

To the blessed memory of Vitaly Lazarevich Ginzburg

Graphene as a Quantum Surface with Curvature-Strain Preserving Dynamics

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Abstract. We discuss how the curvature and the strain density of an atomic lattice generate the quantization of graphene sheets as well as the dynamics of geometric quasiparticles propagating along the constant curvature/strain levels. The internal kinetic momentum of a Riemannian oriented surface (a vector field preserving the Gaussian curvature and the area) is determined.

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1. INTRODUCTION

Graphene and other “one-atom thick” giant 2D-molecules materialize in a sense the mathematical notion of abstract surfaces [1]. However, to be geometrically stable, materialized surfaces of this kind, freely suspended in 3D Euclidean space, are to be curved and strained [2]–[7]. The curvature and strain generate some intrinsic fields which act on charge carriers similarly to magnetic and electric fields [8]–[12]. The pseudo-magnetic field arising in this way forces the trajectories of charge carriers to form cycles analogous to the Larmor ones. These cycle currents can be regarded as intrinsically generated “geometric” quasiparticles, whose size (de Broglie wavelength) correlates with the curvature radius and effective length of the lattice strain. Certainly, the size of geometric quasiparticles is smaller than the size of the geometrically stable area in which they live.

Quasiparticles of this kind can propagate as a whole along the surface in the absence of any external fields, just due to the inhomogeneity of the Riemann metric and strain. This dynamics, in the basic semiclassical approximation, preserves the state density as well as a curvature/strain symplectic form (Poisson brackets) on the surface [13]–[16]. At the quantum level, this leads to the occurrence of quantum structure making the surface coordinates noncommutative (like in the case of Landau–Peierls guiding center coordinates on the plane [17]–[19]). The surface area is to be treated as a quantum “phase space,” where the role of “Planck scale” is played by the inverse scale of the curvature or/and of the lattice strain density. Thus, *2D-materials attaining geometric stability become quantum surfaces.*

Note that, in general, the strength of the intrinsic pseudo-magnetic field in graphene is composed of two sources as

$$s \pm K/2, \tag{1.1}$$

where the signs \pm reflect the direction of the *pseudospin*, K stands for the Gaussian curvature of the graphene surface, and the function s can be referred to as the *strain density* of the atomic lattice on the surface. The strain density is determined by using a linear combination of the first derivatives of the strain tensor components in a specific coordinate system attached to the axes of the lattice [8].

The geometric quasiparticles on the graphene surface exist if and only if at least one of two magnitudes (1.1) is essentially nonzero. The bands on the surface on which both the strain density

s and the curvature K are zero or small are areas of quantum instability, where the carbon $2D$ -lattice is flat, not stretched, and therefore is not going to keep its surface geometry and may be transformed to another shape (tube, fullerene, or schwarzite) or just crumpling to somewhat not two-dimensional. These unstable bands can be treated as “articulations” joining geometrically stable and quantized pieces of the graphene surface.

Using this viewpoint, we have considered in [16] the charge carriers spectrum on graphene-like surfaces taking into account the strength of the external magnetic field or the internal strain and ignoring the curvature contribution. In the present note, we complete this consideration by including the graphene curvature. We especially look at regions at which the curvature contribution dominates over the strain, and obtain the dynamics of geometric quasiparticles preserving the curvature and the area of the surface. The generator of this flow is a vector field which can physically be treated as an *internal kinetic momentum of the Riemannian surface due to its curvature inhomogeneity*.

This classical picture of geometric quasiparticle dynamics in graphene sheets is essentially corrected by the quantum topological condition, à la Planck. The number of quantum states of the quasiparticle turns out to be proportional to the integral of the density (1.1) over the graphene area in question. In the case of small strain $s \approx 0$, as we prove below by using the Gauss–Bonnet theorem, the only way to obtain a sufficiently large number of quantum states, more than 1, is to assume that $K < 0$. Thus, one can conclude that *nonstrained graphene areas of positive Gaussian curvature repel geometrical quasiparticles; these objects can naturally live in areas of negative curvature or, alternatively, they need a strong enough strain of the atomic lattice*.

2. GRAPHENE ALGEBRA

The charge carriers in graphene at energies near the bottom of the conductivity zone mimic the Dirac fermions [20], [21]. The simplest version of quantum Hamiltonian is as follows (for details and generalizations, see, e.g., [6], [22], [23]):

$$\widehat{H} = v\gamma \cdot \widehat{p}, \quad v \simeq 10^8 \text{ cm/sec}, \quad (2.1)$$

where γ is a pseudospin and \widehat{p} is the kinetic momentum. In each local coordinate system $q = (q^1, q^2)$ on the graphene (orientable) surface, the following relations hold between components of γ and \widehat{p} :

$$[\gamma^j, \gamma^m]_+ = 2g^{jm}(q), \quad [\widehat{p}_j, \gamma^m] = i\hbar\Gamma_{ji}^m(q)\gamma^i, \quad (2.2)$$

and also relations involving coordinates:

$$[q^j, \widehat{p}_m] = i\hbar\delta_m^j, \quad [q^j, q^m] = 0, \quad [q^j, \gamma^m] = 0. \quad (2.3)$$

Here $[\cdot, \cdot]$ stands for the ordinary commutator and $[\cdot, \cdot]_+$ for the anticommutator.

The mutual relations between components of the kinetic momentum \widehat{p} are as follows [16]:

$$[\widehat{p}_j, \widehat{p}_m] = i\hbar^2(S_{jm}(q) + \frac{1}{4}R_{sljm}(q)\gamma^{sl}), \quad (2.4)$$

where

$$\gamma^{sl} \stackrel{\text{def}}{=} \frac{i}{2}[\gamma^s, \gamma^l]. \quad (2.5)$$

The tensor $g^{-1} = ((g^{jm}))$ in (2.2) represents the inverse metric on the surface. The tensor

$$R_{jmsl} = g_{jr}R_{msl}^r$$

in (2.4) is the curvature of the metric connection with the Christoffel symbols Γ_{jl}^m of (2.2):

$$R_{msl}^r = \partial_s \Gamma_{lm}^r - \partial_l \Gamma_{sm}^r + \Gamma_{sk}^r \Gamma_{lm}^k - \Gamma_{lk}^r \Gamma_{sm}^k, \tag{2.6}$$

$$\Gamma_{jl}^m = \frac{1}{2} g^{mr} (\partial_l g_{jr} + \partial_j g_{lr} - \partial_r g_{jl}). \tag{2.7}$$

One can also include the torsion terms occurring due to dislocations in the atomic lattice [24] into the connection coefficients Γ_{jl}^m .

The tensor $S = ((S_{jm}))$ in (2.4) is generated by the internal strain density s of the atomic lattice on the surface. More precisely, S has the form $S = sJ$, where the skew-symmetric covariantly constant tensor J is defined by the relation

$$J_{12} = \pm \sqrt{\det g},$$

where the sign (\pm) detects the consistency or inconsistency of the given local coordinate system with the chosen orientation of the surface.

Note that, for the Hamiltonian (2.1) to be self-adjoint, it is to be considered on the Hilbert space where all coordinate operators q^j are self-adjoint and the inner product is given by the Riemannian measure over the given surface. This follows from the second relation in (2.2) and from the identities

$$\partial_l \ln \sqrt{\det g} = \Gamma_{lj}^j,$$

which follow from (2.7).

Relations (2.2)–(2.4) generate an associative algebra indeed, since the Jacobi identities for double commutators and anticommutators hold. Namely, the Jacobi identity for the triple \hat{p}, \hat{p}, γ follows from definition (2.6) and from the property $R_{sljm} = R_{jmsl}$; the Jacobi identity for the triple \hat{p}, γ, γ follows because the connection (2.7) preserves the metric $g = ((g_{jm}))$; the Jacobi identity for the triple $\hat{p}, \hat{p}, \hat{p}$ follows from the closedness of the form

$$\mathfrak{S} = (1/2) S_{jm}(q) dq^j \wedge dq^m = \pm s \sqrt{\det g} dq^1 \wedge dq^2 \tag{2.8}$$

and from the second Bianchi identity (actually, in our two-dimensional case, it holds automatically).

Note that the commutators γ^{sl} (2.5) occurring in (2.4), together with γ^m , generate the “Lie algebra”

$$\begin{aligned} [\gamma^m, \gamma^{sl}] &= 2i(g^{ms}\gamma^l - g^{ml}\gamma^s), \\ [\gamma^{sl}, \gamma^{jm}] &= 2i(g^{lm}\gamma^{sj} + g^{ms}\gamma^{jl} + g^{sj}\gamma^{lm} + g^{jl}\gamma^{ms}). \end{aligned} \tag{2.9}$$

We also note that

$$(\gamma^{sl})^2 = \frac{1}{\det g}. \tag{2.10}$$

This follows from the anticommutation relations (2.2).

Our analysis of the algebra (2.2)–(2.4) is based, first of all, on the key relation (2.4). This shows that, near the bottom levels, we have

$$\hat{p} \sim \frac{\hbar}{l_*}, \tag{2.11}$$

where l_* stands for the characteristic scale of the strain-curvature field $S + \frac{1}{4}R\gamma$. In this case,

$$[q', \hat{p}'] \sim \frac{l_*}{l}, \tag{2.12}$$

by the first relation (2.3), where l stands for the characteristic scale of “large” inhomogeneity of the graphene sheet and $q' = q/l$ and $\hat{p}' = (l_*/\hbar)\hat{p}$ are the normalized coordinates and momenta, $q' \sim 1$ and $\hat{p}' \sim 1$. If the parameter $\varepsilon = l_*/l$ in (2.12) is small, then one can separate the “slow variables” q' (for which $[q', \hat{p}'] \sim \varepsilon$) from the “fast variables” γ, \hat{p}' (for which $[\gamma, \gamma]_+ \sim 1, [\hat{p}', \hat{p}'] \sim 1$) by using the standard adiabatic approximation.

The spectrum of the Hamiltonian (2.1) $\hat{H} = (\hbar v/l_*)\gamma \cdot \hat{p}'$ can be readily computed in the subalgebra of fast variables γ, \hat{p}' producing a series of pseudo-Landau levels, i.e., energies of different size circular currents. Each pseudo-Landau level, except for the zero one, is actually a function depending on the slow variables. These variables are not just q' , but are chosen from the additional condition that they commute with the fast variables up to ε^2 . This condition can be achieved only if the slow variables are admitted to be noncommutative (the way in which the “leading center” coordinates occurred). Finally, one obtains a series of Hamiltonians over the surface with nontrivial commutators between coordinates. These Hamiltonians determine quantum states and the classical dynamics of geometric quasiparticles on the graphene quantum surface, which could be very useful in the “strain electronics” [25].

3. STRAIN QUASIPARTICLES

Recall results of [16] (for details, see [13–15]) related to the case in which the strain dominates the curvature on the right-hand side of (2.4).

Theorem 3.1. *Assume that the tensor $S = ((S_{jm}))$ does not degenerate on the area in question of the graphene surface and dominates the curvature field. Then the following statements hold.*

(i) *The Hamiltonian (2.1) in the low-energy approximation is equivalent to the direct sum of the pseudo-Landau level Hamiltonians*

$$\mathcal{H}(\hat{Q}) = \pm \hbar v \sqrt{k \cdot 4\pi N(\hat{Q})}, \quad k = 0, 1, 2, \dots \quad (3.1)$$

where

$$N \stackrel{\text{def}}{=} (1/(2\pi)) \sqrt{|\det S|/|\det g|} = |s|/2\pi \quad (3.2)$$

on the quantum surface with nontrivial commutation relations between coordinates:

$$[\hat{Q}^j, \hat{Q}^m] = iS^{-1jm}(\hat{Q}) + \text{small corrections}. \quad (3.3)$$

The “small correction” summands in (3.3) are chosen to provide the correct behavior (invariance) after the change of local quantum coordinates Q^j in higher orders with respect to the semiclassical small parameter l_*/l , where l_* characterizes the scale of the strain density,

$$|s| \sim 1/l_*^2. \quad (3.4)$$

(ii) *For $k \neq 0$, the classical dynamics of quasiparticles on the surface generated by the k th Hamiltonian (3.1) reads*

$$\frac{dQ}{dt} = \{\mathcal{H}, Q\} \quad (3.5)$$

with respect to the Poisson brackets $\{\cdot, \cdot\}$ on the surface corresponding to relations (3.3), i.e.,

$$\{A, B\} = \hbar^{-1} S^{-1jm} \partial_m A \partial_j B.$$

The Hamilton-type system (3.5) reads as an equation of Maxwell–Lorentz type:

$$\text{curl } S = \pm \frac{4\pi}{v} j, \quad (3.6)$$

where the “current density” j is defined by

$$j \stackrel{\text{def}}{=} \sqrt{\frac{\pi}{k}} N(Q)^{3/2} \frac{dQ}{dt}, \tag{3.7}$$

and the circulation vector field

$$(\text{curl } S)^m \stackrel{\text{def}}{=} \nabla_j S^{mj}$$

is defined by the Levi-Civita covariant derivative ∇ applied to the tensor S whose indices are raised by means of the metric.

(iii) The function N (3.2) and the form \mathfrak{S} (2.8) are preserved by the flow generated by the dynamical system (3.6). The Planck-type discretization rule for the symplectic form (2.8), or, equivalently, the discretization rule for the integral strain

$$\pm \frac{1}{2\pi} \int_{\Sigma[N]} s \, d\sigma = n + \frac{1}{2} \quad \implies \quad N = N_n, \quad n = 0, 1, 2, \dots, \tag{3.8}$$

(where

$$\partial\Sigma[N] \stackrel{\text{def}}{=} \{N(Q) = N\}$$

and $d\sigma$ stands for the Riemannian measure on the surface), implies the semiclassical asymptotics of the near-zero eigenvalues of the Hamiltonian (2.1):

$$E_{k,n} \approx \pm \hbar v \sqrt{k \cdot 4\pi N_n}. \tag{3.9}$$

Note that, in view of (3.8), the function N (3.2) determines the state density¹ of quasiparticles on pseudo-Landau levels. The quasiparticles propagate along the curves $\{N(Q) = N_n\}$ surrounding the ares with the discrete flux (3.8) on the graphene surface. If the energy (3.9) is about the Fermi energy ε_F , the bound (3.4) for the strain density s correlates with the value of the Fermi momentum $p_F = \varepsilon_F/v$ and with the estimate for the kinetic momentum (2.11); the spatial quasiparticle size l_* then correlates with the Fermi wavelength $l_F = \hbar/p_F$.

4. CURVATURE QUASIPARTICLES

In contrast to the previous section, assume that the curvature contribution dominates the lattice strain contribution in (2.4). In such a situation, one can replace the commutation relation (2.4) of the graphene algebra by the relation

$$[\hat{p}_j, \hat{p}_m] = (i\hbar^2/4) R_{sljm} \gamma^{sl}.$$

The entire algebra (2.2), (2.3), (2.4a) is very interesting from the mathematical point of view, because it is *generated by the metric tensor exclusively*. Therefore, any consequences derivable from representation theory and spectral theory for this algebra and for the Hamiltonian (2.1) contain information on geometrical properties of Riemannian surfaces. For instance, the procedure of adiabatic separation of variables and of the reduction to the pseudo-Landau levels, briefly described at the end of Sec. 2, produces a curvature-preserving flow on the Riemannian surface which can physically be interpreted as the Hamiltonian flow of quasiparticles in graphene. Let us now go into details.

¹The actual density of states in graphene is equal to $4N$ because of two possible values of a pseudospin and two possible choices of a valley (corners of the Brillouin zone).

The principal difference of the purely “curvature case” from the purely “strain case” is the presence of generators γ^{sl} on the right-hand side of (2.4a). However, since we deal with a 2-dimensional surface, the only nonzero generator is $\gamma^{12} = -\gamma^{21}$, and (2.4a) reads

$$[\hat{p}_1, \hat{p}_2] = \frac{i\hbar^2}{2} R_{1212} \gamma^{12}.$$

It follows from (2.10) that the spectrum of γ^{12} consists of numbers $\pm 1/\sqrt{\det g}$ at each point of the surface. Thus, on the eigenspaces of γ^{12} , one obtains a scalar right-hand side in the relation for the momentum components,

$$[\hat{p}_1, \hat{p}_2] = \pm \frac{i\hbar^2}{2} R_{1212} / \sqrt{\det g} = \pm \frac{i\hbar^2}{2} K \sqrt{\det g},$$

where K stands for the Gaussian curvature of the surface,

$$K \stackrel{\text{def}}{=} \frac{1}{2} R_{sljm} g^{sj} g^{lm}.$$

Now we can just apply the results of [16] claimed in the previous section (Sec. 3) by choosing the value $\frac{1}{2}K$ there, instead of the strain density s . For instance, the function (3.2) is

$$N = \frac{|K|}{4\pi} \quad (4.1)$$

in this case. Thus, we obtain the following “curvature copy” of Theorem 3.1.

Theorem 4.1. *Let the surface be oriented. Assume that the Gaussian curvature K does not vanish on the area in question of the surface. Then the Hamiltonian (2.1) over the metric generated algebra (2.2), (2.3), (2.4a), in the low-energy approximation, is equivalent to the direct sum of the Hamiltonians*

$$\mathcal{H} = \pm \hbar v \sqrt{k|K|}(\hat{Q}), \quad k = 0, 1, 2, \dots \quad (4.2)$$

In (4.2), the noncommutative coordinates $\hat{Q} = (\hat{Q}^1, \hat{Q}^2)$ on the quantum surface obey the relation

$$[\hat{Q}^j, \hat{Q}^m] = \mp i \left(\frac{2}{K} J^{-1jm} \right) (\hat{Q}) + \text{small corrections}. \quad (4.3)$$

The pair of signs \mp on the right-hand side of (4.3) corresponds to the pair of opportunities to choose the direction of the pseudospin (i.e., some eigensubspace of the generator γ^{12}). The notion of “small corrections” was explained in Theorem 3.2 (a).

For $k \neq 0$, the classical dynamics of quasiparticles on the surface generated by the k th Hamiltonian (4.2) and by relations (4.3) is given by

$$m_* \frac{dQ}{dt} = \pm k \hbar (J^{-1} \partial \ln |K|)(Q), \quad (4.4)$$

where the effective mass m_* is determined by the relation $\mathcal{H} = m_* v^2$.

The flow generated by the dynamical system (4.4) preserves the Gaussian curvature K and the surface area $d\sigma$. The Planck-type discretization rule, corresponding to relations (4.3) or, equivalently, the discretization rule for the integral curvature

$$\pm \frac{1}{4\pi} \int_{\Sigma[N]} K d\sigma = n + \frac{1}{2} \quad \implies \quad N = N_n, \quad n = 0, 1, 2, \dots, \quad (4.5)$$

where

$$\partial\Sigma[N] \stackrel{\text{def}}{=} \{|K| = 4\pi N\},$$

implies the semiclassical asymptotics (3.9) for the near-zero eigenvalues of the Hamiltonian (2.1).

Note that, by (4.2), one can estimate the nonzero pseudo-Landau levels,

$$\mathcal{H} \sim \frac{\hbar v}{l_*} \sqrt{k},$$

where l_* stands for the effective curvature radius,

$$K \sim 1/l_*^2.$$

Comparing with

$$\varepsilon_F = \hbar v/l_F,$$

we see that the Fermi wavelength

$$l_F \sim l_*/\sqrt{k}$$

correlates with the curvature radius. We also see from (4.5) that the quasiparticles propagate along the curves surrounding areas with discrete values of the integral Gaussian curvature. This dynamics is controlled by the system (4.4) whose right-hand side $k\hbar J \partial \ln |K|$ can be referred to as the *internal kinetic momentum of the Riemannian surface* (on the k th pseudo-Landau level).

The dynamics of strain quasiparticles which is caused by the lattice stretch was described in (3.6) by using the electromagnetic terminology, including “current density” and “Maxwell–Lorentz equation”. In contrast to this language, we represent the dynamics of curvature quasiparticles in a “mechanical” form (4.4) by introducing the notion of effective mass

$$m_* = \frac{\hbar}{v} \sqrt{k|K|}.$$

However, it should be noted that equation (4.4) is not of Newton or Einstein type and, as it seems, has no direct analog in mechanics or general relativity. The flow generated by the internal momentum of the surface is due to the spinor framing (2.2), (2.3), (2.4a) originated from the metric field on the surface.

Remark 4.1. Note that, by the Gauss–Bonnet theorem, the integral curvature of any piece Σ of surface on which $K > 0$ is estimated as

$$\frac{1}{4\pi} \int_{\Sigma} K d\sigma \leq 1, \tag{4.6}$$

and equality in (4.7) is realized only for the closed sphere (fullerene) $\Sigma \sim \mathbb{S}^2$. Thus, the discretization rule (4.6) either fails to hold or holds only for the single index $n = 0$. This means that, on such pieces of the graphene surface, quantum states cannot exist at all or only one state exists (for each pseudo-Landau level $k = 1, 2, \dots$). We conclude that *the areas on the graphene sheet with $K > 0$ repel the states of curvature quasiparticles*².

However, on pieces with $K < 0$, the discretization rule (4.6) can hold for many values of n , and therefore many quantum states of the quasiparticles can exist in these areas. Hence the purely curvature quasiparticles, as quantum objects, naturally live in negatively curved graphene areas.

²This is in a good correspondence with the statement of [26] that the pentagon rings in the carbon lattice repel the charge density (recall that pentagons give the cases of positive curvature and heptagons those of negative curvature in graphene).

Except for trivial saddle surface or one-sheet hyperboloids (e.g., “worm-holes” [27]) one can mention “schwartzites” ([28], [29]) and the carbon foam [30] as interesting cases. For this type of surfaces Σ , the total integral curvature (in the compact case) is given by the Gauss–Bonnet relation,

$$\frac{1}{4\pi} \int_{\Sigma} K d\sigma = 1 - \mathfrak{g}(\Sigma),$$

where \mathfrak{g} stands for the topological genus. Some phenomena observed in topologically complicated graphene-type structures [31] probably can be related to the existence of currents of the curvature quasiparticles on negatively curved surfaces.

Remark 4.2. In the general situation, the pseudo-Landau level Hamiltonians in graphene sheets look as

$$\mathcal{H} = \pm \hbar v \sqrt{2k|s \pm \frac{1}{2}K|}, \quad k = 1, 2, \dots,$$

where the mixture of the strain density and the curvature controls the dynamics and the spectrum of geometric quasiparticles. The integral of the mixed magnitude $|s \pm \frac{1}{2}K|$, presented in the general discretization rule similar to (3.8) or (4.5), due to a large contribution of strain, can take values in a wide interval, and therefore many quantum states of geometric quasiparticles can exist even on positively curved graphene pieces (see, for instance, [32]). On the other hand, on negatively curved graphene sheets, the contribution of strain is small enough [33], and so the curvature effects probably dominate indeed.

5. “ZERO” QUASIPARTICLES

In the previous sections, we dealt with excited pseudo-Landau levels and the corresponding geometric quasiparticles in graphene. In the present section (the last one, and very short), we discuss a conjecture related to the zero-level quasiparticles.

An ordinary-type atomic lattices whose connectedness is determined by short-range electron orbitals (up to valence zone) cannot be two-dimensional geometrically in tens nanometers or more scale in view of the Landau–Peierls theorem [34]. That is why one could try to explain the large-scale geometric stabilization of the graphene surfaces by the existence of interlacing long electron orbitals staying not inside the ordinary molecular shells but just at the boundary between the valence zone and the conductivity zone (at the Dirac point). In other words, our conjecture is that the electron orbitals taking part in the geometric stabilization of freely suspended graphene could be just the geometric quasiparticles, i.e., the pseudo-Larmor circular currents, at the zero pseudo-Landau level with the index $k = 0$, in our notation.

Such “zero” quasiparticles are generated by the effective pseudo-magnetic fields, i.e., by the lattice strain and the curvature, whereas they are indifferent to the inhomogeneity of these fields. Moreover, the existence of zero modes for the Dirac operator (2.1) is indifferent to relative magnitude of the strain-curvature radius (de Broglie wave length) $l_* \sim (|s| + |K|/2)^{-1/2}$. In the semiclassical regime, the dynamics like (3.6), (4.4) for the “zero” quasiparticles is

$$\frac{dQ}{dt} = 0.$$

Thus, in average, the “zero” quasiparticles generate no actual current, as is to be for molecular orbitals.

If one agrees that a part of the energy of “zero” quasiparticles is spent to the geometric stabilization of the lattice, then it could explain the anomaly small width of the zero pseudo-Landau level in free graphene [35] (unusually small values of the Stoner parameter).

Further, let the graphene sheet be placed into a weak electric field \mathcal{E} , of several volts. Then the general dynamics of geometric quasiparticles like (3.6), (4.4) admits an additional pseudo-Hall drift directed perpendicularly to \mathcal{E} . This pseudo-Hall current interferes and competes with the geometric current considered in Secs. 3 and 4 (see [16]). However, specifically, for the “zero” quasiparticles which have no geometric current component, the only contribution to dynamics is given by the pseudo-Hall component,

$$\frac{dQ}{dt} = \frac{e}{\hbar} \cdot \frac{1}{s \pm K/2} J^{-1} \mathcal{E}.$$

The pseudo-Hall dynamics is generated by the lattice strain, the surface curvature, and the electric field $\mathcal{E} = \nabla \mathcal{U}$. This semiclassical dynamics preserves the strain-curvature area $|s \pm K/2|d\sigma$ on the graphene surface, as well as the electric potential \mathcal{U} , i.e., the current lines of “zero” quasiparticles on the graphene sheet placed into electric field are equipotential.

If one accepts that a part of the “zero” quasiparticles energy is spent to the geometric stabilization of the graphene lattice, then the pseudo-Hall current actually has to be less than that pointed above. This reduction agrees with the fact that, on the zero pseudo-Landau level, the Hall resistance is observed to have an anomalous behavior [36] under a weak electric field.

During the Hall drift, the interaction thus conjectured between “zero” quasiparticles and the lattice would imply the deformation of the lattice in the direction perpendicular to the electric field. This deformation changes the lattice strain. As a result, by increasing the electric field, one pushes the system to jump to the next (nonzero) pseudo-Landau level, and thus to change its pseudo-Hall resistance sharply (as in [36]).

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