WEIGHTED NORM INEQUALITIES FOR
DE BRANGES–ROVNYAK SPACES AND THEIR APPLICATIONS

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ABSTRACT. Let $\mathcal{H}(b)$ denote the de Branges–Rovnyak space associated with a function b in the unit ball of $H^\infty(\mathbb{C}_+)$. We study the boundary behavior of the derivatives of functions in $\mathcal{H}(b)$ and obtain weighted norm estimates of the form $\|f^{(n)}\|_{L^2(\mu)} \leq C\|f\|_{\mathcal{H}(b)}$, where $f \in \mathcal{H}(b)$ and μ is a Carleson-type measure on $\mathbb{C}_+ \cup \mathbb{R}$. We provide several applications of these inequalities. We apply them to obtain embedding theorems for $\mathcal{H}(b)$ spaces. These results extend Cohn and Volberg–Treil embedding theorems for the model (star-invariant) subspaces which are special classes of de Branges–Rovnyak spaces. We also exploit the inequalities for the derivatives to study stability of Riesz bases of reproducing kernels $\{k^b_{\lambda_n}\}$ in $\mathcal{H}(b)$ under small perturbations of the points λ_n.

1. INTRODUCTION

Let $\mathbb{C}_+$ denote the upper half-plane in the complex plane and let $H^2(\mathbb{C}_+)$ denote the usual Hardy space on $\mathbb{C}_+$. For $\varphi \in L^\infty(\mathbb{R})$, let T_φ stand for the Toeplitz operator defined on $H^2(\mathbb{C}_+)$ by

$$T_\varphi f := P_+(\varphi f), \qquad f \in H^2(\mathbb{C}_+),$$

where P_+ denotes the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Then, for $\varphi \in L^\infty(\mathbb{R})$, $\|\varphi\|_\infty \leq 1$, the de Branges–Rovnyak space $\mathcal{H}(\varphi)$ associated to φ consists of those functions in $H^2(\mathbb{C}_+)$ which are in the range of the operator $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. It is a Hilbert space

when equipped with the inner product

$$\langle (Id - T_\varphi T_{\overline{\varphi}})^{1/2} f, \ (Id - T_\varphi T_{\overline{\varphi}})^{1/2} g \rangle_\varphi = \langle f, g \rangle_2,$$

where $f, g \in H^2(\mathbb{C}_+) \ominus \ker (Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. In what follows we always assume that $\varphi = b$ is an analytic function in the unit ball of $H^\infty(\mathbb{C}_+)$. In this case, if

$$(1.1) \qquad\qquad k_\omega^b(z) := \frac{1 - \overline{b(\omega)}b(z)}{z - \overline{\omega}}, \qquad \omega \in \mathbb{C}_+,$$

then we have $\langle f, k_\omega^b \rangle_b = 2\pi i f(\omega)$ for all $f \in \mathcal{H}(b)$. In other words, $\mathcal{H}(b)$ is a reproducing kernel Hilbert space.

These spaces (and, more precisely, their general vector-valued version) were introduced by de Branges and Rovnyak [14, 15] as universal model spaces for Hilbert space contractions. Thanks to the pioneer works of Sarason, we know that de Branges–Rovnyak spaces play an important role in numerous questions of complex analysis and operator theory (e.g. see [4, 23, 37, 38, 39]). For the general theory of $\mathcal{H}(b)$ spaces we refer to [37].

In the special case where $b = \Theta$ is an inner function (that is, $|\Theta| = 1$ a.e. on $\mathbb{R}$), the operator $(Id - T_\Theta T_{\overline{\Theta}})^{1/2}$ is an orthogonal projection and $\mathcal{H}(\Theta)$ becomes a closed (ordinary) subspace of $H^2(\mathbb{C}_+)$ which coincides with the so-called model subspace

$$K_\Theta^2 = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+) = H^2(\mathbb{C}_+) \cap \Theta \overline{H^2(\mathbb{C}_+)}$$

(for the model space theory see [29]). We mention one important particular class of model spaces. If $\Theta(z) = \exp(iaz)$, $a > 0$, then $\mathcal{H}(\Theta) = K_\Theta^2 = H^2(\mathbb{C}_+) \cap PW_a^2$, where PW_a^2 stands for the Paley–Wiener space of all entire functions of exponential type at most a, whose restrictions to $\mathbb{R}$ belong to $L^2(\mathbb{R})$. Then the famous Bernstein's inequality asserts that

$$\|f'\|_2 \le a\|f\|_2, \qquad f \in PW_a^2.$$

This classical and important inequality was extended by many authors in many different directions. It is impossible to give an exhaustive list of references, but we would like to mention [9, 22, 32, 34, 35, 40] and [26, Lecture 28].

Notably, one natural direction is to extend Bernstein's inequality to general model subspaces. In [27], Levin showed that if Θ is an inner function and $|\Theta'(x)| < \infty$, $x \in \mathbb{R}$, then for each function $f \in K_\Theta^\infty = H^\infty(\mathbb{C}_+) \cap \Theta \overline{H^\infty(\mathbb{C}_+)}$, the derivative $f'(x)$ exists in the sense of nontangential boundary values and

$$|f'(x)/\Theta'(x)| \leq \|f\|_\infty.$$

Differentiation in the model spaces $K_\Theta^p := H^p(\mathbb{C}_+) \cap \Theta \overline{H^p(\mathbb{C}_+)}$, $1 < p < \infty$, was studied extensively by Dyakonov [16, 17], who showed that the Bernstein-type inequality $\|f'\|_p \leq C\|f\|_p$, $f \in K_\Theta^p$, holds if and only if $\Theta' \in L^\infty(\mathbb{R})$. Recently, Baranov [5, 6, 8] has obtained weighted Bernstein-type inequalities for the model subspaces K_Θ^p, which generalized previous results of Levin and Dyakonov. More precisely, for a general inner function Θ, he proved estimates of the form

$$(1.2) \qquad \|f^{(n)} w_{p,n}\|_{L^p(\mu)} \leq C \, \|f\|_p, \qquad f \in K_\Theta^p,$$

where $n \geq 1$, μ is a Carleson measure in the closed upper half-plane and $w_{p,n}$ is some weight related to the norm of reproducing kernels of the space K_Θ^2 which compensates possible growth of the derivative near the boundary.

One of the main ingredients in the results of Dyakonov and Baranov was an integral formula for the derivatives of functions in K_Θ^p. Using Cauchy formula, it is easy to see that if Θ is inner, $\omega \in \mathbb{C}_+$, n is a non-negative integer and $f \in K_\Theta^p$, then we have

$$(1.3) \qquad f^{(n)}(\omega) = \frac{1}{2\pi i} \int_\mathbb{R} f(t) \, \overline{k_{\omega,n}^\Theta(t)} \, dt,$$

where

$$(1.4) \qquad \frac{k_{\omega,n}^\Theta(z)}{n!} := \frac{1 - \Theta(z) \sum_{p=0}^n \dfrac{\overline{\Theta^{(p)}(\omega)}}{p!}(z - \overline{\omega})^p}{(z - \overline{\omega})^{n+1}}, \qquad z \in \mathbb{C}_+.$$

A natural question is whether one can extend the formula (1.3) to boundary points x_0. If $x_0 \in \mathbb{R}$ does not belong to the boundary spectrum $\sigma(\Theta)$ of Θ (see the definition in Section

5), then Θ and all functions of K_Θ^p are analytic through a neighborhood of x_0 and then it is obvious that (1.3) is valid for $z = x_0$. More generally, if x_0 satisfies

$$(1.5) \qquad \sum_k \frac{\operatorname{Im} z_k}{|x_0 - z_k|^{(n+1)q}} + \int_{\mathbb{R}} \frac{d\mu(t)}{|t - x_0|^{(n+1)q}} < +\infty,$$

then, by the results of Ahern and Clark [1] (for $p = 2$) and Cohn [12] (for $p > 1$), the formula (1.3) is still valid at the point $x_0 \in \mathbb{R}$ for any $f \in K_\Theta^p$ (here $\{z_k\}$ is the sequence of zeros of Θ and μ is the singular measure associated to Θ). Recently Fricain and Mashreghi studied the boundary behavior of functions in de Branges–Rovnyak spaces $\mathcal{H}(b)$ and obtained a generalization of representation (1.3) [20, 21].

In the present paper de Branges–Rovnyak spaces are studied from the point of view of function theory. Namely, we are interested in boundary properties of the elements of $\mathcal{H}(b)$ and of their derivatives, and we establish a number of weighted Bernstein-type inequalities. Our first goal is to exploit the generalization of representation (1.3) and obtain an analogue of Bernstein-type inequality (1.2) for the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is an *arbitrary function* in the unit ball of $H^\infty(\mathbb{C}_+)$ (not necessarily inner). It should be noted that the inner product in $\mathcal{H}(b)$ is not given by a usual integral formula. This fact causes certain difficulties. For example, we will see that one has to add one more term to formula (1.3) in the general case. In what follows we try to emphasize the points where there is a difference with the inner case, and suggest a few open questions.

Our second goal is to provide several applications of these Bernstein-type inequalities. The classical Carleson embedding theorem gives a simple geometrical condition on a measure μ in the closed upper half-plane such that the embedding $H^p(\mathbb{C}_+) \subset L^p(\mu)$ holds. A similar question for model subspaces K_Θ^p was studied by Cohn [11] and then by Volberg and Treil [42]. An approach based on the (weighted norm) Bernstein inequalities for model subspaces K_Θ^p was suggested in [6]. Given b in the unit ball of $H^\infty(\mathbb{C}_+)$, we describe a class of Borel measures μ in $\mathbb{C}_+ \cup \mathbb{R}$ such that $\mathcal{H}(b) \subset L^2(\mu)$. We obtain

a geometric condition on μ sufficient for such embedding. This result generalizes the previous results of Cohn and Volberg–Treil.

Another application concerns the problem of stability of Riesz bases consisting of reproducing kernels of $\mathcal{H}(b)$. This problem is connected with the famous problem of bases of exponentials in L^2 on an interval which goes back to Paley and Wiener [30]. Exponential bases were described by Pavlov [31] and by Hruschev, Nikolski and Pavlov in [24], where functional model methods have been used. This approach has been proved fruitful; it has allowed both to recapture all the classical results and to extend them to general model spaces (for a detailed presentation of the subject see [29]). Fricain has pursued this investigation with respect to bases of reproducing kernels in vector-valued model spaces [18] and in de Branges–Rovnyak spaces [19] where some criteria for a family of reproducing kernels to be a Riesz basis were obtained. However, the criteria mentioned above involve some properties of a given family of reproducing kernel that are rather difficult to verify. On the other hand, in many cases, the given family is a slight perturbation of another family of reproducing kernels that is known to be a basis. This gives rise to the following stability problem: *Given a Riesz basis of reproducing kernels $(k_{\lambda_n}^b)_{n \geq 1}$ of $\mathcal{H}(b)$, characterize perturbations of frequencies $(\lambda_n)_{n \geq 1}$ which preserve the property to be a Riesz basis.*

This problem was also studied by many authors in the context of exponential bases (see e.g. [25, 36]) and of model subspaces K_Θ^2 [7, 13, 18]. In the present paper, using the weighted norm inequalities (1.2) we extend the results about stability in pseudohyperbolic metrics from [7, 18] to de Branges-Rovnyak spaces.

The paper is organized as follows. Sections 2 and 3 contain some preliminaries concerning integral representations for the n-th derivative of functions in de Branges–Rovnyak spaces. In Section 4 we prove our first main result, a Bernstein-type inequality for $\mathcal{H}(b)$. Section 5 contains some estimates relating the weight $w_{p,n}$ involved in Bernstein inequalities to the distances to the level sets of $|b|$. Section 6 is devoted to embedding theorems.

Finally, in Section 7 we apply the Bernstein inequality to the problem of stability of Riesz basis of reproducing kernels in $\mathcal{H}(b)$.

In what follows, the letter C will denote a positive constant and we assume that its value may change. We write $f \asymp g$ if $C_1 g \leq f \leq C_2 g$ for some positive constants C_1, C_2. The set of integers $1, 2, \cdots$ will be denoted by $\mathbb{N}$.

2. PRELIMINARIES

Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$ and let $b = B I_\mu O_b$ be its canonical factorization, where

$$B(z) = \prod_r e^{i\alpha_r} \frac{z - z_r}{z - \bar{z}_r}$$

is a Blaschke product, the singular inner function I_μ is given by

$$I_\mu(z) = \exp\left(iaz - \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{t^2 + 1} \right) d\mu(t) \right)$$

with a positive singular measure μ and $a \geq 0$, and O_b is the outer function

$$O_b(z) = \exp\left(\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{t^2 + 1} \right) \log |b(t)| \, dt \right).$$

Then the modulus of the angular derivative of b at a point $x \in \mathbb{R}$ is given by

$$(2.1) \qquad |b'(x)| = a + \sum_r \frac{2 \operatorname{Im} z_r}{|x - z_r|^2} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{|x - t|^2} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\left| \log |b(t)| \right|}{|x - t|^2} \, dt.$$

Hence, we are motivated to define

$$(2.2) \qquad S_n(x) := \sum_{r=1}^{+\infty} \frac{\operatorname{Im} z_r}{|x - z_r|^n} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x - t|^n} + \int_{\mathbb{R}} \frac{\left| \log |b(t)| \right|}{|x - t|^n} \, dt,$$

and

$$E_n(b) := \{ x \in \mathbb{R} : S_n(x) < +\infty \}.$$

The formula (2.1) explains why the quantity S_2 is of special interest.

We will need the following simple estimate.

Lemma 2.1. *For any $x \in \mathbb{R}$, $y > 0$, we have $|b'(x + iy)| \leq |b'(x)|$.*

Proof. Let $z = x + iy$, $y > 0$, and assume that b is outer,

$$b(z) = \exp\left(\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{t^2+1}\right) \log|b(t)|\, dt\right).$$

Then

$$b'(z) = -b(z)\frac{i}{\pi} \int_{\mathbb{R}} \frac{\log|b(t)|}{(t-z)^2}\, dt,$$

and clearly

$$|b'(z)| \leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{|\log|b(t)||}{|t-z|^2}\, dt \leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{|\log|b(t)||}{|t-x|^2}\, dt = |b'(x)|,$$

by (2.1). The estimates for inner factors are analogous and left to the reader (recall that $|b'(x)| = |O_b'(x)| + |I_\mu'(x)| + |B'(x)|$, $x \in \mathbb{R}$).

$\square$

Ahern and Clark [2] showed that if $x_0 \in E_n(b)$, then b and all its derivatives up to order $n-1$ have (finite) nontangential limits at x_0. In [20], we showed that if $x_0 \in E_{2n+2}(b)$ where $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, then, for each $f \in \mathcal{H}(b)$ and for each $0 \leq j \leq n$, the nontangential limit

$$f^{(j)}(x_0) := \lim_{\substack{z \to x_0 \\ \lhd}} f^{(j)}(z)$$

exists. This is a generalization of the Ahern–Clark theorem [1] for the elements of model subspaces K_Θ^2, i.e. for the case when $b = \Theta$ is an inner function. Moreover, for every $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$ and for every function $f \in \mathcal{H}(b)$, we obtained in [21] the following integral representation for $f^{(n)}(z_0)$. Let $\rho(t) = 1 - |b(t)|^2$ and let $H^2(\rho)$ be the span of the Cauchy kernels k_z, $z \in \mathbb{C}_+$, in $L^2(\rho)$ (recall that $k_z(\omega) = (\omega - \bar{z})^{-1}$). Consider the operator

$$\widetilde{T}_\rho : L^2(\rho) \longrightarrow H^2(\mathbb{C}_+)$$
$$q \longmapsto P_+(q\rho).$$

We know from [37, II-3, III-2] that if $f \in \mathcal{H}(b)$ then there exists a (unique) function g in $H^2(\rho)$ such that $T_{\bar{b}}f = \widetilde{T}_\rho g$. It was shown in [21] that, for $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, $n \in \mathbb{Z}_+$,

we have

$$(2.3) \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i} \left(\int_{\mathbb{R}} f(t)\overline{k^b_{z_0,n}(t)}\, dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{k^\rho_{z_0,n}(t)}\, dt \right),$$

where $k^b_{z_0,n}$ is the function in $\mathcal{H}(b)$ defined by

$$(2.4) \qquad k^b_{z_0,n}(z) := \frac{1 - b(z) \displaystyle\sum_{j=0}^{n} \frac{\overline{b^{(j)}(z_0)}}{j!}(z - \overline{z_0})^j}{(z - \overline{z_0})^{n+1}}, \qquad z \in \mathbb{C}_+,$$

and $k^\rho_{z_0,n}$ is the function in $L^2(\rho)$ defined by

$$(2.5) \qquad k^\rho_{z_0,n}(t) := \frac{\displaystyle\sum_{j=0}^{n} \frac{\overline{b^{(j)}(z_0)}}{j!}(t - \overline{z_0})^j}{(t - \overline{z_0})^{n+1}}, \qquad t \in \mathbb{R}.$$

Note that if b is inner, then $\rho \equiv 0$ and thus (2.3) reduces to (1.3) which was the key representation formula used in [5, 6, 16, 17] to obtain Bernstein-type inequalities for model subspaces K^p_Θ. If $n = 0$ then $k^b_{z_0,0}$ corresponds to the reproducing kernel of $\mathcal{H}(b)$ defined in (1.1).

3. A new representation formula for the derivatives

We start with a slight modification of the representation (2.3) for $n \in \mathbb{N}$.

Proposition 3.1. *Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$. Let $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, $n \in \mathbb{N}$, and let*

$$(3.1) \qquad \mathfrak{K}^\rho_{z_0,n}(t) := \overline{b(z_0)} \frac{\sum_{j=0}^{n} \binom{n+1}{j+1}(-1)^j \overline{b^j(z_0)}\, b^j(t)}{(t - \overline{z_0})^{n+1}}, \qquad t \in \mathbb{R}.$$

Then $(k^b_{z_0})^{n+1} \in H^2(\mathbb{C}_+)$ and $\mathfrak{K}^\rho_{z_0,n} \in L^2(\rho)$. Moreover, for every function $f \in \mathcal{H}(b)$, we have

$$(3.2) \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i} \left(\int_{\mathbb{R}} f(t)\overline{(k^b_{z_0})^{n+1}(t)}\, dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{\mathfrak{K}^\rho_{z_0,n}(t)}\, dt \right),$$

where $g \in H^2(\rho)$ is such that $T_{\overline{b}}f = \widetilde{T}_\rho g$.

Proof. Let $a_j = b^{(j)}(z_0)/j!$. Then

$$
k_{z_0,\ell}^b(z) = \frac{1 - \overline{b(z_0)}b(z) - b(z)\sum_{j=1}^{\ell}\overline{a_j}(z - \overline{z_0})^j}{(z - \overline{z_0})^{\ell+1}}
$$

$$
= \frac{1 - \overline{b(z_0)}b(z)}{(z - \overline{z_0})^{\ell+1}} - b(z)\sum_{j=1}^{\ell}\frac{\overline{a_j}}{(z - \overline{z_0})^{\ell+1-j}}.
$$

Hence, multiplying by $(1 - \overline{b(z_0)}b(z))^\ell$, we obtain

$$
(3.3) \quad (k_{z_0}^b)^{\ell+1}(z) = (1 - \overline{b(z_0)}b(z))^\ell\, k_{z_0,\ell}^b(z) + b(z)\sum_{j=1}^{\ell}\overline{a_j}(1 - \overline{b(z_0)}b(z))^{j-1}(k_{z_0}^b)^{\ell+1-j}(z).
$$

Since $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, according to [21, Proposition 3.1 and Lemma 3.2], the functions $k_{z_0}^b$ and $k_{z_0,\ell}^b$ $(1 \le \ell \le n)$ belong to $\mathcal{H}(b)$. Hence, using the recurrence relation (3.3) and that $1 - \overline{b(z_0)}b(z) \in H^\infty(\mathbb{C}_+)$, we see immediately by induction that $(k_{z_0}^b)^{n+1} \in H^2(\mathbb{C}_+)$.

We prove now that $\mathcal{K}_{z_0,n}^\rho \in L^2(\rho)$. Write $\mathcal{K}_{z_0,n}^\rho(t) = (t - \overline{z_0})^{-(n+1)}\varphi(t)$, with

$$
\varphi(t) = \overline{b(z_0)}\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\overline{b^j(z_0)}b^j(t).
$$

Since $\varphi \in L^\infty(\mathbb{R})$, it is sufficient to prove that $(t - \overline{z_0})^{-(n+1)} \in L^2(\rho)$. If $z_0 \in \mathbb{C}_+$, this fact is trivial and if $z_0 \in E_{2n+2}(b)$, the inequality $1 - x \le |\log x|$, $x \in (0,1]$, implies

$$
\int_{\mathbb{R}}\frac{1 - |b(t)|^2}{|t - z_0|^{2n+2}}\, dt \le 2\int_{\mathbb{R}}\frac{\big|\log|b(t)|\big|}{|t - z_0|^{2n+2}}\, dt < +\infty
$$

which is the required result.

It remains to prove (3.2). Let ψ be any element of $H^2(\mathbb{C}_+)$. According to (2.3), we have

$$
\frac{2\pi i}{n!}f^{(n)}(z_0) = \langle f, k_{z_0,n}^b \rangle_2 + \langle \rho g, k_{z_0,n}^\rho \rangle_2
$$

$$
= \langle f, k_{z_0,n}^b - b\psi \rangle_2 + \langle \overline{b}f, \psi \rangle_2 + \langle \rho g, k_{z_0,n}^\rho \rangle_2.
$$

But we have $T_{\bar{b}} f = \widetilde{T}_\rho g$, which means that $\bar{b} f - \rho g \perp H^2(\mathbb{C}_+)$. Since $\psi \in H^2(\mathbb{C}_+)$, it follows that $\langle \bar{b} f, \psi \rangle_2 = \langle \rho g, \psi \rangle_2$. Hence the identity

$$(3.4) \qquad \frac{2\pi i}{n!} f^{(n)}(z_0) = \langle f, k^b_{z_0,n} - b\psi \rangle_2 + \langle \rho g, k^\rho_{z_0,n} + \psi \rangle_2$$

holds for each $\psi \in H^2(\mathbb{C}_+)$. A very specific ψ gives us the required representation. To find ψ note that, on one hand, we have

$$
\begin{aligned}
k^b_{z_0,n}(t) - (k^b_{z_0})^{n+1}(t) &= \frac{1 - b(t)\sum_{j=0}^{n} \overline{a_j}(t - \overline{z_0})^j - (1 - \overline{b(z_0)}b(t))^{n+1}}{(t - \overline{z_0})^{n+1}} \\
&= \frac{1 - (1 - \overline{b(z_0)}b(t))^{n+1}}{(t - \overline{z_0})^{n+1}} - b(t)\frac{\sum_{j=0}^{n} \overline{a_j}(t - \overline{z_0})^j}{(t - \overline{z_0})^{n+1}} = b(t)\psi(t),
\end{aligned}
$$

where

$$\psi(t) = \frac{\sum_{j=1}^{n+1}(-1)^{j+1}\binom{n+1}{j}(\overline{b(z_0)})^j (b(t))^{j-1}}{(t - \overline{z_0})^{n+1}} - \frac{\sum_{j=0}^{n} \overline{a_j}(t - \overline{z_0})^j}{(t - \overline{z_0})^{n+1}}.$$

On the other hand, we easily see that

$$
\begin{aligned}
k^\rho_{z_0,n}(t) + \psi(t) &= \frac{\sum_{j=1}^{n+1}(-1)^{j+1}\binom{n+1}{j}(\overline{b(z_0)})^j (b(t))^{j-1}}{(t - \overline{z_0})^{n+1}} \\
&= \overline{b(z_0)}\frac{\sum_{j=0}^{n}(-1)^{j}\binom{n+1}{j+1}(\overline{b(z_0)})^j (b(t))^{j}}{(t - \overline{z_0})^{n+1}} = \mathfrak{K}^\rho_{z_0,n}(t).
\end{aligned}
$$

Therefore, (3.2) follows immediately from (3.4).

$\square$

We now introduce the weight involved in our Bernstein-type inequalities. Let $1 < p \leq 2$ and let q be its conjugate exponent. Let $n \in \mathbb{N}$. Then, for $z \in \overline{\mathbb{C}_+}$, we define

$$w_{p,n}(z) := \min\left\{ \|(k^b_z)^{n+1}\|_q^{-pn/(pn+1)}, \ \|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q^{-pn/(pn+1)} \right\};$$

we assume $w_{p,n}(x) = 0$, whenever $x \in \mathbb{R}$ and at least one of the functions $(k^b_x)^{n+1}$ or $\rho^{1/q}\mathfrak{K}^\rho_{x,n}$ is not in $L^q(\mathbb{R})$. In what follows we will write w_p for $w_{p,1}$.

The choice of the weight is motivated by representation (3.2) which shows that the quantity $\max\left\{ \|(k^b_z)^{n+1}\|_2, \|\rho^{1/2}\mathfrak{K}^\rho_{z,n}\|_2 \right\}$ is related to the norm of the functional $f \mapsto f'(z)$ on $\mathcal{H}(b)$. Moreover, we strongly believe that the norms of reproducing kernels are an

important characteristic of the space $\mathcal{H}(b)$ which captures many geometric properties of b (see Section 5 for certain estimates confirming this point).

Using similar arguments as in the proof of Proposition 3.1, it is easy to see that $\rho^{1/q}\mathfrak{K}_{x,n}^{\rho} \in L^q(\mathbb{R})$ if $x \in E_{q(n+1)}(b)$. It is also natural to expect that $(k_x^b)^{n+1} \in L^q(\mathbb{R})$ for $x \in E_{q(n+1)}(b)$. This is true when b is an inner function, by a result of Cohn [12], and for a general function b with $q = 2$ by (3.3) and [20, Lemma 3.2]. However, it seems that the methods of [12] and [20] do not apply in the general case.

Question 3.2. Is it true that for $x \in \mathbb{R}$, $(k_x^b)^{n+1} \in L^q(\mathbb{R})$ if $x \in E_{q(n+1)}(b)$?

Remark 3.3. If $f \in \mathcal{H}(b)$ and $1 < p \leq 2$, then $(f^{(n)}w_{p,n})(x)$ is well-defined on $\mathbb{R}$. It follows from the [20] that $f^{(n)}(x)$ and $w_{p,n}(x)$ are finite if $S_{2n+2}(x) < +\infty$. If $S_{2n+2}(x) = +\infty$, then $\|(k_x^b)^{n+1}\|_2 = +\infty$. Hence, $\|(k_x^b)^{n+1}\|_q = +\infty$ which, by definition, implies $w_{p,n}(x) = 0$, and thus we may assume $(f^{(n)}w_{p,n})(x) = 0$.

Remark 3.4. In the inner case, we have $\rho(t) \equiv 0$ and the second term in the definition of the weight $w_{p,n}$ disappears. It should be emphasized that in the general case both terms are essential: below we show (Example 4.2) that the norm $\|\rho^{1/q}\mathfrak{K}_{z,n}^{\rho}\|_q$ can not be majorized uniformly by the norm $\|(k_z^b)^{n+1}\|_q$.

Lemma 3.5. *For $1 < p \leq 2$, $n \in \mathbb{N}$, there is a constant $A = A(p,n) > 0$ such that*

$$w_{p,n}(z) \geq A \frac{(\operatorname{Im} z)^n}{(1 - |b(z)|)^{\frac{pn}{q(pn+1)}}}, \qquad z \in \mathbb{C}_+.$$

Proof. On one hand, note that

$$\|(k_z^b)^{n+1}\|_q^q = \int_{\mathbb{R}} \left| \frac{1 - \overline{b(z)}b(t)}{t - \bar{z}} \right|^{(n+1)q} dt \leq \frac{C}{(\operatorname{Im} z)^{(n+1)q-2}} \int_{\mathbb{R}} \left| \frac{1 - \overline{b(z)}b(t)}{t - \bar{z}} \right|^2 dt$$

$$= \frac{C}{(\operatorname{Im} z)^{(n+1)q-2}} \|k_z^b\|_b^2 \leq C \frac{1 - |b(z)|}{(\operatorname{Im} z)^{(n+1)q-1}}.$$

On the other hand, we have

$$\|\rho^{1/q}\mathfrak{K}^{\rho}_{z,n}\|^q_q = \int_{\mathbb{R}} \left| \frac{b(z)\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\overline{b(z)}^j b^j(t)}{(t-\overline{z})^{n+1}} \right|^q (1-|b(t)|^2)\,dt$$

$$\leq \frac{C}{(\operatorname{Im} z)^{(n+1)q-2}} \int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}\,dt.$$

If $|b(z)| < 1/2$, then we obviously have

$$\int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}dt \leq C\frac{1-|b(z)|}{\operatorname{Im} z},$$

and if $|b(z)| \geq 1/2$, using $1-|b(t)| \leq \big|\log|b(t)|\big|$, we get

$$\operatorname{Im} z \int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}dt \leq \operatorname{Im} z \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-z|^2}dt = \pi \log \frac{1}{|O_b(z)|} \asymp 1-|O_b(z)|,$$

since $|O_b(z)| \geq |b(z)| \geq 1/2$. We recall that O_b is the outer part of b. Therefore, in any case we have

$$\int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}dt \leq C\frac{1-|b(z)|}{\operatorname{Im} z},$$

and we get

$$\|\rho^{1/q}\mathfrak{K}^{\rho}_{z,n}\|^q_q \leq C\,\frac{1-|b(z)|}{(\operatorname{Im} z)^{(n+1)q-1}}.$$

To complete the proof, it suffices to note that $\frac{(n+1)q-1}{q} = n + \frac{1}{p} = \frac{np+1}{p}$.

$\square$

Representation formulae discussed above reduce the study of differentiation in de Branges–Rovnyak spaces $\mathcal{H}(b)$ to the study of certain integral operators.

4. BERNSTEIN-TYPE INEQUALITIES

A Borel measure μ in the closed upper half-plane $\overline{\mathbb{C}_+}$ is said to be a Carleson measure if there is a constant $C_\mu > 0$ such that

(4.1) $$\mu(\,S(x,h)\,) \leq C_\mu\,h,$$

for all squares $S(x, h) = [x, x + h] \times [0, h]$, $x \in \mathbb{R}$, $h > 0$, with the lower side on the real axis. We denote the class of Carleson measures by $\mathcal{C}$. Recall that, according to a classical theorem of Carleson, $\mu \in \mathcal{C}$ if and only if $H^p(\mathbb{C}_+) \subset L^p(\mu)$ for some (all) $p > 0$.

One of our main results in this paper is the following Bernstein-type inequality.

Theorem 4.1. *Let $\mu \in \mathcal{C}$, let $n \in \mathbb{N}$, let $1 < p \leq 2$, and let*

$$(T_{p,n}f)(z) = f^{(n)}(z)w_{p,n}(z), \qquad f \in \mathcal{H}(b).$$

If $1 < p < 2$, then $T_{p,n}$ is a bounded operator from $\mathcal{H}(b)$ to $L^2(\mu)$, that is, there is a constant $C = C(\mu, p, n) > 0$ such that

$$(4.2) \qquad \|f^{(n)}w_{p,n}\|_{L^2(\mu)} \leq C\|f\|_b, \qquad f \in \mathcal{H}(b).$$

If $p = 2$, then $T_{2,n}$ is of weak type $(2, 2)$ as an operator from $\mathcal{H}(b)$ to $L^2(\mu)$.

Proof. According to Proposition 3.1, for all $z \in \overline{\mathbb{C}_+}$ and any function $f \in \mathcal{H}(b)$, we have

$$(4.3) \quad \frac{2\pi i}{n!}f^{(n)}(z)w_{p,n}(z) = w_{p,n}(z)\int_{\mathbb{R}} f(t)\overline{(k_{z_0}^b)^{n+1}(t)}\,dt + w_{p,n}(z)\int_{\mathbb{R}} g(t)\rho(t)\overline{\mathfrak{K}_{z,n}^\rho(t)}\,dt.$$

Let

$$w_{p,n}^{(1)}(z) := \|(k_z^b)^{n+1}\|_q^{-pn/(pn+1)}, \qquad w_{p,n}^{(2)}(z) := \|\rho^{1/q}\mathfrak{K}_{z,n}^\rho\|_q^{-pn/(pn+1)},$$

where we assume that $w_{p,n}^{(i)}(z) = 0$ if the corresponding integrand is not in $L^q(\mathbb{R})$, and put $h_i(z) = (w_{p,n}^{(i)}(z))^{1/n}$, $i = 1, 2$. We remind that

$$w_{p,n}(z) = \min\{ w_{p,n}^{(1)}(z),\ w_{p,n}^{(2)}(z)\}.$$

We split each of the two integrals in (4.3) into two parts, i.e.

$$\frac{2\pi i}{n!}f^{(n)}(z)w_{p,n}(z) = I_1f(z) + I_2f(z) + I_3g(z) + I_4g(z),$$

where

$$I_1f(z) = w_{p,n}(z)\int_{|t-z|\geq h_1(z)} f(t)\overline{(k_z^b)^{n+1}(t)}\,dt,$$

$$I_2f(z) = w_{p,n}(z)\int_{|t-z|<h_1(z)} f(t)\overline{(k_z^b)^{n+1}(t)}\,dt,$$

$$I_3 g(z) = w_{p,n}(z) \int_{|t-z| \geq h_2(z)} g(t) \rho(t) \overline{\mathfrak{K}^{\rho}_{z,n}(t)} \, dt,$$

$$I_4 g(z) = w_{p,n}(z) \int_{|t-z| < h_2(z)} g(t) \rho(t) \overline{\mathfrak{K}^{\rho}_{z,n}(t)} \, dt.$$

Note that by Lemma 3.5, $h_i(z) \geq A \operatorname{Im} z$, $z \in \mathbb{C}_+$, $i = 1, 2$. Hence,

$$|I_1 f(z)| \leq C h_1^n(z) \int_{|t-z| \geq h_1(z)} \frac{|f(t)|}{|t-z|^{n+1}} \, dt$$

$$\leq C h_1(z) \int_{|t-z| \geq h_1(z)} \frac{|f(t)|}{|t-z|^2} \, dt,$$

and

$$|I_3 g(z)| \leq C h_2^n(z) \int_{|t-z| \geq h_2(z)} \frac{|g(t)| \rho^{1/2}(t)}{|t-z|^{n+1}} \, dt$$

$$\leq C h_2(z) \int_{|t-z| \geq h_2(z)} \frac{|g(t)| \rho^{1/2}(t)}{|t-z|^2} \, dt.$$

Using [6, Theorem 3.1], we see that $I_1 : L^2(\mathbb{R}) \longrightarrow L^2(\mu)$ and $I_3 : L^2(\rho) \longrightarrow L^2(\mu)$ are bounded operators. To estimate the integral $I_2 f$, put

$$K(z,t) := h_1^n(z) |(k_z^b)^{n+1}(t)|.$$

Then

$$\|K(z, \cdot)\|_q^{-p} = (h_1(z))^{-pn} \|(k_z^b)^{n+1}\|_q^{-p}$$

$$= (h_1(z))^{-pn} (w_{p,n}^{(1)}(z))^{(pn+1)/n} = h_1(z).$$

Thus

$$|I_2 f(z)| \leq h_1^n(z) \int_{|t-z| < h_1(z)} |f(t)| |(k_z^b)^{n+1}(t)| \, dt = \int_{|t-z| < \|K(z, \cdot)\|_q^{-p}} |f(t)| K(z,t) \, dt.$$

Since $\|K(z, \cdot)\|_q^{-p} = h_1(z) \geq A \operatorname{Im} z$, we may apply [6, Theorem 3.2]. Therefore, the operator I_2 is of weak type $(2, 2)$ as an operator from $L^2(\mathbb{R})$ to $L^2(\mu)$ if $p = 2$ and it is a

bounded operator from $L^2(\mathbb{R})$ to $L^2(\mu)$ if $1 < p < 2$. To estimate the integral $I_4 g$, we use the same technique and put

$$\kappa(z,t) := \frac{\rho^{1/q}(t)|\mathfrak{K}_{z,n}^{\rho}(t)|}{\|\rho^{1/q}\mathfrak{K}_{z,n}^{\rho}\|_q^{pn/(pn+1)}}.$$

In other words, $\kappa(z,t) = w_{p,n}^{(2)}(z)\rho^{1/q}(t)|\mathfrak{K}_{z,n}^{\rho}(t)|$. Thus

$$|I_4 g(z)| \leq w_{p,n}^{(2)}(z) \int_{|t-z|<h_2(z)} |g(t)|\rho(t)|\mathfrak{K}_{z,n}^{\rho}(t)| \, dt$$

$$= \int_{|t-z|<h_2(z)} |g(t)|\rho^{1/p}(t)\kappa(z,t) \, dt.$$

But $\|\kappa(z,\cdot)\|_q^{-p} = (w_{p,n}^{(2)}(z))^{-p}\|\rho^{1/q}\mathfrak{K}_{z,n}^{\rho}\|_q^{-p} = h_2(z)$. Hence, we get

$$|I_4 g(z)| \leq \int_{|t-z|<\|\kappa(z,\cdot)\|_q^{-p}} |g(t)|\rho^{1/p}(t)\kappa(z,t) \, dt.$$

Since $p \leq 2$ and $\rho(t) \leq 1$, we have

$$|I_4 g(z)| \leq \int_{|t-z|<\|\kappa(z,\cdot)\|_q^{-p}} |g(t)|\rho^{1/2}(t)\kappa(z,t) \, dt,$$

and since $\|\kappa(z,\cdot)\|_q^{-p} = h_2(z) \geq A \operatorname{Im} z$, we may apply again [6, Theorem 3.2]. Therefore, the operator I_4 is of weak type $(2,2)$ as an operator from $L^2(\rho)$ to $L^2(\mu)$ if $p = 2$ and it is a bounded operator from $L^2(\rho)$ to $L^2(\mu)$ if $1 < p < 2$.

To conclude it remains to note that

$$\|f\|_b^2 = \|f\|_2^2 + \|g\|_\rho^2,$$

which implies that the operators $f \mapsto f$ from $\mathcal{H}(b)$ to $H^2(\mathbb{C}_+)$ and $f \mapsto g$ from $\mathcal{H}(b)$ to $L^2(\rho)$ are contractions.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Example 4.2. We show that for a general function b both terms in the definition of the weight $w_{p,n}$ are important. Obviously, for an inner b the norm $\|\rho^{1/q}\mathfrak{K}_{z,n}^{\rho}\|_q$ vanishes. However, for some outer functions b it may be essentially larger than $\|(k_z^b)^{n+1}\|_q$.

Let $\varepsilon \in (0,1)$ and let b be an outer function such that $|b(t)| = \varepsilon$ for $|t| < 1$ and $|b(t)| = 1$ for $|t| > 1$. Note that $b(z) = \exp\left(-\frac{i}{\pi}\log\varepsilon \log\frac{z-1}{z+1}\right)$, where $\log$ is the main branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$. We show that

$$(4.4) \qquad \sup_{y>0} \frac{\|\rho^{1/q}\mathcal{K}^{\rho}_{iy,1}\|_q}{\|(k^b_{iy})^2\|_q} \longrightarrow \infty \quad \text{as} \quad \varepsilon \longrightarrow 1-,$$

and so, the second term in the weight $w_{p,1}$ can be dominating. Note that $b(iy) \to \varepsilon$ and $b(t) \to \varepsilon$, as $y \to 0+$ and $|t| \leq \sqrt{y}$. Hence, for a fixed ε and sufficiently small $y > 0$ we have

$$\int_{|t|\leq\sqrt{y}} |k^b_{iy}(t)|^{2q}dt = \int_{|t|\leq\sqrt{y}} \left|\frac{1-\overline{b(iy)}b(t)}{t+iy}\right|^{2q} dt \leq C(1-\varepsilon)^{2q}\int_{|t|\leq\sqrt{y}} \frac{dt}{|t+iy|^{2q}}.$$

Thus

$$(4.5) \qquad \int_{|t|\leq\sqrt{y}} \left|\frac{1-\overline{b(iy)}b(t)}{t+iy}\right|^{2q} dt \leq C\frac{(1-\varepsilon)^{2q}}{y^{2q-1}},$$

whereas

$$(4.6) \qquad \int_{|t|>\sqrt{y}} \left|\frac{1-\overline{b(iy)}b(t)}{t+iy}\right|^{2q} dt \leq Cy^{-q+1/2}.$$

On the other hand,

$$\mathcal{K}^{\rho}_{iy,1}(t) = \overline{b(iy)}\frac{2-\overline{b(iy)}b(t)}{(t+iy)^2},$$

and so

$$\|\rho^{1/q}\mathcal{K}^{\rho}_{iy,1}\|^q_q \asymp |b(iy)|^q \int_{\mathbb{R}} \frac{1-|b(t)|}{|t+iy|^{2q}} \asymp \frac{1-\varepsilon}{y^{2q-1}}.$$

Combining the last estimate with (4.5) and (4.6), we obtain (4.4).

Remark 4.3. It should be emphasized that the constants in the Bernstein-type inequalities corresponding to Theorem 4.1 depend only on p, n and the Carleson constant C_μ of the measure μ, but not on b (the properties of b are contained in the weight $w_{p,n}$ in the left-hand side of (4.2)).

Remark 4.4. All the results stated above have their natural analogues for the spaces $\mathcal{H}(b)$ in the unit disc. In particular, Theorem 4.1 remains true when we replace the kernels for the half-plane by the kernels for the disc. The case of inner functions in the disc is considered in detail in [8].

Remark 4.5. An important feature of the de Branges–Rovnyak spaces theory is the difference between the extreme (i.e. b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$) and the non-extreme cases. Our Bernstein inequality applies to both cases. However, in the extreme case one can expect more regularity near the boundary and this situation is more interesting for us.

5. Distances to the level sets

To apply Theorem 4.1, one should have effective estimates for the weight $w_{p,n}$, that is, for the norms of the reproducing kernels. In this section we relate the weight $w_{p,n}$ to the distances to the level sets of $|b|$. We start with some notations. Denote by $\sigma(b)$ the boundary spectrum of b, i.e.

$$\sigma(b) := \left\{ x \in \mathbb{R} : \liminf_{\substack{z \longrightarrow x \\ z \in \mathbb{C}_+}} |b(z)| < 1 \right\}.$$

Then, for $b = BI_\mu O_b$, $\operatorname{Clos} \sigma(b)$ is the smallest closed subset of $\mathbb{R}$ containing the limit points of the zeros of the Blaschke product B and the supports of the measures μ and $\log |b(t)| \, dt$. It is well known and easy to see that b and any element of $\mathcal{H}(b)$ has an analytic extension through any interval from the open set $\mathbb{R} \setminus \operatorname{Clos} \sigma(b)$.

For $\varepsilon \in (0, 1)$, we put

$$\Omega(b, \varepsilon) := \{ z \in \mathbb{C}_+ : |b(z)| < \varepsilon \},$$

and

$$\widetilde{\Omega}(b, \varepsilon) := \sigma(b) \cup \Omega(b, \varepsilon),$$

where $\sigma(b)$ is the boundary spectrum of b. Finally, for $x \in \mathbb{R}$, we introduce the following three distances

$$
\begin{aligned}
d_0(x) &:= \operatorname{dist}(x, \sigma(b)), \\
d_\varepsilon(x) &:= \operatorname{dist}(x, \Omega(b, \varepsilon)), \\
\tilde{d}_\varepsilon(x) &:= \operatorname{dist}(x, \widetilde{\Omega}(b, \varepsilon)).
\end{aligned}
$$

Note that whenever $b = \Theta$ is an inner function, for all $x \in \sigma(\Theta)$, we have

$$
\liminf_{\substack{z \longrightarrow x \\ z \in \mathbb{C}_+}} |\Theta(z)| = 0,
$$

and thus $d_\varepsilon(t) = \tilde{d}_\varepsilon(t)$, $t \in \mathbb{R}$. However, for an arbitrary function b in the unit ball of $H^\infty(\mathbb{C}_+)$, we have to distinguish between the distance functions d_ε and $\tilde{d}_\varepsilon$.

Lemma 5.1. *There exists a positive constant $C = C(\varepsilon)$ such that, for all $x \in \mathbb{R} \setminus \sigma(b)$,*

$$
|b'(x)| \le C\big(\tilde{d}_\varepsilon(x)\big)^{-1}.
$$

Proof. For the case of an inner function the inequality is proved in [6, Theorem 4.9]. For the general case, let $b = I_b O_b$ be the inner-outer factorization of b. Since $|b'(x)| = |I_b'(x)| + |O_b'(x)|$, $x \in \mathbb{R} \setminus \sigma(b)$, we may assume, without loss of generality, that b is outer. Recall that in this case

$$
|b'(x)| = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t - x|^2} \, dt.
$$

Fix $x \in \mathbb{R} \setminus \sigma(b)$ and suppose $0 < y < d_0(x)$. Let $z = x + iy$. Then

$$
\log \frac{1}{|b(z)|} = \frac{y}{\pi} \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t - z|^2} \, dt = \frac{y}{\pi} \int_{|t-x| \ge d_0(x)} \frac{\big|\log|b(t)|\big|}{|t - z|^2} \, dt.
$$

Since $|t - z| \le |t - x| + y \le 2|t - x|$ whenever $|t - x| \ge d_0(x)$, we have

$$
\log \frac{1}{|b(z)|} \ge \frac{y}{4\pi} \int_{|t-x| \ge d_0(x)} \frac{\big|\log|b(t)|\big|}{|t - x|^2} \, dt = \frac{y|b'(x)|}{4}.
$$

Hence

(5.1) $$
|b(x + iy)| \le \exp\big(-y|b'(x)|/4\big),
$$

provided that $0 < y < d_0(x)$.

Let $C = 4\log\varepsilon^{-1}$. If $|b'(x)| \le C/|d_0(x)|$, then the statement is valid since $\tilde{d}_\varepsilon(x) \le d_0(x)$. On the other hand, if $|b'(x)| > C/|d_0(x)|$, then we consider the point $z = x + iC/|b'(x)|$ for which $\operatorname{Im} z = C/|b'(x)| < d_0(x)$. Hence, by (5.1), we have $|b(z)| \le \varepsilon$ which immediately implies $\tilde{d}_\varepsilon(x) \le C/|b'(x)|$.

$\square$

Lemma 5.2. *For each $p > 1$, $n \ge 1$ and $\varepsilon \in (0, 1)$, there exists $C = C(\varepsilon, p, n) > 0$ such that*

$$(5.2) \qquad \left(\tilde{d}_\varepsilon(x)\right)^n \le C\, w_{p,n}(x + iy),$$

for all $x \in \mathbb{R}$ and $y \ge 0$.

Proof. Let $z = x + iy$, $y \ge 0$. Assume that $x \in \mathbb{R} \setminus \sigma(b)$ (otherwise $\tilde{d}_\varepsilon(x) = 0$ and (5.2) is trivial). Since $-(n+1)q + 1 = -q\frac{np+1}{p}$, the estimate (5.2) is equivalent to

$$(5.3) \qquad \int_{\mathbb{R}} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt \le C(\tilde{d}_\varepsilon(x))^{-(n+1)q+1},$$

and

$$(5.4) \qquad \int_{\mathbb{R}} \left| \frac{\overline{b(z)} \sum_{j=0}^n \binom{n+1}{j+1}(-1)^j \overline{b(z)}^j b^j(t)}{(t - \overline{z})^{n+1}} \right|^q \rho(t)\, dt \le C(\tilde{d}_\varepsilon(x))^{-(n+1)q+1}.$$

Inequality (5.4) is obvious, since $\rho(t) = 0$ if $|t - x| < \tilde{d}_\varepsilon(x)$. To prove (5.3), we estimate separately the integrals over $\{t : |t-x| \le \tilde{d}_\varepsilon(x)/2\}$ and $\{t : |t-x| > \tilde{d}_\varepsilon(x)/2\}$. Obviously,

$$\int_{|t-x|>\tilde{d}_\varepsilon(x)/2} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt \le C(\tilde{d}_\varepsilon(x))^{-(n+1)q+1}.$$

Since $|b(t)| = 1$ if $|t - x| \le \tilde{d}_\varepsilon(x)/2$, for the second integral we have

$$\int_{|t-x|\le\tilde{d}_\varepsilon(x)/2} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt = \int_{|t-x|\le\tilde{d}_\varepsilon(x)/2} \left| \frac{b(t) - b(z)}{t - z} \right|^{(n+1)q} dt$$

$$\le \tilde{d}_\varepsilon(x) \max |b'(u)|^{(n+1)q},$$

where the maximum is taken over $u \in [t, z]$ with $|t - x| \leq \tilde{d}_\varepsilon(x)/2$ (by $[t, z]$ we denote the straight line segment with the endpoints t and z). Note that for such u we have $|\operatorname{Re} u - x| \leq \tilde{d}_\varepsilon(x)/2$. By Lemma 5.2, $|b'(u)| \leq |b'(\operatorname{Re} u)|$, and hence,

$$\int_{|t-x| \leq \tilde{d}_\varepsilon(x)/2} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt \leq \tilde{d}_\varepsilon(x) \max_{|t-x| \leq \tilde{d}_\varepsilon(x)/2} |b'(t)|^{(n+1)q}.$$

According to Lemma 5.1, $|b'(t)| \leq C_1(\tilde{d}_\varepsilon(t))^{-1} \leq C_2(\tilde{d}_\varepsilon(x))^{-1}$ whenever $|t - x| < \tilde{d}_\varepsilon(x)/2$ which leads to the required estimate.

$\square$

Corollary 5.3. *For each $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, there exists $C = C(\varepsilon, n)$ such that*

$$\|f^{(n)} \tilde{d}_\varepsilon^n\|_2 \leq C\|f\|_b, \qquad f \in \mathcal{H}(b).$$

Proof. The statement follows immediately from Lemma 5.2 and Theorem 4.1.

$\square$

We conclude this section with a a corollary of our Bernstein inequalities, concerning the regularity on the boundary for functions in $\mathcal{H}(b)$. This technical result will be used later.

Corollary 5.4. *Let $I = [x_0, x_0 + y_0]$ be a bounded interval on $\mathbb{R}$, $1 < p < 2$. Assume that*

$$(5.5) \qquad \int_I w_p(x)^{-2} dx < +\infty.$$

Then we have

    a) $]x_0, x_0 + y_0[\cap \sigma(b) = \emptyset$. *In particular, each function f in $\mathcal{H}(b)$ is differentiable on $]x_0, x_0 + y_0[$.*

    b) b *is continuous on the Carleson square $S(I) = [x_0, x_0 + y_0] \times [0, y_0]$.*

Proof. a) According to Theorem 4.1, there is a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f'(x)w_p(x)|^2 \, dx \leq C\|f\|_b^2, \qquad f \in \mathcal{H}(b).$$

Then, using (5.5) and the Cauchy–Schwartz inequality, we get $f' \in L^1(I)$ for any $f \in \mathcal{H}(b)$. Now choose $z \in \mathbb{C}_+$ such that $b(z) \neq 0$ and take $f = k_z^b$. We have

$$f'(x) = -\overline{b(z)} \frac{b'(x)}{x - \overline{z}} - \frac{k_z^b(x)}{x - \overline{z}}$$

and, since $k_z^b \in L^1(I)$, we conclude that

(5.6)
$$\int_{x_0}^{x_0 + y_0} |b'(x)| \, dx < +\infty.$$

Now it follows immediately from the formula (2.1) for $|b'(x)|$ that (5.6) implies $]x_0, x_0 + y_0[\, \cap \, \sigma(b) = \emptyset$. As a matter of fact, this is obvious for the outer and the singular inner factors since $\int_I (x - t)^{-2} dt = \infty$ for any $x \in I$; and if b is a Blaschke product with zeros z_r tending to $x \in]x_0, x_0 + y_0[$, then, for sufficiently large r,

$$\int_{x_0}^{x_0 + y_0} \frac{2 \, \mathrm{Im} \, z_r}{|x - z_r|^2} dx \geq \pi,$$

and so the integral in (5.6) diverges.

   b) By statement a), b is continuous on $S(I)$ except possibly at the points x_0 and $x_0 + y_0$. It remains to show that b is continuous at x_0 and $x_0 + y_0$. Fix $x_1 \in]x_0, x_0 + y_0[$ and define

$$b(x_0) := b(x_1) - \int_{x_0}^{x_1} b'(x) \, dx.$$

(Note that this definition of $b(x_0)$ does not seem to correspond to the classical one with non-tangential limits but, in fact, as we will see at the end, they coincide). Since b is differentiable on $]x_0, x_0 + y_0[$, this definition does not depend on the choice of x_1 and we see from (5.6) that $b(x)$ tends to $b(x_0)$ as $x \to x_0$ along I. Now let $z = x + iy \in S(I)$, with $x \in [x_0, x_0 + y_0/2[$, $y \in]0, y_0/2[$. Write $b(z) - b(x_0) = b(x + iy) - b(x + y) + b(x + y) - b(x_0)$. Using the continuity of b at x_0 along I, we have $b(x + y) - b(x_0) \to 0$, as $x \to x_0$ and $y \to 0$. Moreover, since b is analytic on $\mathbb{C}_+ \cup]x_0, x_0 + y_0[$, we can write

$$b(x + y) - b(x + iy) = (1 - i)y \int_0^1 b'(t(x + y) + (1 - t)(x + iy)) \, dt.$$

Applying Lemma 2.1, we get

$$|b(x+y) - b(x+iy)| \leq \sqrt{2} \int_x^{x+y} |b'(u)| \, du.$$

According to (5.6), we deduce that $b(x+y) - b(x+iy) \to 0$, as $x \to x_0$ and $y \to 0$. Therefore, $b(z) \to b(x_0)$, as $z \to x_0$, $z \in S(I)$. □

6. Carleson-type embedding theorems

Weighted Bernstein-type inequalities of the form (1.2) turned out to be an efficient tool for the study of the so-called Carleson-type embedding theorems for the shift-coinvariant subspaces K_Θ^p. More precisely, given an inner function Θ, we want to describe the class of Borel measure μ in the closed upper half-plane $\overline{\mathbb{C}_+}$ such that the embedding $K_\Theta^p \subset L^p(\mu)$ takes place. In other words, we are interested in the class of Borel measure μ in $\overline{\mathbb{C}_+}$ such that there is a constant C satisfying

$$\|f\|_{L^p(\mu)} \leq C\|f\|_p,$$

for all $f \in K_\Theta^p$. This problem was posed by Cohn in [11]. In spite of a number of beautiful results (see, e.g., [11, 12, 28, 42]), the question still remains open in the general case. Compactness of the embedding operator is also of interest and is considered in [10, 13, 41].

Methods based on the Bernstein-type inequalities allow to give unified proofs and essentially generalize almost all known results concerning these problems (see [6, 8]). Here we obtain an embedding theorem for de Branges–Rovnyak spaces. In the case of an inner function the first statement coincides with a well-known theorem due to Volberg and Treil [42].

A Carleson measure for the closed upper half-plane is called a *vanishing Carleson measure* if $\mu(S(x,h))/h \to 0$ whenever $h \to 0$ or $\operatorname{dist}(S(x,h), 0) \to \infty$. Vanishing Carleson measures in the closed unit disc are discussed, e.g., in [33]. An equivalent definition for a

vanishing Carleson measure ν in the disc is that

$$\int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \overline{z}\zeta|^2} d\nu(\zeta) \longrightarrow 0, \qquad \text{as } |z| \to 1.$$

Changing the variables to the upper half-plane with $|w + i|^{-2} d\mu(w) = d\nu(\zeta)$, we obtain

$$\int_{\mathbb{C}_+} \frac{\operatorname{Im} z}{|w - \overline{z}|^2} d\mu(w) \longrightarrow 0,$$

whenever either $\operatorname{Im} z \to 0$ or $|z| \to +\infty$. It is easily seen that this condition is equivalent to the above definition of a vanishing Carleson measure. It is well known that an embedding $H^p(\mathbb{C}_+) \subset L^p(\mu)$ is compact if and only if μ is a vanishing Carleson measure.

Theorem 6.1. *Let μ be a Borel measure in $\overline{\mathbb{C}_+}$, and let $\varepsilon \in (0, 1)$.*

(a) *Assume that $\mu(S(x, h)) \leq Kh$ for all Carleson squares $S(x, h)$ satisfying*

$$S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset.$$

   Then $\mathcal{H}(b) \subset L^2(\mu)$, that is, there is a constant $C > 0$ such that

$$\|f\|_{L^2(\mu)} \leq C\|f\|_b, \qquad f \in \mathcal{H}(b).$$

(b) *Assume that μ is a vanishing Carleson measure for $\mathcal{H}(b)$, that is, $\mu(S(x, h))/h \to 0$ whenever $S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$ and $h \to 0$ or $\operatorname{dist}(S(x, h), 0) \to +\infty$. Then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.*

In Theorem 6.1 we need to verify the Carleson condition only on a *special* subclass of squares. Geometrically this means that when we are far from the spectrum $\sigma(b)$, the measure μ in Theorem 6.1 can be essentially larger than standard Carleson measures. The reason is that functions in $\mathcal{H}(b)$ have much more regularity at the points $x \in \mathbb{R} \setminus \operatorname{Clos} \sigma(b)$ where $|b(x)| = 1$. On the other hand, if $|b(x)| \leq \delta < 1$, almost everywhere on some interval $I \subset \mathbb{R}$, then the functions in $\mathcal{H}(b)$ behave on I essentially the same as a general element of $H^2(\mathbb{C}_+)$ on that interval, and for any Carleson measure for $\mathcal{H}(b)$ its restriction to the square $S(I)$ is a standard Carleson measure.

We will see that, for a class of functions b, the sufficient condition of Theorem 6.1 is also necessary. However, it may be far from being necessary for certain functions b even in the model space setting.

By *a closed square* in $\overline{\mathbb{C}}_+$, we mean a set of the form

$$(6.1) \qquad S(x_0, y_0, h) := \{x + iy : x_0 \le x \le x_0 + h, \ y_0 \le y \le y_0 + h\},$$

where $x_0 \in \mathbb{R}$, $y_0 \ge 0$ and $h > 0$; by the *lower side* of the closed square $S(x_0, y_0, h)$ we mean the interval $\{x + iy_0 : x_0 \le x \le x_0 + h\}$.

We deduce Theorem 6.1 from the following more general result. Recall that

$$w_p(z) = w_{p,1}(z) = \min(\|(k_z^b)^2\|_q^{-p/(p+1)}, \|\rho^{1/q}\mathfrak{K}_{z,1}^\rho\|_q^{-p/(p+1)}).$$

Theorem 6.2. *Let $\{S_k\}_{k \ge 1}$ be a sequence of closed squares in $\overline{\mathbb{C}}_+$, let I_k denote the lower side of the square S_k, and let δ_{I_k} be the Lebesgue measure on I_k. Assume that the squares S_k satisfy the following two conditions:*

$$(6.2) \qquad \sum_k \delta_{I_k} \in \mathcal{C},$$

and, for some p, $1 < p < 2$,

$$(6.3) \qquad \sup_{k \ge 1, \, y \ge 0} |I_k| \int_{S_k \cap \{\mathrm{Im}\, z = y\}} w_p^{-2}(u)|du| < \infty.$$

Let μ be a Borel measure with $\mathrm{supp}\, \mu \subset \bigcup_k S_k$. Then

   (a) *if $\mu(S_k) \le C|I_k|$, then $\mathcal{H}(b) \subset L^2(\mu)$.*
   (b) *if, moreover, $I_k \cap \mathrm{Clos}\, \sigma(b) = \emptyset$, $k \ge 1$, and $\mu(S_k) = o(|I_k|)$, $k \to \infty$, then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.*

For the model subspaces a result, analogous to Theorem 6.2, was obtained in [6, Theorem 2.2]. For the sake of completeness, we include the proof.

Proof. (a) The idea of the proof is to replace the measure μ with some Carleson measure ν, and to estimate the difference between the norms $\|f\|_{L^2(\mu)}$ and $\|f\|_{L^2(\nu)}$ using the Bernstein-type inequality of Section 4.

It follows from Corollary 5.4 (b) that the set of functions $f \in \mathcal{H}(b)$ which are continuous on each of S_k is dense in $\mathcal{H}(b)$ (take the reproducing kernels k_z^b, $z \in \mathbb{C}^+$). Thus it is sufficient to prove the estimate $\|f\|_{L^2(\mu)} \leq C\|f\|_b$ only for $f \in \mathcal{H}(b)$ continuous on $\bigcup_k S_k$. Now let $f \in \mathcal{H}(b)$ be continuous on each of S_k. Then there exist $w_k \in S_k$ such that

$$(6.4) \qquad \|f\|_{L^2(\mu)}^2 \leq \sum_k |f(w_k)|^2 \mu(S_k) \leq \sup_k \frac{\mu(S_k)}{|I_k|} \cdot \sum_k |f(w_k)|^2 |I_k|.$$

Statement (a) will be proved as soon as we show that

$$(6.5) \qquad \sum_k |f(w_k)|^2 |I_k| \leq C\|f\|_b^2$$

where the constant C does not depend on f and on the choice of $w_k \in S_k$.

Consider the intervals $J_k = S_k \cap \{\operatorname{Im} z = \operatorname{Im} w_k\}$. Let $\nu = \sum_k \delta_{J_k}$. Then it follows from (6.2) that $\nu \in \mathcal{C}$ (and the Carleson constants C_ν of such measures ν are uniformly bounded). We have

$$(6.6) \qquad \left(\sum_k |f(w_k)|^2 |I_k| \right)^{1/2} \leq \|f\|_{L^2(\nu)} + \left(\sum_k \int_{J_k} |f(z) - f(w_k)|^2 |dz| \right)^{1/2},$$

and $\|f\|_{L^2(\nu)} \leq C_1 \|f\|_2 \leq C_1 \|f\|_b$.

We estimate the last term in (6.6). For $z \in J_k$ denote by $[z, w_k]$ the straight line interval with the endpoints z and w_k. Then $f(z) - f(w_k) = \int_{[z,w_k]} f'(u) du$ (in the case $J_k \subset \mathbb{R}$ note that, by Corollary 5.4 (a), any $f \in \mathcal{H}(b)$ is differentiable on J_k except, may be, at the endpoints). So, by the Cauchy–Schwartz inequality,

$$\sum_k \int_{J_k} |f(z) - f(w_k)|^2 |dz| \leq \sum_k \int_{J_k} \left| \int_{J_k} |f'(u)||du| \right|^2 |dz|$$

$$\leq \sum_k |J_k| \left(\int_{J_k} w_p^{-2}(u)|du| \right) \left(\int_{J_k} |f'(u)|^2 w_p^2(u)|du| \right).$$

By (6.3), we obtain

$$\sum_k \int_{J_k} |f(z) - f(w_k)|^2 |dz| \leq C_2 \sum_k \int_{J_k} |f'(u)|^2 w_p^2(u) |du|$$

$$= C_2 \|f' w_p\|_{L^2(\nu)}^2 \leq C_3 \|f\|_b^2,$$

where the last inequality follows from Theorem 4.1.

(b) For a Borel set $E \subset \overline{\mathbb{C}_+}$ define the operator $\mathcal{I}_E : \mathcal{H}(b) \to L^2(\mu)$ by $\mathcal{I}_E f = \chi_E f$ where χ_E is the characteristic function of E. For $N \in \mathbb{N}$ put $F_N = \bigcup_{k=1}^{N} S_k$ and $\widehat{F}_N = \overline{\mathbb{C}_+} \setminus F_N$. As above we assume that $f \in \mathcal{H}(b)$ is continuous on $\bigcup_k S_k$. Then it follows from (6.4) and (6.5) that

$$\int_{\widehat{F}_N} |f|^2 d\mu \leq C \sup_{k > N} \frac{\mu(S_k)}{|I_k|} \|f\|_b^2,$$

and so $\|\mathcal{I}_{\widehat{F}_N}\| \to 0$, $N \to \infty$. Statement (b) will be proved as soon as we show that $\mathcal{I}_{F_N}$ is a compact operator for any N (thus, our embedding operator $\mathcal{I}_{F_N} + \mathcal{I}_{\widehat{F}_N}$ may be approximated in the operator norm by compact operators $\mathcal{I}_{F_N}$). Clearly, it suffices to prove the compactness of $\mathcal{I}_{S_k}$ for each fixed k.

We approximate $\mathcal{I}_{S_k}$ by finite rank operators. For a given $\epsilon > 0$, partition the square S_k into finite union of squares $\{\tilde{S}_l\}_{l=1}^{L}$ with pairwise disjoint interiors so that

$$(6.7) \qquad \left(\int_{[\zeta, z]} w_p^{-2}(u) |du| \right) < \epsilon$$

for any l, $1 \leq l \leq L$, and any $\zeta, z \in \tilde{S}_l$. Such a partition exists since $I_k \cap \operatorname{Clos} \sigma(b) = \emptyset$, $k \geq 1$. Indeed, b is analytic in a neighborhood of S_k, and the norms involved in the definition of $w_p(z)$ are continuous on S_k.

Now fix $\zeta_l \in \tilde{S}_l$ and consider the finite rank operator $T : \mathcal{H}(b) \to L^2(\mu)$, $(Tf)(z) = \sum_{l=1}^{L} f(\zeta_l) \chi_{\tilde{S}_l}(z)$. We show that $\|\mathcal{I}_{S_k} - T\|^2 \leq C\epsilon$. As in the proof of (a), we have

$$\|(\mathcal{I}_{S_K} - T)f\|_{L^2(\mu)}^2 = \sum_{l=1}^{L} \int_{\tilde{S}_l} |f(z) - f(\zeta_l)|^2 d\mu(z)$$

$$\leq \sum_{l=1}^{L} \int_{\tilde{S}_l} \left(\int_{[\zeta_l,z]} |f'(u)|^2 w_p^2(u)|du| \right) \cdot \left(\int_{[\zeta_l,z]} w_p^{-2}(u)|du| \right) d\mu(z).$$

By Theorem 4.1,

$$\int_{[\zeta_l,z]} |f'(u)|^2 w_p^2(u)|du| \leq C_1\|f\|_b^2$$

where C_1 does not depend on $f \in \mathcal{H}(b)$, $1 \leq l \leq L$ and $z \in \tilde{S}_l$. Hence, by (6.7),

$$\|(\mathcal{I}_{S_K} - T)f\|_{L^2(\mu)}^2 \leq C_1\epsilon\|f\|_b^2 \sum_{l=1}^{L} \mu(\tilde{S}_l) = C_1\epsilon\mu(S_k)\|f\|_b^2.$$

We conclude that $\mathcal{I}_{S_K}$ may be approximated by finite rank operators and is, therefore, compact. $\qquad\square$

We comment now on a couple of details of the proof where the situation differs from the inner case.

Remark 6.3. In the inner case $b = \Theta$ one can prove the estimate $\|f\|_{L^2(\mu)} \leq C\|f\|_2$ for functions f in K_Θ^2 which are continuous on the closed upper half-plane $\overline{\mathbb{C}_+}$ and then use a result of Aleksandrov [3] which says that such functions are dense in K_Θ^2. We do not know if this result is still valid in $\mathcal{H}(b)$. To avoid this difficulty, in the proof of Theorem 6.2, we used the density in $\mathcal{H}(b)$ of the functions continuous on all squares S_k.

Question 6.4. Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$. Is it true that the set of functions f in $\mathcal{H}(b)$, continuous on $\overline{\mathbb{C}_+}$, is dense in $\mathcal{H}(b)$?

Remark 6.5. In the inner case, in Theorem 6.2, the assumption (6.3) can be replaced by the weaker assumption (only for the lower side of the square)

$$(6.8) \qquad \sup_{k \geq 1} |I_k| \int_{I_k} w_p^{-2}(u)|du| < \infty.$$

It was noticed in [6, Corollary 4.7] that in the inner case, for $q > 1$, there exists $C = C(q) > 0$ such that, for any $x \in \mathbb{R}$ and $0 \leq y_2 \leq y_1$, we have

$$(6.9) \qquad \|k_{x+iy_1}^b\|_q \leq C(q)\|k_{x+iy_2}^b\|_q.$$

Thus, it follows from (6.9) that if the sequence $\{S_k\}$ satisfies (6.8), then it also satisfies (6.3).

Question 6.6. Does the monotonicity property (6.9) of the norms of the reproducing kernels along the rays parallel to imaginary axis remains true for a general b? (It is true for $q = 2$, but this is not the interesting case for us.)

Proof. of Theorem 6.1. (a) Consider the open set $E = \mathbb{R} \setminus \mathrm{Clos}\,\widetilde{\Omega}(b, \varepsilon)$. If $E = \emptyset$, then μ is a Carleson measure and $\mathcal{H}(b) \subset H^2(\mathbb{C}_+) \subset L^2(\mu)$. So we may assume that $E \neq \emptyset$ and we can write it as a union of disjoint intervals Δ_l. Note that $\int_{\Delta_l} (\tilde{d}_\varepsilon(t))^{-1} dt = \infty$. Hence, partitioning the intervals Δ_l, we may represent E as a union of intervals I_k with mutually disjoint interiors such that

$$\int_{I_k} \left[\tilde{d}_\varepsilon(t)\right]^{-1} dt = \frac{1}{2}.$$

It follows that there exists $x_k \in I_k$ such that $\tilde{d}_\varepsilon(x_k) = 2|I_k|$. Hence, for any $x \in I_k$, $\tilde{d}_\varepsilon(x) \geq \tilde{d}_\varepsilon(x_k) - |I_k| = |I_k|$ and $\tilde{d}_\varepsilon(x) \leq 3|I_k|$. This implies

$$|I_k| \int_{I_k} \left[\tilde{d}_\varepsilon(t)\right]^{-2} dt \leq 1,$$

and using Lemma 5.2, we conclude that the intervals I_k satisfy (6.3). Condition (6.2) is obvious.

Let $S_k = S(I_k)$ be the Carleson square with the lower side I_k, let $F = \bigcup_k S_k$, and let $G = \overline{\mathbb{C}_+} \setminus F$. Put $\mu_1 = \mu|_F$ and $\mu_2 = \mu|_G$. We show that the measure μ_1 satisfies the conditions of Theorem 6.2 whereas μ_2 is a usual Carleson measure (and, thus, $\mathcal{H}(b) \subset H^2(\mathbb{C}_+) \subset L^2(\mu_2)$).

Let us show that $\mu_1(S_k) \leq C_2|I_k|$. Indeed, it follows from the estimate $|I_k| \leq \tilde{d}_\varepsilon(x) \leq 3|I_k|$, $x \in I_k$, that $S(6I_k) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$ (by $6I_k$ we denote the 6 times larger interval with the same center as I_k). By the hypothesis, $\mu_1(S_k) \leq \mu(S(6I_k)) \leq C|I_k|$. Hence, μ_1 satisfies the conditions of Theorem 6.2 (a), and so $\mathcal{H}(b) \subset L^2(\mu_1)$.

Now we show that $\mu_2 \in \mathcal{C}$. Assume that $S(I) \cap G \neq \emptyset$ for some interval $I \subset \mathbb{R}$, and let $z = x + iy \in S(I) \cap G$. If $x \in \operatorname{Clos} \widetilde{\Omega}(b, \varepsilon)$, then $S(2I) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$. Otherwise, if $x \in I_k$ for some k, then $\tilde{d}_\varepsilon(x) \leq 3|I_k| \leq 3|I|$ since $z \in S(I) \setminus S(I_k)$. Thus

$$(6.10) \qquad\qquad S(6I) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset.$$

By the hypothesis, $\mu_2(S(I)) \leq \mu(S(6I)) \leq C|I|$, and so μ_2 is a Carleson measure.

(b) Let F, G, μ_1 and μ_2 be the same as above. We show that μ_1 satisfies the conditions of Theorem 6.2 (b), whereas μ_2 is a vanishing Carleson measure. Indeed, we can split the family $\{S_k\}$ into two families $\{S_k\}_{k \in K_1}$ and $\{S_k\}_{k \in K_2}$ such that $|I_k| \to 0$, $k \to \infty$, $k \in K_1$, whereas $\operatorname{dist}(I_k, 0) \to \infty$ when $k \to \infty$, $k \in K_2$. Since $S(6I_k) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$ we conclude that Theorem 6.2 (b) applies to μ_1 and the embedding $\mathcal{H}(b) \subset L^2(\mu_1)$ is compact. Finally, any Carleson square $S(I)$ with $S(I) \cap G \neq \emptyset$ satisfies (6.10), and so, by the assumptions of Theorem 6.1 (b), μ_2 is a vanishing Carleson measure.

$\square$

We state an analogous result for the spaces in the unit disc (for the case of inner functions statement (b) is proved in [8]; it answers a question posed in [10]).

Theorem 6.7. *Let μ be a Borel measure in the closed unit disc $\overline{\mathbb{D}}$, and let $\varepsilon \in (0, 1)$.*

(a) *Assume that $\mu(S(x, h) \leq Ch$ for all Carleson squares $S(x, h)$ such that $S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$. Then $\mathcal{H}(b) \subset L^2(\mu)$.*

(b) *If, moreover, $\mu(S(x, h))/h \to 0$ when $h \to 0$ and $S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$, then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.*

For a class of functions b the converse to Theorem 6.1 is also true. We say that b satisfies the *connected level set condition* if the set $\Omega(b, \varepsilon)$ is connected for some $\varepsilon \in (0, 1)$. Our next result is analogous to certain results from [11] and to [42, Theorem 3].

Theorem 6.8. *Let b satisfy the connected level set condition for some $\varepsilon \in (0, 1)$. Assume that $\Omega(b, \varepsilon)$ is unbounded and $\sigma(b) \subset \operatorname{Clos} \Omega(b, \varepsilon)$. Let μ be a Borel measure on $\overline{\mathbb{C}_+}$. Then the following statements are equivalent:*

(a) $\mathcal{H}(b) \subset L^2(\mu)$.

(b) *There exists $C > 0$ such that $\mu(S(x, h)) \leq Ch$ for all Carleson squares $S(x, h)$ such that $S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$.*

(c) *There exists $C > 0$ such that*

$$(6.11) \qquad \int_{\overline{\mathbb{C}_+}} \frac{\operatorname{Im} z}{|\zeta - \overline{z}|^2} d\mu(\zeta) \leq \frac{C}{1 - |b(z)|}, \qquad z \in \mathbb{C}_+.$$

Proof. The implication (b) $\Longrightarrow$ (a) holds for any b by Theorem 6.1, and the implication (a) $\Longrightarrow$ (c) is trivial (apply the inequality $\|f\|_{L^2(\mu)} \leq C\|f\|_b$ to $f = k_z^b$). To prove that (c) $\Longrightarrow$ (b), we use an argument from [42]. Let $S(x, h)$ be a Carleson square such that $S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$. Since $\sigma(b) \subset \operatorname{Clos} \Omega(b, \varepsilon)$ it follows that $S(x, 2h) \cap \Omega(b, \varepsilon) \neq \emptyset$. Choose $z_1 \in S(x, 2h) \cap \mathbb{C}_+$ with $|b(z_1)| < \varepsilon$. Now consider $S(x, 3h)$. Since $\Omega(b, \varepsilon)$ is connected and unbounded, there exists a point z_2 on the boundary of $S(x, 3h)$ such that $|b(z_2)| < \varepsilon$. Hence, there exists a continuous curve γ connecting z_1 and z_2 and such that $|b| < \varepsilon$ on γ. Now let $z = x + ih$. Applying the theorem on two constants to the domain $\operatorname{Int} S(x, 3h) \setminus \gamma$ we conclude that $|b(z)| \leq \delta$ where $\delta \in (0, 1)$ depends only on ε. Then inequality (6.11) implies

$$h \int_{S(x,h)} \frac{d\mu(\zeta)}{|\zeta - \overline{z}|^2} \leq C(1 - \delta)^{-1}.$$

It remains to note that $|\zeta - \overline{z}| \leq C_1 h$, $\zeta \in S(x, h)$ to obtain $\mu(S(x, h)) \leq C_2 h$.

$\square$

Example 6.9. Examples are known of inner functions satisfying the connected level set condition. We would like to emphasize that there are also many outer functions satisfying the conditions of Theorem 6.8. For example, let $b(z) = \exp(\frac{i}{\pi} \log z)$, where $\log z$ is the main branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$.

Remark 6.10. We see that if b satisfies the conditions of Theorem 6.8, then it suffices to verify the inequality $\|f\|_{L^2(\mu)} \leq C\|f\|_b$ for the reproducing kernels of the space $\mathcal{H}(b)$ to get it for all functions f in $\mathcal{H}(b)$. Recently, Nazarov and Volberg [28] showed that it is no longer true in the general case.

7. Stability of bases of reproducing kernels

Another application of Bernstein inequalities for model subspaces K_Θ^p is considered in [7]; it is connected with stability of Riesz bases and frames of reproducing kernels $(k_{\lambda_n}^\Theta)$ under small perturbations of the points λ_n. Riesz bases of reproducing kernels in de Branges–Rovnyak spaces $\mathcal{H}(b)$ were studied in [19]. Making use of Theorem 4.1 we extend the results of [7] to the spaces $\mathcal{H}(b)$.

For $\lambda \in \mathbb{C}_+ \cup E_2(b)$, we denote by κ_λ^b the normalized reproducing kernel at the point λ, that is, $\kappa_\lambda^b = k_\lambda^b/(2\pi i \, \|k_\lambda^b\|_b)$. Let $(\kappa_{\lambda_n}^b)_{n \geq 1}$ be a Riesz basis in $\mathcal{H}(b)$, let $\lambda_n \in G_n$ and let $G = \bigcup_n G_n \subset \overline{\mathbb{C}_+}$ satisfy the following properties.

(i) There exist positive constants c and C such that

$$c \leq \frac{\|k_{z_n}^b\|_b}{\|k_{\lambda_n}^b\|_b} \leq C, \qquad z_n \in G_n.$$

(ii) For any $z_n \in G_n$, the measure $\nu = \sum_n \delta_{[\lambda_n, z_n]}$ is a Carleson measure and, moreover, the Carleson constants C_ν of such measures (see (4.1)) are uniformly bounded with respect to z_n. Here $[\lambda_n, z_n]$ is the straight line interval with the endpoints λ_n and z_n, and $\delta_{[\lambda_n, z_n]}$ is the Lebesgue measure on the interval.

Remark 7.1. As in the inner case, it should be noted that for $\lambda_n \in \mathbb{C}_+$, there always exist non-trivial sets G_n satisfying (i) and (ii). More precisely, we can take

$$G_n := \{z \in \mathbb{C}_+ : |z - \lambda_n| < r \operatorname{Im} \lambda_n\},$$

for sufficiently small $r > 0$. Indeed, we know [19] that if $(\kappa_{\lambda_n}^b)_{n \geq 1}$ is a Riesz basis in $\mathcal{H}(b)$, then $(\lambda_n)_{n \geqslant 1}$ is a Carleson sequence, that is,

$$\inf_{k \geqslant 1} \prod_{n \neq k} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \overline{\lambda}_k} \right| > 0.$$

In particular, the measure $\nu := \sum_n \operatorname{Im} \lambda_n \, \delta_{\lambda_n}$ is a Carleson measure. Therefore, we see that G_n satisfy (ii). Moreover, using Lemma 7.3 below, we see that G_n satisfy also the condition (i).

Recall that $w_p(z) = \min(\|(k_z^b)^2\|_q^{-p/(p+1)}, \|\rho^{1/q} \mathfrak{K}_{z,1}^\rho\|_q^{-p/(p+1)})$.

Theorem 7.2. *Let $(\lambda_n)_{n \geq 1} \subset \mathbb{C}_+ \cup E_2(b)$ be such that $(\kappa_{\lambda_n}^b)_{n \geq 1}$ is a Riesz basis in $\mathcal{H}(b)$ and let $p \in [1, 2)$. Then for any set $G = \bigcup_n G_n$ satisfying (i) and (ii), there is $\varepsilon > 0$ such that the system of reproducing kernels $(\kappa_{\mu_n}^b)_{n \geq 1}$ is a Riesz basis whenever $\mu_n \in G_n$ and*

$$(7.1) \qquad \sup_{n \geq 1} \frac{1}{\|k_{\lambda_n}^b\|_b^2} \int_{[\lambda_n, \mu_n]} w_p(z)^{-2} |dz| < \varepsilon.$$

Proof. Since $\mu_n \in G_n$, the condition (i) implies that $\|k_{\mu_n}^b\|_b \asymp \|k_{\lambda_n}^b\|_b$ and thus $(\kappa_{\mu_n}^b)_{n \geq 1}$ is a Riesz basis if and only if $(\widetilde{\kappa}_{\mu_n}^b)_{n \geq 1}$ is a Riesz basis where

$$\widetilde{\kappa}_{\mu_n}^b = \frac{k_{\mu_n}^b}{2\pi i \, \|k_{\lambda_n}^b\|_b}.$$

In view of [7, Lemma 2.3], it suffices to check the estimate

$$(7.2) \qquad \sum_{n=1}^{\infty} |\langle f, \kappa_{\lambda_n}^b - \widetilde{\kappa}_{\mu_n}^b \rangle_b|^2 \leq \varepsilon \|f\|_b^2, \qquad f \in \mathcal{H}(b),$$

for sufficiently small $\varepsilon > 0$. Now it follows from (7.1) and Corollary 5.4 (a) that any f in $\mathcal{H}(b)$ is differentiable in $]\lambda_n, \mu_n[$. Moreover, the set of functions in $\mathcal{H}(b)$ which are continuous on $[\lambda_n, \mu_n]$ is dense in $\mathcal{H}(b)$ (take the set of reproducing kernels). Therefore, we can prove (7.2) only for functions $f \in \mathcal{H}(b)$ continuous on $[\lambda_n, \mu_n]$. Then

$$|\langle f, \kappa_{\lambda_n}^b - \widetilde{\kappa}_{\mu_n}^b \rangle_b|^2 = \frac{|f(\lambda_n) - f(\mu_n)|^2}{\|k_{\lambda_n}^b\|_b^2} = \frac{1}{\|k_{\lambda_n}^b\|_b^2} \left| \int_{[\lambda_n, \mu_n]} f'(z) \, dz \right|^2.$$

By the Cauchy–Schwartz inequality and (7.1), we get

$$|\langle f, \kappa_{\lambda_n}^b - \widetilde{\kappa}_{\mu_n}^b \rangle_b|^2 \leq \varepsilon \int_{[\lambda_n, \mu_n]} |f'(z)w_p(z)|^2 |dz|.$$

It follows from assumption (ii) that $\nu := \sum_n \delta_{[\lambda_n, \mu_n]}$ is a Carleson measure with a constant C_ν which does not exceed some absolute constant depending only on G. Hence, according to Theorem 4.1, we have

$$\sum_{n=1}^{\infty} |\langle f, \kappa_{\lambda_n}^b - \widetilde{\kappa}_{\mu_n}^b \rangle_b|^2 \leq \varepsilon \sum_{n=1}^{\infty} \int_{[\lambda_n, \mu_n]} |f'(z)w_p(z)|^2 |dz|$$

$$= \varepsilon \|f'w_p\|_{L^2(\nu)}^2 \leq C \varepsilon \|f\|_b^2,$$

for a constant C which depends on G, (λ_n) and p. Then Lemma 2.3 of [7] implies that we can choose a sufficiently small $\varepsilon > 0$ such that $(\widetilde{\kappa}_{\mu_n}^b)_{n \geq 1}$ is a Riesz basis in $\mathcal{H}(b)$.

$\square$

Denote by $\rho(z, \omega)$ the pseudohyperbolic distance between z and ω,

$$\rho(z, \omega) := \left| \frac{z - \omega}{z - \overline{\omega}} \right|.$$

For the proof of the next corollary we need the following well-known property.

Lemma 7.3. *Let* $b \in H^\infty(\mathbb{C}_+)$ *with* $\|b\|_\infty \leq 1$ *and* $\varepsilon_0 \in (0, 1)$. *Then there exist constants* $C_1, C_2 > 0$ *(depending only on* ε_0*) such that for any* $z, \omega \in \mathbb{C}_+$ *satisfying* $\rho(z, \omega) < \varepsilon_0$, *we have*

(7.3)
$$C_1 \leq \frac{1 - |b(z)|}{1 - |b(\omega)|} \leq C_2.$$

Proof. For the case of an inner function, the proof can be found, e.g., in [7, Lemma 4.1]. Since for $0 \leq t_1, t_2, s_1, s_2 < 1$, we have

$$\frac{1 - t_1 t_2}{1 - s_1 s_2} \leq \frac{1 - t_1}{1 - s_1} + \frac{1 - t_2}{1 - s_2},$$

the inner and outer factors of b can be treated separately and we can assume that b is outer. It follows easily from $\rho(z, \omega) < \varepsilon_0$ that

$$(7.4) \qquad\qquad |z - \omega| < \frac{2\varepsilon_0}{1 - \varepsilon_0} \operatorname{Im} \omega$$

and

$$\frac{1 - \varepsilon_0}{1 + \varepsilon_0} < \frac{\operatorname{Im} z}{\operatorname{Im} \omega} < \frac{1 + \varepsilon_0}{1 - \varepsilon_0}.$$

Hence

$$\frac{\operatorname{Im} z}{\pi} \int_{\mathbb{R}} \frac{\big| \log |b(t)| \big|}{|t - z|^2} \, dt \asymp \frac{\operatorname{Im} \omega}{\pi} \int_{\mathbb{R}} \frac{\big| \log |b(t)| \big|}{|t - \omega|^2} \, dt.$$

Since b is outer, we have

$$(7.5) \qquad\qquad \log |b(z)| = -\frac{\operatorname{Im} z}{\pi} \int_{\mathbb{R}} \frac{\big| \log |b(t)| \big|}{|t - z|^2} \, dt \asymp \log |b(\omega)|,$$

which implies $1 - |b(z)| \asymp 1 - |b(\omega)|$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 7.4. *Let $(\lambda_n) \subset \mathbb{C}_+$, let $(\kappa_{\lambda_n}^b)_{n \geq 1}$ be a Riesz basis in $\mathcal{H}(b)$, and let $\gamma > 1/3$. Then there is $\varepsilon > 0$ such that the system $(\kappa_{\mu_n}^b)_{n \geq 1}$ is a Riesz basis whenever*

$$(7.6) \qquad\qquad \left| \frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}} \right| \leq \varepsilon (1 - |b(\lambda_n)|)^\gamma.$$

Proof. By Remark 7.1, for sufficiently small $r > 0$, the sets $G_n = \{z : |z - \lambda_n| \leq r \operatorname{Im} \lambda_n\}$ satisfy the conditions (i) and (ii). Let $(\mu_n)_{n \geq 1}$ satisfy (7.6). Then, by (7.4), we have

$$(7.7) \qquad\qquad |\lambda_n - \mu_n| \leq \frac{2\varepsilon}{1 - \varepsilon} (1 - |b(\lambda_n)|)^\gamma \operatorname{Im} \lambda_n.$$

Therefore, if ε is sufficiently small, then $\mu_n \in G_n$. Without loss of generality, we can assume that $\gamma < 1$ and since $\gamma > 1/3$, there exists $1 < p < 2$ such that $2\frac{p-1}{p+1} = 1 - \gamma$. Let q be the conjugate exponent of p and note that $\frac{2p}{q(p+1)} = 1 - \gamma$.

Then it follows from Lemma 3.5 that there is a constant $C = C(p) > 0$ such that

$$w_p(z) \geq C \frac{\operatorname{Im} z}{(1 - |b(z)|)^{\frac{p}{q(p+1)}}}, \qquad z \in \mathbb{C}_+.$$

Therefore, by Lemma 7.3, we have

$$w_p^{-2}(z) \leq C_1 \frac{(1 - |b(\lambda_n)|)^{1-\gamma}}{(\operatorname{Im} \lambda_n)^2}$$

for $z \in [\lambda_n, \mu_n]$. Hence,

$$\frac{1}{\|k_{\lambda_n}^b\|_b^2} \int_{[\lambda_n, \mu_n]} w_p(z)^{-2} |dz| \leq C_2 \frac{\operatorname{Im} \lambda_n}{1 - |b(\lambda_n)|} |\lambda_n - \mu_n| \frac{(1 - |b(\lambda_n)|)^{1-\gamma}}{(\operatorname{Im} \lambda_n)^2}$$

and using (7.7), we obtain

$$\frac{1}{\|k_{\lambda_n}^b\|_b^2} \int_{[\lambda_n, \mu_n]} w_p(z)^{-2} |dz| \leq C_3 \varepsilon.$$

To complete the proof, take a sufficiently small ε and apply Theorem 7.2.

$\square$

Remark 7.5. It should be noted that all the statements remain valid if we are interested in the stability of Riesz sequences of reproducing kernels, that is, of systems of reproducing kernels which constitute Riesz bases in their closed linear spans.

Remark 7.6. In the case where

(7.8)
$$\sup_{n \geq 1} |b(\lambda_n)| < 1,$$

the stability condition (7.6) is equivalent to

$$\left| \frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}} \right| \leq \varepsilon,$$

and we essentially get the result of stability obtained in the inner case in [18]. Moreover, if b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$ and if (7.8) is satisfied, then a criterion for $(\kappa_{\lambda_n}^n)$ to be a Riesz basis of $\mathcal{H}(b)$ is given in [19]. On the other hand, in the non-extreme case, there are no Riesz bases of $\mathcal{H}(b)$ and the previous results (Theorem 7.2 and Corollary 7.4) apply only for Riesz sequences.

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