{i=1}^{m/2} u_{2i} = f(y) + g(y) + \sum_{i=1}^{m/2} u_{2i-1} \quad \text{and} \quad v + \sum_{i=1}^{m/2} v_{2i} = g(y) + f(y) + \sum_{i=1}^{m/2} v_{2i-1},$$

which imply equality (5.9). $\square$

Since the total joint variation (2.5) is defined via truncated maps with the base at the point a, in our next lemma we present a counterpart of Chistyakov's equality [9, Part I, Lemma 7] exhibiting the relation between the mixed difference $\mathrm{md}_{|\alpha|}(f_\alpha^x, I_x^y\lfloor\alpha)$ and certain mixed differences of maps f_β^a with the base at a for some $0 \neq \beta \leq 1$ (see also [13, Part II, Theorem 3]).

Lemma 2. *If $f, g : I_a^b \to M$, $0 \neq \alpha \leq 1$ and $x, y \in I_a^b$ with $x \leq y$, then*

$$md_{|\alpha|}(f_\alpha^x, g_\alpha^x, I_x^y\lfloor\alpha) \leq \sum_{\alpha \leq \beta \leq 1} md_{|\beta|}\big(f_\beta^a, g_\beta^a, I_{a+\alpha(x-a)}^{x+\alpha(y-x)}\lfloor\beta\big).$$

Proof. The inequality (actually, equality) is clear if $\alpha = 1$, and so, we assume that $\alpha \neq 1$. The joint mixed difference at the left-hand side is given by (5.4) with $z = x$, while given $\alpha \leq \beta \leq 1$, noting that $\alpha\beta = \alpha$ and applying equality (5.4), we get the following expression for the joint mixed difference at the right-hand side (cf. [9, Part I, expression (3.7)]):

$$\mathrm{md}_{|\beta|}\big(f^a_\beta, g^a_\beta, I^{x+\alpha(y-x)}_{a+\alpha(x-a)}\lfloor\beta\big) = d\Bigg(\sum_{\mathrm{ev}\,\theta\le\beta} h_f(\theta) + \sum_{\mathrm{od}\,\theta\le\beta} h_g(\theta), \sum_{\mathrm{od}\,\theta\le\beta} h_f(\theta) + \sum_{\mathrm{ev}\,\theta\le\beta} h_g(\theta)\Bigg),$$

where $h_f(\theta) = f\big(a+(\alpha\vee\theta)(x-a)+\alpha\theta(y-x)\big)$, and likewise for $h_g(\theta)$, and $\alpha\vee\theta = \alpha+\theta-\alpha\theta$. Changing the summation multiindex $\beta\mapsto\beta-\alpha$ in the sum at the right of the inequality in Lemma 2, we find that it is equivalent to

$$d(u_f+v_g, v_f+u_g) \le \sum_{0\le\beta\le 1-\alpha} d\big(u_f(\beta)+v_g(\beta), v_f(\beta)+u_g(\beta)\big), \tag{5.13}$$

where $u_f = \sum_{\mathrm{ev}\,\theta\le\alpha} f(x+\theta(y-x))$, $v_f = \sum_{\mathrm{od}\,\theta\le\alpha} f(x+\theta(y-x))$, u_g and v_g are defined similarly, $u_f(\beta) = \sum_{\mathrm{ev}\,\theta\le\alpha+\beta} h_f(\theta)$, $v_f(\beta) = \sum_{\mathrm{od}\,\theta\le\alpha+\beta} h_f(\theta)$, and likewise for $u_g(\beta)$ and $v_g(\beta)$. In order to establish (5.13), we apply inequality (5.7) with $m = |1-\alpha|+1 = n-|\alpha|+1$, $u = u_f+v_g$, $v = v_f+u_g$, $u_j = u_{f,j}+v_{g,j}$ and $v_j = v_{f,j}+u_{g,j}$ for $j\in\{1,\dots,m\}$, where $u_{f,j} = \sum_{|\beta|=j-1} u_f(\beta)$, $v_{f,j} = \sum_{|\beta|=j-1} v_f(\beta)$, and $u_{g,j}$ and $v_{g,j}$ are defined similarly. Suppose that we have already verified equalities (5.8) and (5.9). Following (5.6), by virtue of (2.1), we find (note that the β's below satisfy also $0\le\beta\le 1-\alpha$)

$$d(u_j, v_j) \le \sum_{|\beta|=j-1} d\big(u_f(\beta)+v_g(\beta), v_f(\beta)+u_g(\beta)\big),$$

which, after summing over $j = 1,\dots,m$, gives

$$\sum_{j=1}^{m} d(u_j, v_j) \le \sum_{i=0}^{|1-\alpha|}\sum_{|\beta|=i} d\big(u_f(\beta)+v_g(\beta), v_f(\beta)+u_g(\beta)\big),$$

and so, the desired inequality (5.13) is now a consequence of (5.7).

It remains to verify (5.8) and (5.9). Let $1-\alpha$ be even. Then $m = |1-\alpha|+1$ is odd. Let us verify (5.8). It was shown in [13, Part II, p. 87] that

$$u_f + \sum_{i=1}^{(m-1)/2} u_{f,2i} = \sum_{i=1}^{(m+1)/2} u_{f,2i-1} \quad\text{and}\quad v_g + \sum_{i=1}^{(m-1)/2} v_{g,2i} = \sum_{i=1}^{(m+1)/2} v_{g,2i-1}, \tag{5.14}$$

and similar equalities hold if we interchange f and g. Summing the two equalities in (5.14) and then summing the two equalities corresponding to the interchanged f and g, we get the equalities in (5.11), respectively, from which (5.8) follows.

Now, let $1-\alpha$ be odd. Then $m = |1-\alpha|+1$ is even. Let us verify (5.9). It was proved in [13, Part II, p. 88] that

$$\sum_{i=1}^{m/2} u_{f,2i} = v_f + \sum_{i=1}^{m/2} u_{f,2i-1} \quad\text{and}\quad \sum_{i=1}^{m/2} v_{g,2i} = u_g + \sum_{i=1}^{m/2} v_{g,2i-1}, \tag{5.15}$$

and similar equalities hold with interchanged f and g. Summing the two equalities in (5.15) and then summing the two equalities corresponding to the interchanged f and g, we find, respectively,

$$\sum_{i=1}^{m/2} u_{2i} = v + \sum_{i=1}^{m/2} u_{2i-1} \quad\text{and}\quad \sum_{i=1}^{m/2} v_{2i} = u + \sum_{i=1}^{m/2} v_{2i-1},$$

which imply equality (5.9). □

The *additivity* property of $|\alpha|$-th joint variation $V_{|\alpha|}$ for each $0\ne\alpha\le 1$, mentioned on p. 651, is expressed by the following

Lemma 3. *Given $f, g : I^b_a \to M$, $x, y \in I^b_a$ with $x < y$, $z\in I^b_a$ and $0\ne\alpha\le 1$, if $\{x[\sigma]\}^{\kappa}_{\sigma=0}$ is a net partition of I^y_x, then*

$$V_{|\alpha|}(f^z_\alpha, g^z_\alpha, I^y_x\lfloor\alpha) = \sum_{1\lfloor\alpha\le\sigma\lfloor\alpha\le\kappa\lfloor\alpha} V_{|\alpha|}(f^z_\alpha, g^z_\alpha, I^{x[\sigma]}_{x[\sigma-1]}\lfloor\alpha),$$

where the summation is taken only over those σ_i in the range $1\le\sigma_i\le\kappa_i$ with $i\in\{1,\dots,n\}$, for which $\alpha_i = 1$.

This lemma is a counterpart of Theorem 1 from [13, Part I] (which is well known for $M=\mathbb{R}$ [18]), and its proof is similar to the proof of the cited theorem. We only need a variant of [13, Part I, Lemma 5], which is as follows.

Lemma 4. *If $f, g : I^b_a \to M$, $x, y\in I^b_a$ with $x<y$ and $x'\in I^y_x$, then*

$$md_n(f, g, I^y_x) \le \sum_{0\le\alpha\le 1} md_n\big(f, g, I^{x'+\alpha(y-x')}_{x+\alpha(x'-x)}\big).$$

Proof. It was shown in [13, Part I, p. 67] that $I^y_x = \bigcup_{0\le\alpha\le 1} I^{x'+\alpha(y-x')}_{x+\alpha(x'-x)}$ is the union of non-overlapping (possibly, degenerated) rectangles.

By virtue of (2.3), we have $\mathrm{md}_n(f, g, I_x^y) = d(u_f + v_g, v_f + u_g)$, where $u_f = \sum_{\mathrm{ev}\,\theta\le 1} f(x+\theta(y-x))$ and $v_f = \sum_{\mathrm{od}\,\theta\le 1} f(x+\theta(y-x))$, and likewise for u_g and v_g. Again, (2.3) implies that $\mathrm{md}_n\big(f, g, I_{x+\alpha(x'-x)}^{x'+\alpha(y-x')}\big)$ is equal to

$$D(\alpha) \equiv d\Bigg(\sum_{\mathrm{ev}\,\beta\le 1} h_f(\alpha,\beta) + \sum_{\mathrm{od}\,\beta\le 1} h_g(\alpha,\beta), \sum_{\mathrm{od}\,\beta\le 1} h_f(\alpha,\beta) + \sum_{\mathrm{ev}\,\beta\le 1} h_g(\alpha,\beta)\Bigg),$$

where $h_f(\alpha,\beta) = f(x + (\alpha\vee\beta)(x'-x) + \alpha\beta(y-x'))$, likewise for $h_g(\alpha,\beta)$, and $\alpha\vee\beta = \alpha+\beta-\alpha\beta$. It was shown in [13, Part I, Lemma 5] that

$$\sum_{0\le\alpha\le 1}\sum_{\mathrm{ev}\,\beta\le 1} h_f(\alpha,\beta) = \sum_{\mathrm{ev}\,\beta\le 1}\sum_{\substack{0\le\alpha\le 1\\ \alpha\ne\beta}} h_f(\alpha,\beta) + u_f \equiv U_f + u_f,$$

$$\sum_{0\le\alpha\le 1}\sum_{\mathrm{od}\,\beta\le 1} h_f(\alpha,\beta) = \sum_{\mathrm{od}\,\beta\le 1}\sum_{\substack{0\le\alpha\le 1,\\ \alpha\ne\beta}} h_f(\alpha,\beta) + v_f \equiv V_f + v_f,$$

and, moreover, $U_f = V_f$, and similar equalities hold with f replaced by g. By the translation invariance of d and (2.1), these equalities yield:

$$d(u_f+v_g, v_f+u_g) = d(U_f+u_f+V_g+v_g, V_f+v_f+U_g+u_g) \le \sum_{0\le\alpha\le 1} D(\alpha),$$

which was to be proved. $\square$

Acknowledgment

The study of the first author was carried out within "The National Research University Higher School of Economics' Academic Fund Program" in 2010–2012, research grant No. 10-01-0071.

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