Projections of Orbital Measures, Gelfand–Tsetlin Polytopes, and Splines

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Abstract. The unitary group $U(N)$ acts by conjugations on the space $\mathcal{H}(N)$ of $N \times N$ Hermitian matrices, and every orbit of this action carries a unique invariant probability measure called an orbital measure. Consider the projection of the space $\mathcal{H}(N)$ onto the real line assigning to an Hermitian matrix its $(1,1)$-entry. Under this projection, the density of the pushforward of a generic orbital measure is a spline function with $N$ knots. This fact was pointed out by Andrei Okounkov in 1996, and the goal of the paper is to propose a multidimensional generalization. Namely, it turns out that if instead of the $(1,1)$-entry we cut out the upper left matrix corner of arbitrary size $K \times K$, where $K = 2, \ldots, N - 1$, then the pushforward of a generic orbital measure is still computable: its density is given by a $K \times K$ determinant composed from one-dimensional splines. The result can also be reformulated in terms of projections of the Gelfand–Tsetlin polytopes.

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1. Introduction

Orbital measures. Let $\mathcal{H}(N)$ be the space of $N \times N$ Hermitian matrices. For $K = 1, \ldots, N - 1$, we denote by $p^K_N : \mathcal{H}(N) \to \mathcal{H}(K)$ the linear projection consisting in deleting from the matrix $H \in \mathcal{H}(N)$ its last $N - K$ rows and columns. We call $p^K_N(H)$, the image of $H$ under this projection, the $K \times K$ corner of $H$.

The unitary group $U(N)$ acts on $\mathcal{H}(N)$ by conjugations, and because $U(N)$ is compact, each orbit of this action carries a unique invariant probability measure, which we call the orbital measure. Given an orbital measure $\mu$ on $\mathcal{H}(N)$, denote by $p^K_N(\mu)$ its pushforward under projection $p^K_N$. Our goal is to describe $p^K_N(\mu)$.

The orbits in $\mathcal{H}(N)$ (and hence the orbital measures) can be indexed by $N$-tuples of weakly increasing real numbers $X = (x_1 \leq \cdots \leq x_N)$, the matrix

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eigenvalues. Let $\mathcal{X}(N) \subset \mathbb{R}^N$ denote the set of all such $X$’s. Given $X \in \mathcal{X}(N)$, we write $O_X$ and $\mu_X$ for the corresponding orbit and orbital measure, respectively.

Since $p^N_N(\mu_X)$ is a $U(K)$-invariant probability measure on $\mathcal{X}(K)$, it can be uniquely decomposed into a continual convex combination of orbital measures, governed by a probability measure $\nu_{X,K}$ on the parameter space $\mathcal{X}(K)$. That is, $\nu_{X,K}$ is characterized by the property that, for an arbitrary Borel subset $S \subseteq \mathcal{X}(N)$,

$$(p^N_K(\mu_X))(S) = \int_{Y \in \mathcal{X}(K)} \mu_Y(S)\nu_{X,K}(dY).$$

The measure $\nu_{X,K}$ can be called the radial part of measure $p^N_K(\mu_X)$.

**Main result.** Denote by $\mathcal{X}^0(N)$ the interior of $\mathcal{X}(N)$; that is, $\mathcal{X}^0(N)$ consists of $N$-tuples of strictly increasing real numbers. If $X \in \mathcal{X}^0(N)$, then $\nu_{X,K}$ is absolutely continuous with respect to Lebesgue measure on $\mathcal{X}(K) \subset \mathbb{R}^K$, and the main result, Theorem 3.3, gives an explicit formula for the density of $\nu_{X,K}$.

In the case $K = 1$ the target space of the projection is the real line, and the density in question coincides with a $B$-spline, a certain piecewise polynomial function on $\mathbb{R}$ (this fact was observed by Andrei Okounkov). In the general case, it turns out that the density of $\nu_{X,K}$ is expressed through a $K \times K$ determinant composed from some $B$-splines.

As the reader will see, the proof of Theorem 3.3 is straightforward and elementary. The main reason why I believe the result may be of interest is the very appearance of splines, which are objects of classical and numerical analysis, in a problem of representation-theoretic origin.

**Gelfand–Tsetlin polytopes.** Before explaining a connection with representation theory I want to give a different interpretation of the measure $\nu_{X,K}$.

For $X \in \mathcal{X}(N)$ and $Y \in \mathcal{X}(N-1)$, write $Y \prec X$ or $X \succ Y$ if the coordinates of $X$ and $Y$ *interlace*, that is

$$x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{N-1} \leq y_{N-1} \leq x_N.$$

Given $X \in \mathcal{X}(N)$, the corresponding *Gelfand–Tsetlin polytope* $P_X$ is the compact convex subset in the vector space

$$\mathbb{R}^{N-1} \times \mathbb{R}^{N-2} \times \cdots \times \mathbb{R} = \mathbb{R}^{N(N-1)/2},$$

formed by triangular arrays subject to the interlacement constraints:

$$P_X := \{(Y^{(N-1)}, \ldots, Y^{(1)}) \in R^{N(N-1)/2} : X \succ Y^{(N-1)} \succ \cdots \succ Y^{(1)}\}.$$

Consider the map assigning to a matrix $H \in O_X$ the array formed by the collections of eigenvalues of its corners $p_{N-1}^N(H), p_{N-2}^N(H), \ldots, p_1^N(H)$. It is well known (see Corollary 3.2 below) that this map projects the orbit $O_X$ onto the polytope $P_X$ and takes $\mu_X$ to the uniform measure on $P_X$ (that is, the normalized Lebesgue measure). Next, given $K = 1, \ldots, N-1$, consider the natural projection $P_X \rightarrow \mathcal{X}(K)$ extracting from the array $(Y^{(N-1)}, \ldots, Y^{(1)})$ its $K$th component $Y^{(K)}$. The measure $\nu_{X,K}$ is nothing else than the pushforward of the uniform measure under the latter projection.
Discrete version of the problem: relative dimension in Gelfand–Tsetlin graph. Let $\mathcal{GT}_N := \mathcal{X}(N) \cap \mathbb{Z}^N$ be the set of weakly increasing $N$-tuples of integers. The elements of $\mathcal{GT}_N$ are in bijection with the irreducible representations of the group $U(N)$: with $X = (x_1, \ldots, x_N) \in \mathcal{GT}_N$ we associate the irreducible representation $T_X$ with signature (=highest weight) $\hat{X} := (x_N, \ldots, x_1)$. Here we pass from $X$ to $\hat{X}$, because the coordinates of signatures are usually written in the descending order, see Weyl \[13\].

Let $X \in \mathcal{GT}_N$ and consider the finite set $P_X^{\mathbb{Z}} := P_X \cap \mathbb{Z}^{N(N-1)/2}$ consisting of integral points in the polytope $P_X$. Let us replace the uniform measure on $P_X$ by the uniform measure on $P_X^{\mathbb{Z}}$ (that is, the normalized counting measure). Next, given $K = 1, \ldots, N-1$, we consider again the same projection $P_X \to \mathcal{X}(K)$ as before and denote by $\nu_{X,K}^{\mathbb{Z}}$ the pushforward of the uniform measure on $P_X^{\mathbb{Z}}$. Evidently, $\nu_{X,K}^{\mathbb{Z}}$ is a probability measure with finite support.

Elements of $P_X^{\mathbb{Z}}$ are the Gelfand–Tsetlin schemes (also called Gelfand–Tsetlin patterns) with top row $X$; they parameterize the elements of Gelfand–Tsetlin basis in $T_X$. By the very definition of $\nu_{X,K}^{\mathbb{Z}}$, for $Y \in \mathcal{GT}_K$, the quantity $\nu_{X,K}^{\mathbb{Z}}(Y)$ (the mass assigning by $\nu_{X,K}^{\mathbb{Z}}$ to $Y$) equals the fraction of the schemes with the $K$th row equal to $Y$. This quantity is the same as the relative dimension of the isotypic component of $T_Y$ in the restriction of $T_X$ to the subgroup $U(K) \subset U(N)$.

The Gelfand–Tsetlin graph has the vertex set $\mathcal{GT}_1 \sqcup \mathcal{GT}_2 \sqcup \ldots$ and the edges formed by couples $Y \prec X$. In the terminology of Borodin–Olshanski \[3\], $\nu_{X,K}^{\mathbb{Z}}(Y)$ is the relative dimension of the vertex $Y \in \mathcal{GT}_K$ with respect to the vertex $X \in \mathcal{GT}_N$. In \[3\], we derived a determinantal formula for the relative dimension (see also Petrov \[11\] for a different proof). That formula can be viewed as a discrete version of the formula of Theorem 3.3.

I first guessed the formula of Theorem 3.3 by degenerating the “discrete” formula of \[3\]. However, this is not an optimal way of derivation, because the discrete case is much more difficult than the continuous one. I am grateful to Alexei Borodin for the suggestion to study the degeneration of the “discrete” formula. Note that from the comparison of the measures $\nu_{N,K}$ and $\nu_{X,K}^{\mathbb{Z}}$ it is seen that the former should be related to the latter by a scaling limit transition.

2. Preliminaries

The fundamental spline with $n \geq 2$ knots $y_1 < \cdots < y_n$ can be characterized as the only function $a \mapsto M(a; y_1, \ldots, y_n)$ on $\mathbb{R}$ of class $C^{n-3}$, vanishing outside the interval $(y_1, y_n)$, equal to a polynomial of degree $\leq n-2$ on each interval $(y_i, y_{i+1})$, and normalized by the condition

$$\int_{-\infty}^{+\infty} M(a; y_1, \ldots, y_n) da = 1.$$  

Here is an explicit expression:

$$M(a; y_1, \ldots, y_n) := (n-1) \sum_{\substack{i: y_i > a \geq \min \{r: y_i \neq y_r\}}} (y_i - a)^{n-2} \frac{\prod_{r: r \neq i} (y_i - y_r)}{\prod_{r: r \neq i} (y_i - y_r)}.$$  \hspace{1cm} (1)
In particular, for $n = 2$

$$M(a; y_1, y_2) = \frac{1_{y_1 < a < y_2}}{y_2 - y_1}.$$

**Remark 2.1.** The above definition is taken from Curry–Schoenberg [4]. In the subsequent publications, Schoenberg changed the term to $B$-spline. The latter term became commonly used. However, in the modern literature, it more often refers to the function

$$B(a; y_1, \ldots, y_n) := (y_n - y_1) \sum_{i: y_i > a} \left( \frac{(y_i - a)^{n-2}}{\prod_{r: r \neq i} (y_i - y_r)} \right),$$

which differs from $M(a; y_1, \ldots, y_n)$ by the numerical factor $(y_n - y_1)/(n - 1)$; see, e.g., de Boor [2] or Phillips [12]. The normalization in (2) has its own advantages, but we will not use it. Note also that $M(a; y_1, \ldots, y_n)$ is a special case of Peano kernel, see Davis [5], Faraut [7].

We need two well-known formulas relating $M(a; y_1, \ldots, y_n)$ to divided differences (see, e.g., [4], [7]).

Recall that the divided difference of a function $f(x)$ on points $y_1, \ldots, y_n$ is defined recursively by

$$f[y_1, y_2] = \frac{f(y_2) - f(y_1)}{y_2 - y_1}, \quad f[y_1, y_2, y_3] = \frac{f[y_2, y_3] - f[y_1, y_2]}{y_3 - y_1},$$

and so on; the final step is

$$f[y_1, \ldots, y_n] = \frac{f[y_2, \ldots, y_n] - f[y_1, \ldots, y_{n-1}]}{y_n - y_1}.$$  

(3)

Next, set

$$x^+_s = \begin{cases} x^s, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

In this notation, the first formula in question is

$$M(a; y_1, \ldots, y_n) = (n - 1)f[y_1, \ldots, y_n], \quad \text{where } f(x) := (x - a)^{n-2},$$

(4)

and the second formula is

$$f[y_1, \ldots, y_n] = \frac{1}{(n - 1)!} \int M(a; y_1, \ldots, y_n)f^{(n-1)}(a)da.$$  

(5)

In (5), $f$ is assumed being a function on $\mathbb{R}$ with piecewise continuous derivative of order $n - 1$. In particular, (5) is applicable to $f(x) = (x - t)^{n-1}$, which is used in the lemma below.

To shorten the notation, let us abbreviate $Y := (y_1 < \cdots < y_n)$. 

Lemma 2.2. Fix \( n = 2, 3, \ldots \) and an \( n \)-tuple \( Y = (y_1 < \cdots < y_n) \in \mathcal{X}^0(N) \). For an arbitrary \( b \in \mathbb{R} \) set
\[
f_b(x) := (x - b)^{n-1}, \quad x \in \mathbb{R}.
\]
One has
\[
\int_{-\infty}^{c} M(a; Y) da = 1 - f_c[Y], \quad c \in \mathbb{R},
\]
(6)
\[
\int_{b}^{c} M(a; Y) da = f_{b}[Y] - f_{c}[Y], \quad b < c,
\]
(7)
\[
\int_{b}^{+\infty} M(a; Y) da = f_b[Y], \quad b \in \mathbb{R}.
\]
(8)

Proof. To check (7), we apply (5) to \( f(x) = f_b(x) - f_c(x) \), which is justified, see the comment just after (5). Then in the left-hand side of (5) we get \( f_b[Y] - f_c[Y] \).

Next, observe that the \((n-1)\)th derivative of \( f_t(x) \) equals \((n-1)! x^{n-1-1} \) so that
\[
f^{(n-1)}(a) = (n-1)! (1_{a \geq b} - 1_{a \geq c}) = (n-1)! 1_{b \leq a < c}.
\]

Therefore, in the right-hand side we get \( \int_{c}^{b} M(a; Y) da \), which proves (7).

Now (8) follows from (7) by setting \( c = +\infty \), and (6) follows from (8), because the total integral of the \( M(a; Y) \) equals 1. \( \blacksquare \)

3. Projections of orbital measures

We keep to the notation of Sections 1 and 2.

Given \( X \in \mathcal{X}(N) \), the pushforward of the orbital measure \( \mu_X \) under the map
\[
O_X \ni H \mapsto \text{the spectrum of } p_{N-1}^{N}(H)
\]
can be viewed as a probability measure on \( \mathcal{X}(N-1) \) depending on \( X \) as a parameter; let us denote it by \( \Lambda_{N-1}^N(X, \cdot) \) or \( \Lambda_{N-1}^N(X, dY) \). We regard \( \Lambda(X, dY) \) as a Markov kernel.

By classical Rayleigh’s theorem, the eigenvalues of a matrix \( H \in \mathcal{X}(N) \) and its corner \( p_{N-1}^{N}(H) \) interlace. Therefore, the measure \( \Lambda_{N-1}^N(X, \cdot) \) is concentrated on the subset
\[
\{ Y \in \mathcal{X}(N-1) : Y \prec X \} \subset R^{N-1}.
\]
(9)

Proposition 3.1. Assume \( X = (x_1, \ldots, x_N) \in \mathcal{X}^0(N) \). Then the measure \( \Lambda_{N-1}^N(X, \cdot) \) is absolutely continuous with respect to Lebesgue measure on the set \( \{ Y \in \mathcal{X}(N-1) : Y \prec X \} \), and the density of \( \Lambda_{N-1}^N(X, \cdot) \), denoted by \( \Lambda_{N-1}^N(X, Y) \), is given by
\[
\Lambda_{N-1}^N(X, Y) = (N-1)! \frac{V(Y)}{V(X)} 1_{Y \prec X},
\]
(10)
where we use the notation
\[
V(X) = \prod_{j>i} (x_j - x_i)
\]
and the symbol $1_{Y \prec X}$ equals 1 or 0 depending on whether $Y \prec X$ or not.

**Proof.** To the best of my knowledge, a published proof first appeared in Baryshnikov [1, Proposition 4.2]. However, the argument given in [1] was known earlier: it is hidden in the first computation of the spherical functions of the groups $SL(N, \mathbb{C})$ due to Gelfand and Naimark, see [8, §9]. Note also that a more general result can be found in Neretin [9].

Here is a different proof. Consider the Laplace transform of the orbital measure $\mu_X$:

$$\hat{\mu}_X(Z) := \int e^{\text{Tr}(ZH)} \mu_X(dH),$$

where $Z$ is a complex $N \times N$ matrix. The integral in the right-hand side is often called the Harish-Chandra–Itzykson–Zuber integral. Its value is given by a well-known formula (see, e.g., Olshanski–Vershik [10, Corollary 5.2]):

$$\hat{\mu}_X(Z) = c_N \prod_{j>i} \left( z_j - z_i \right) \left( x_j - x_i \right),$$

where $z_1, \ldots, z_N$ are the eigenvalues of $Z$ and

$$c_N = (N-1)!(N-2)! \ldots 0!$$

(note that the right-hand side of (12) does not depend on the enumeration of the eigenvalues of $Z$).

The claim of the proposition is equivalent to the following equality: Assume that the entries in the last row and column of $Z$ equal 0, so that $Z$ has the form

$$Z = \begin{bmatrix} \tilde{Z} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{Z}$ is a complex matrix of size $(N-1) \times (N-1)$; then

$$\hat{\mu}_X(Z) = \frac{(N-1)!}{V(X)} \int_{Y \prec X} V(Y) \hat{\mu}_Y(\tilde{Z}) dY.$$  

(14)

To prove (14), consider the matrix $T := [e^{z_i x_j}]$ in the right-hand side of (12). Since $Z$ has the form (13), at least one of the eigenvalues $z_1, \ldots, z_N$ equals 0. It is convenient to slightly change the enumeration and denote the eigenvalues as $z_0 = 0, z_1, \ldots, z_{N-1}$. In accordance to this we will assume that the row number $i$ of $T$ ranges over $\{0, \ldots, N-1\}$ while the column index $j$ ranges over $\{1, \ldots, N\}$. Since $z_0 = 0$, the 0th row of $T$ is $(1, \ldots, 1)$. Let us subtract the $(N-1)$th column from the $N$th one, then subtract the $(N-2)$th column from the $(N-1)$th one, etc. This gives $\det T = \det \tilde{T}$, where $\tilde{T}$ stands for the matrix of order $N-1$ with the entries

$$\tilde{T}_{i,j} = e^{z_i x_{j+1}} - e^{z_i x_j} = z_i \int_{x_i}^{x_j} e^{z_i y} dy, \quad i, j = 1, \ldots, N-1.$$
It follows
\[ \det \tilde{T} = z_1 \ldots z_N \int_{Y < X} dY \det [e^{z_i y_j}]_{i,j=1}^{N-1}, \]
so that
\[ \hat{\mu}_X(Z) = c_N \frac{z_1 \ldots z_N \int_{Y < X} dY \det [e^{z_i y_j}]_{i,j=1}^{N-1}}{\prod_{N-1 \geq j > i \geq 0} (z_j - z_i) \cdot V(X)} . \]

Next, because \( z_0 = 0 \), the product over \( j > i \) in the denominator equals
\[ z_1 \ldots z_N \prod_{N-1 \geq j > i \geq 1} (z_j - z_i), \]
so that the product \( z_1 \ldots z_N \) is cancelled out. Taking into account the fact that \( \hat{\mu}_Y(\tilde{Z}) \) is given by the determinantal formula similar to (12) and using the obvious relation \( c_N = (N - 1)!c_{N-1} \) we finally get the desired equality (14).

From Proposition 3.1 it is easy to deduce the following corollary (see also [1, Proposition 4.7] and Defosseux [6]).

**Corollary 3.2.** Fix \( X \in \mathcal{Z}^0(N) \) and let \( H \) range over \( O_X \). The map assigning to \( H \) the collection of the eigenvalues of the corners \( p^N_K(H) \), where \( K = N - 1, N - 2, \ldots, 1 \), projects \( O_X \) onto the Gelfand–Tsetlin polytope \( P_X \) and takes the measure \( \mu_X \) to the Lebesgue measure multiplied by the constant
\[ \frac{(N - 1)! (N - 2)! \ldots 0!}{V(X)} . \]

In particular, the volume of \( P_X \) in the natural coordinates is equal to the inverse of the above quantity.

Recall that \( \nu_{X,K} \) stands for the radial part of the \( K \times K \) corner of the random matrix \( H \in O_X \), driven by the orbital measure \( \mu_X \) (see Section 1), and \( M(a; y_1, \ldots, y_n) \) denotes the fundamental spline with \( n \) knots \( y_1, \ldots, y_n \) (see [1] and (1)).

**Theorem 3.3.** Fix \( X = (x_1, \ldots, x_N) \in \mathcal{Z}^0(N) \). For any \( K = 1, \ldots, N-1 \), the measure \( \nu_{X,K} \) on \( \mathcal{Z}^K \) is absolutely continuous with respect to Lebesgue measure and has the density
\[ M(a_1, \ldots, a_K; x_1, \ldots, x_N) := c_{N,K} \frac{V(A) \det [M(a_j; x_i, \ldots, x_{N-K+i})]_{i,j=1}^{K}}{\prod_{(j,i) : j-i \geq N-K+1} (x_j - x_i)} , \tag{15} \]
where
\[ c_{N,K} = \prod_{i=1}^{K-1} \binom{N - K + i}{i} . \]
Note that for $K = 1$ the right-hand side reduces to the fundamental spline with knots $x_1, \ldots, x_N$. Thus, in the case $K = 1$ the theorem says that the density of the measure $\nu_{N,1}$ on $\mathbb{R}$ coincides with the spline $M(a; x_1, \ldots, x_N)$. This simple but important claim is due to Andrei Okounkov, see [10, Proposition 8.2].

**Proof.** We argue by induction on $K$, starting with $K = N - 1$ and ending at $K = 1$.

**Step 1.** Examine the case $K = N - 1$, which is the base of induction. We have $\nu_{X,N-1}(dA) = \Lambda_{N-1}^N(X, dA)$. By proposition 3.1 the measure $\Lambda_{N-1}^N(X, \cdot)$ on $\mathbb{R}^N$ is absolutely continuous with respect to Lebesgue measure and has density $\Lambda_{N-1}^N(X, A)$ given by [10]. Thus, we have to check that $\Lambda_{N-1}^N(X, A)$ coincides with the quantity $M(a_1, \ldots, a_{N-1}; x_1, \ldots, x_N)$ given by the right-hand side of (15), where we have to take $K = N - 1$. That is, the desired equality has the form

$$(N - 1)! V(A) \frac{V(X)}{V(\mathbb{R})} \mathbf{1}_{A < X} = c_{N,N-1} \frac{V(A) \det [M(a_j; x_i, x_{i+1})]_{i,j=1}^{N-1}}{\prod_{(j,i): j - i \geq 2} (x_j - x_i)}.$$

Since $c_{N,N-1} = (N - 1)!$, the desired equality reduces to

$$\det [M(a_j; x_i, x_{i+1})]_{i,j=1}^{N-1} = \mathbf{1}_{A < X} \frac{1}{(x_2 - x_1)(x_3 - x_2) \cdots (x_N - x_{N-1})}.$$

Observe that the $(i, j)$-entry in the determinant is the quantity

$$M(a_j; x_i, x_{i+1}) = \frac{1}{x_{i+1} - x_i},$$

which vanishes unless $a_j \in [x_i, x_{i+1}]$. Since $a_1 \leq \cdots \leq a_{N-1}$, the determinant vanishes unless $A < X$. Furthermore, if $A < X$, then the matrix under the sign of determinant is diagonal, so the determinant equals the product of the diagonal entries, which equals

$$\frac{1}{(x_2 - x_1)(x_3 - x_2) \cdots (x_N - x_{N-1})},$$

as required.

**Step 2.** Given $K = 1, \ldots, N - 1$, we consider the superposition of Markov kernels

$$\Lambda_K^N := \Lambda_{N-1}^N \Lambda_{N-2}^N \cdots \Lambda_{K+1}^N.$$

In more detail, the result is a Markov kernel on $\mathbb{R}^N \times \mathbb{R}^K$ given by

$$\Lambda_K^N(X, dA) = \int \Lambda_{N-1}^N(x, dY^{(N-1)}) \Lambda_{N-2}^N(Y^{(N-1)}, dY^{(N-2)}) \cdots \Lambda_{K+1}^N(Y^{(K+1)}, dA),$$

where the integral is taken over variables $Y^{(N-1)}, \ldots, Y^{(K+1)}$. Obviously, $\Lambda_K^N(X, dA) = \nu_{X,K}(dA)$, which entails the recurrence relation

$$\nu_{X,K-1} = \nu_{X,K} \Lambda_{K-1}^K, \quad K = N - 1, N - 2, \ldots, 2,$$

(16)
where, by definition, $\nu_{X,K} \Lambda_{K-1}^K$ is the measure on $\mathcal{X}(K-1)$ given by

$$
(\nu_{X,K} \Lambda_{K-1}^K)(dB) = \int_{A \in \mathcal{X}(K)} \nu_{X,K}(dA) \Lambda_{K-1}^K(A,dB). \quad (17)
$$

**Step 3.** Assume now that the claim of the theorem holds for some $K \geq 2$ and deduce from this that it also holds for $K-1$. To do this we employ (16) and (17).

First of all, (16) and (17) imply that $\nu_{X,K} \Lambda_{K-1}^K$ is absolutely continuous with respect to Lebesgue measure on $\mathcal{X}(K-1)$ and has the density

$$
(\nu_{X,K} \Lambda_{K-1}^K)(B) = \int_{A \in \mathcal{X}(K)} \nu_{X,K}(dA) \Lambda_{K-1}^K(A,B), \quad B \in \mathcal{X}(K-1). \quad (18)
$$

Let us compute the integral explicitly. By the induction assumption, $\nu_{X,K}$ is absolutely continuous and has density (15). Therefore, integral (18) can be written in the form

$$
\int_{A \in \mathcal{X}(0)(K)} M(a_1, \ldots, a_K; x_1, \ldots, x_N) \Lambda_{K-1}^K(A,B) da_1 \ldots da_K.
$$

Write $B = (b_1, \ldots, b_K)$. Substituting the explicit expression for $\Lambda_{K-1}^K(A,B)$ given by Proposition 3.1, we rewrite this as

$$
(K-1)! V(B) \int_A \frac{M(a_1, \ldots, a_K; x_1, \ldots, x_N)}{V(A)} da_1 \ldots da_K, \quad (19)
$$

where the integration domain is

$$
-\infty < a_1 \leq b_1, \ldots, b_i \leq a_{i+1} \leq b_{i+1}, \ldots, b_{K-1} \leq a_K < +\infty. \quad (20)
$$

Next, plug into (19) the explicit expression for $M(a_1, \ldots, a_K; x_1, \ldots, x_N)$ given by (15). Then the factor $V(A)$ is cancelled out and we get

$$
\frac{c_{N,K}(K-1)! V(B)}{\prod_{(j,i): j-i \geq N-K+1} (x_j - x_i)} \int_A \det [M(a_j; x_i, \ldots, x_{N-K+1})]_{i,j=1}^K da_1 \ldots da_K \quad (21)
$$

with the same integration domain (20).

Put aside the pre-integral factor in (21) and examine the integral itself. It can be written as a $K \times K$ determinant,

$$
\det[F(i,j)]_{i,j=1}^K,
$$

where

$$
F(i,j) := \int_{b_{j-1}}^{b_j} M(a; Y_i) da
$$

and

$$
Y_i := (x_i, \ldots, x_{N-K+i})
$$

with the understanding that $b_0 = -\infty$ and $b_K = +\infty$. 
We are going to prove that
\[
\det[F(i,j)]_{i,j=1}^{K} = (N - K + 1)^{K-1} \prod_{i=1}^{K-1} (x_{N-K+i+1} - x_i)
\]
\[
\times \det[M(b_j; x_i, \ldots, x_{N-K+i+1})]_{i,j=1}^{K-1}.
\]
This will justify the induction step, because
\[
c_{N,K} = c_{N,K-1} \cdot \frac{(N - K + 1)^{K-1}}{(K-1)!}
\]
and
\[
\prod_{i=1}^{K-1} (x_{N-K+i+1} - x_i) \prod_{(j,i): j-i \geq N-K+1} (x_j - x_i).
\]

**Step 4.** It remains to prove (22). We evaluate the quantities $F(i,j)$ using Lemma 2.2, where we substitute $n = N - K + 1$ and $Y = Y_i$. Then we get that the matrix entries $F(i,j)$ are given by the following formulas:

- The entries of the first column have the form $F(i,1) = 1 - f_{b_i}[Y_i]$ by (6).
- The entries of the $j$th column, $2 \leq j \leq K - 1$, have the form $F(i,j) = f_{b_{j-1}}[Y_i] - f_{b_j}[Y_i]$ by (7).
- The entries of the last column have the form $F(i,K) = f_{b_K}[Y_i]$ by (8).

We have $\det F = \det G$, where the $K \times K$ matrix $G$ is defined by
\[
G(i,j) := F(i,j) + \cdots + F(i,K).
\]
The entries of the matrix $G$ are
\[
G(i,1) = 1, \quad G(i,j) = f_{b_{j-1}}[Y_i], \quad 2 \leq j \leq K.
\]

Next, we get $\det G = \det H$ with the $(K - 1) \times (K - 1)$ matrix $H$ defined by
\[
H(i,j) := F(i + 1, j + 1) - F(i,j), \quad 1 \leq i, j \leq K - 1.
\]
Observe now that
\[
H(i,j) = f_{b_j}[Y_{i+1}] - f_{b_i}[Y_i],
\]
which can be rewritten as
\[
H(i,j) = (x_{N-K+i+1} - x_i) \frac{f_{b_j}[x_{i+1}, \ldots, x_{N-K+i+1}] - f_{b_i}[x_i, \ldots, x_{N-K+i+1}]}{x_{N-K+i+1} - x_i}
\]
\[
= (x_{N-K+i+1} - x_i) f_{b_j}[x_i, \ldots, x_{N-K+i+1}] \quad \text{by (3)}
\]
\[
= \frac{1}{N - K + 1} (x_{N-K+i+1} - x_i) M(b_j; x_i, \ldots, x_{N-K+i+1}) \quad \text{by (4).}
\]
This shows that the determinant $\det H = \det[H(i,j)]_{i,j=1}^{K-1}$ equals the right-hand side of (22). Since $\det H = \det F$, this completes the proof.

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