Characteristic Relations for Quantum Matrices

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Abstract

General algebraic properties of the algebras of vector fields over quantum linear groups $GL_q(N)$ and $SL_q(N)$ are studied. These quantum algebras appears to be quite similar to the classical matrix algebra. In particular, quantum analogues of the characteristic polynomial and characteristic identity are obtained for them. The q-analogues of the Newton relations connecting two different generating sets of central elements of these algebras (the determinant-like and the trace-like ones) are derived. This allows one to express the q-determinant of quantized vector fields in terms of their q-traces.

1 Introduction

Since the time of their discovery quantum groups were presented in several closely related but not strictly equivalent forms. Being originally obtained as the quantized universal enveloping (QUE) algebras [1, 2] they were then reformulated in a matrix form [3]. In this latter approach a quantum group is generated by a pair of upper- and lower-triangular matrix generators L_+ and L_- satisfying quadratic permutation relations. A further variation of this approach is to combine L_+ and L_- into a single matrix generator $L = S(L_-)L_+$. Here $S(\cdot)$ is the usual notation for the antipodal mapping. Following Ref.[4] we will call the algebra generated by the matrix generator L as the reflection equation algebra (REA). After suitable completing this algebra can be related to the quantum group by the (Hopf algebra) isomorphism [5, 6], although the Hopf structure is implicit in REA formulation. This algebra have found several applications (see [4, 7, 8, 9] and references therein). Let us mention here only one of them. Namely, in the construction of the quantum group differential calculus the matrix generators L are used as the basic set of (right-)invariant vector fields [10, 11, 12].

A remarkable property of the REA formulation is that the algebra of the L matrices in several aspects appears to be quite similar to the classical matrix algebra. In particular, both the notions of the matrix trace and that of the matrix determinant admit the generalization to the case of L matrices (see [3, 13] and [6, 14, 15]). In the present paper we intend to establish further similarities of the REA with the classical matrix algebra. We restrict ourselves to considering the REA of the $GL_q(N)$ and/or $SL_q(N)$ type only. For these algebras the recurrent formulas relating two different center generating sets, the determinant-like and the trace-like ones, are obtained. These formulas are the quantum analogues of the classical Newton relations. Further we define the characteristic polynomial and derive the characteristic identities (the analogue of the Cayley-Hamilton theorem) for the L-matrices. Existence of such identities were first mentioned in Ref. [4] where they were pre-

sented for the case $N = 2$. For general N the characteristic identities were obtained in Ref.[16] (see Remark 4.8 of this paper) when studying the algebraic structure of quantum Yangians $Y_q(gl(N))$. We reproduce this result in REA approach. Then, by the joint use of the quantum Newton relations and the characteristic identities one can obtain the expressions for trace-like central elements of higher powers. In the QUE presentation the similar characteristic identities were considered in Ref.[17]. We believe that the REA presentation and the use of R-matrix technique makes all considerations and the final formulas much clearer.

We conclude this Section with a brief remainder on some facts from the classical theory of matrices (see e.g. [18]) which then will be generalized to the quantum case.

Consider $N \times N$ matrix A with complex entries. Its characteristic polynomial is defined as

$$\triangle \, (\, x \,) \equiv \det || \, x \, \mathbf{1} - A \, || \equiv x^N + \sum_{k=1}^{N} (-1)^k \sigma(k) \, x^{N-k} \, . \tag{1.1}$$

The eigenvalues $\{\lambda_i\}$, $i = 1, \ldots, N$, of the matrix A are solutions of the characteristic equation $\triangle(x) = 0$. The coefficients $\sigma(i)$ of the characteristic polynomial if expressed in terms of λ_i form the set of basic symmetric polynomials of N variables:

$$\sigma(1) \equiv \sum_{i=1}^{N} \lambda_i = Tr \, A \, , \quad \sigma(2) \equiv \sum_{i<j} \lambda_i \lambda_j \, , \, \ldots \, , \quad \sigma(N) \equiv \prod_{i=1}^{N} \lambda_i = \det A \, . \tag{1.2}$$

Also one can directly express $\sigma(i)$ in terms of the matrix elements of A. Up to some numerical factor each $\sigma(i)$ is given by the sum of all the principal minors of the i-th order

$$\sigma(i) = \frac{1}{i! \, (N-i)!} \, \epsilon^{1 \ldots N} A_1 \ldots A_i \, \epsilon^{1 \ldots N} \, . \tag{1.3}$$

Here $\epsilon^{1 \ldots N}$ is the antisymmetric Levi-Civita N-tensor. The compressed matrix conventions used in this formula will be explained later (directly in the quantum case).

Another standard set of symmetric polynomials is given by the traces of powers of the matrix A

$$s(i) \equiv \sum_{k=1}^{N} (\lambda_k)^i = Tr \, A^i \, , \quad 1 \leq i \leq N \, . \tag{1.4}$$

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The two basic sets $\{\sigma(i)\}$ and $\{s(i)\}$ are connected by the so-called Newton relations

$$i\,\sigma(i) - s(1)\,\sigma(i-1) + \ldots + (-1)^{i-1}s(i-1)\,\sigma(1) + (-1)^i s(i) = 0. \qquad (1.5)$$

In particular these recurrent relations allow one to express the determinant of the matrix A as a polynomial of its powers traces.

Finally, if one substitutes the matrix A into the characteristic polynomial (1.1) instead of the scalar variable x then the resulting matrix expression vanishes identically. This is the Cayley-Hamilton theorem, and according to it any function of matrix A can be reduced to a polynomial of an order not exceeding $N - 1$.

2 Quantum Newton Relations and Characteristic Polynomial

First of all let us introduce some definitions and notation to be used in what follows. The REA is defined as the algebra generated by matrix generators L subject to the following permutation rules

$$L_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12} \;=\; \hat{R}_{12}\,L_1\,\hat{R}_{12}\,L_1\;. \qquad (2.1)$$

Here the standard conventions for denoting matrix spaces (see [3]) are used. $\hat{R}_{12}$ is the $GL_q(N)$ R-matrix [2] satisfying the Yang-Baxter equation and the Hecke condition respectively

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} \;=\; \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}\;, \qquad (2.2)$$

$$\hat{R}^2 \;=\; \mathbf{1} + \lambda\hat{R}\;, \qquad (2.3)$$

where $\mathbf{1}$ is the unit matrix, and $\lambda = q - 1/q$. Below we will further compress this notations denoting $\hat{R}_{i(i+1)} \equiv \hat{R}_i$ and omitting the index of L-matrix: $L_1 \equiv L$, since it always appears in the first matrix space.

We will also need the notions of the quantum trace and the q-deformed Levi-Civita tensor. The operation Tr_q of taking the quantum trace of $N \times N$ quantum

matrix X looks like

$$Tr_q(X) = Tr(\mathcal{D}X) \quad , \quad \mathcal{D} = diag\{q^{-N+1}, q^{-N+3}, \ldots, q^{N-1}\} . \qquad (2.4)$$

The q-deformed Levi-Civita tensor $\epsilon_q^{i_1 \ldots i_N}$ (or $\epsilon_q^{1 \ldots N}$ in brief notation) is defined, up to a factor by its characteristic property

$$\left(\hat{R}_i + \frac{1}{q} \right) \epsilon_q^{1 \ldots N} = 0 , \quad 1 \leq i \leq N - 1 . \qquad (2.5)$$

The normalization is usually fixed by demanding $\epsilon_q^{i_1 \ldots i_N} = 1$ for $i_1 = 1, \ldots, i_N = N$. Its square is then equal to

$$|\epsilon_q|^2 \equiv \epsilon_q^{1 \ldots N} \epsilon_q^{1 \ldots N} = q^{N(N-1)/2} N_q!,$$

where $p_q = (q^p - q^{-p})/\lambda$ are the usual q-numbers.

In ref. [3] the two generating sets for the center of the REA were presented. One of them is formed by the trace-like elements

$$s_q(i) = q^{1-N} Tr_q L^i \quad 1 \leq i \leq N , \qquad (2.6)$$

where the normalizing factor is chosen for future convenience. Another generating set consists of the determinant-like elements $\sim \epsilon_q^{1 \ldots N} L_{-N} \ldots L_{-(i+1)} L_{+i}, \ldots L_{+1} \epsilon_q^{1 \ldots N}$. For our purposes it is better to express these generators in terms of the L-matrices:

$$\sigma_q(i) = \alpha_i \epsilon_q^{1 \ldots N} (L_1 \hat{R}_1 \ldots \hat{R}_{i-1})^i \epsilon_q^{1 \ldots N}. \qquad (2.7)$$

Here α_i are some normalizing constants. First of them is fixed to be

$$\alpha_1 = q^{1-N} N_q / |\epsilon_q|^2$$

by the natural condition $\sigma_q(1) = s_q(1)$. The other ones will be specified below.

The connection between the two basic sets $\{\sigma_q(k)\}$ and $\{s_q(k)\}$ is provided by quantum analogues of the Newton relations (1.5). At the classical level their derivation is usually based on using the spectral presentations (1.2), (1.4) for $\sigma(k)$ and

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$s(k)$. However this presentation is not available in quantum case. Indeed, the spectrum of the quantum matrix L can be constructed only if the center of REA is algebraically closed. The latter in turn can be treated only on passing to a concrete representation of REA. In order to by-pass this difficulty we shall develop some more technique.

Define an operator S_N which symmetrizes any $N \times N$ matrix X in N matrix spaces:

$$S_N(X) = X_1 + \hat{R}_1 X_1 \hat{R}_1 + \ldots + \hat{R}_{N-1} \ldots \hat{R}_1 X_1 \hat{R}_1 \ldots \hat{R}_{N-1}. \qquad (2.8)$$

The characteristic properties of this symmetrizer

$$\left[S_N(X), \, \hat{R}_i \right] = 0 \,, \quad 1 \le i \le N - 1 \,, \qquad (2.9)$$

are fulfilled due to the relations (2.2), (2.3). Further, the following useful formula

$$\epsilon_q^{1 \ldots N} S_N(X) = s_X \epsilon_q^{1 \ldots N} \qquad (2.10)$$

is a direct consequence of (2.9) and (2.5). Here the scalar factor s_X is

$$s_X = \frac{1}{|\epsilon_q|^2} \epsilon_q^{1 \ldots N} S_N(X) \epsilon_q^{1 \ldots N} = q^{1-N} Tr_q X.$$

For $X = L^i$ this factor coincides with the $s_q(i)$ (2.6). Now we are able to prove

Proposition. *For $1 \le i \le N$ the generators $\sigma_q(i)$ and $s_q(i)$ are connected by the relations*

$$\frac{i_q}{q^{i-1}} \sigma_q(i) - s_q(1) \, \sigma_q(i-1) + \ldots + (-1)^{i-1} s_q(i-1) \, \sigma_q(1) + (-1)^i s_q(i) = 0 \,, \quad (2.11)$$

provided that the numerical factors α_i are fixed as follows

$$\alpha_i = \frac{N_q!}{(N-i)_q! \, i_q!} \frac{q^{-i(N-i)}}{|\epsilon_q|^2} \,. \qquad (2.12)$$

Proof. Consider the quantities $s_q(i-p)\sigma_q(p)$ for $1 \le p \le i - 1$. With the help of (2.10) and the definitions of $s_q(i)$ and $\sigma_q(i)$ one can perform the following transformations:

$$s_q(i-1)\sigma_q(1) \;\; = \;\; \alpha_1 \underline{s_q(i-1)\epsilon_q^{1 \ldots N} L \, \epsilon_q^{1 \ldots N}} = \alpha_1 \epsilon_q^{1 \ldots N} S_N(L^{i-1}) L \, \epsilon_q^{1 \ldots N}$$

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$$= s_q(i) + \alpha_1 \frac{(N-1)_q}{q^{N-2}} \epsilon_q^{1...N} L^{i-1} R_1 L R_1 \epsilon_q^{1...N} \ ;$$

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$$
\begin{aligned}
s_q(i-p)\sigma_q(p) \ = \ & \alpha_p \frac{p_q}{q^{p-1}} \epsilon_q^{1...N} (L^{i-p+1} R_1 \ldots R_{p-1})(L R_1 \ldots R_{p-1})^{p-1} \epsilon_q^{1...N} \\
+ \ & \alpha_p \frac{(N-p)_q}{q^{N-p-1}} \epsilon_q^{1...N} (L^{i-p} R_1 \ldots R_p)(L R_1 \ldots R_p)^p \epsilon_q^{1...N} \ ;
\end{aligned}
$$

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\begin{aligned}
s_q(1)\sigma_q(i-1) \ = \ & \alpha_{i-1} \frac{(i-1)_q}{q^{i-2}} \epsilon_q^{1...N} (L^2 R_1 \ldots R_{i-2})(L R_1 \ldots R_{i-2})^{i-2} \epsilon_q^{1...N} \\
+ \ & \alpha_{i-1} \frac{(N-i+1)_q}{q^{N-i}} \frac{\sigma_q(i)}{\alpha_i} \ .
\end{aligned}
$$

Now the arbitrary coefficients α_p should be fixed in such a way that the last term in $s_q(i-p+1)\sigma_q(p-1)$ and the first one in $s_q(i-p)\sigma_q(p)$ be equal. This is the case if α_p satisfy the relations

$$\alpha_p = q^{2p-1-N} \frac{(N-p+1)_q}{p_q} \alpha_{p-1} \ . \tag{2.13}$$

Then, on taking the alternating sum $\sum_p (-1)^{p-1} s_q(i-p)\sigma_q(p)$ we find that the only terms which survive are the first one in $s_q(i-1)\sigma_q(1)$ and the last one in $s_q(1)\sigma_q(i-1)$ and, thus, we obtain relations (2.11). Finally, given the value of α_1 one easily shows that recursion (2.13) is solved by (2.12). ∎

Few remarks are in order here:

1. Up to now we always consider the $GL_q(N)$ matrices. Specification to the $SL_q(N)$ case can be achieved by fixing quantum determinant of the L-matrix (see [6, 14, 15]): $Det \, L = q^{1-N} \sigma_q(N) = 1$.

2. It is worth mentioning that the singular points where the connection between $\{s_q(i)\}$ and $\{\sigma_q(i)\}$ breaks down are the roots of unity: $k_q = 0$ for $1 \le k \le N$. This is apparently related to the fact that the isomorphism of the Hecke algebra of A_{N-1} type and the group algebra of symmetric group $\mathbf{C}S_N$ is also destroyed at these points (see e.g. [19]).

Now let us turn to deriving the quantum characteristic identity for the matrix L. It can be found in a way quite similar to that of the classical case. Namely, we

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should find a matrix polynomial B of $(N-1)$-th order of L obeying the relation

$$(L - x\mathbf{1}) B(L,x) \, \epsilon_q^{1...N} = \epsilon_q^{1...N} \triangle (x) \,. \qquad (2.14)$$

Here x is a $\mathbf{C}$-number variable and $\triangle(x)$ is the scalar polynomial of x – the characteristic polynomial of the L-matrix. The following is a generalization of the Cayley-Hamilton theorem to the quantum case.

Theorem. *The matrix polynomial $B(L,x)$ when defined as*

$$B(L,x) = \hat{R}_1 \dots \hat{R}_{N-1} \prod_{i=1}^{N-1} \left[(L - q^{2i}x\mathbf{1})\hat{R}_1 \dots \hat{R}_{N-1} \right] \,, \qquad (2.15)$$

satisfies the relation (2.14). The characteristic polynomial of the matrix L looks like

$$\triangle (x) = \sum_{i=0}^{N} (-x)^i \sigma_q(N-i) \,, \qquad (2.16)$$

and for the L-matrix the following characteristic identity is satisfied

$$\triangle (L) = \sum_{i=0}^{N} (-L)^i \sigma_q(N-i) \equiv 0 \,. \qquad (2.17)$$

Proof: The relation (2.14) is fulfilled if and only if its left hand side is totally q-antisymmetric, i.e. if it obeys the characteristic relations (2.5) of the q-antisymmetric tensor. This in turn is valid if the matrix quantity $(L - x\mathbf{1}) B(L,x)$ commutes with all the $\hat{R}_i$, $1 \leq i \leq N-1$ up to terms proportional to the q-symmetric projectors $P_{+i} \equiv (\hat{R}_i + 1/q)/2_q$, which vanish when contracting with $\epsilon_q^{1...N}$. Now the key observation is that the commutator $[\hat{R}_1 \,, (L - x\mathbf{1})\hat{R}_1(L - \beta x\mathbf{1})\hat{R}_1]$ is proportional to P_+ only if $\beta = q^2$. With this observation the construction of B becomes clear and one immediately checks that chosen as in (2.15) B does satisfy the relation (2.14).

Then as a direct consequence of (2.14) and (2.15) we get the following expression for $\triangle(x)$:

$$\triangle(x) = \frac{1}{|\epsilon|^2} \epsilon_q^{1...N} \prod_{i=0}^{N-1} \left[(L - q^{2i}x\mathbf{1})\hat{R}_1 \dots \hat{R}_{N-1} \right] \epsilon_q^{1...N} \,.$$

This expression can be further simplified with the use of (2.5), (2.1), (2.2) together with the q-combinatorial relations. The calculations are straightforward but rather lengthy and we omit them here presenting the result in (2.16).

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To prove the characteristic identity we shall contract relation (2.14) with $\epsilon_q^{2...N+1}$:

$$(L - x\mathbf{1})\,\epsilon_q^{2...N+1}B(L,x)\,\epsilon_q^{1...N} = (\epsilon_q^{2...N+1}\epsilon_q^{1...N})\,\triangle(x)\,.$$

The right hand side of this relation is proportional to the unit matrix and, hence, $\epsilon_q^{2...N+1}B\epsilon_q^{1...N}$ is proportional to $(L - x\mathbf{1})^{-1}$. The classical limit of this relation is the standard base for proving the Cayley-Hamilton theorem [18]. In quantum case all the considerations are completely the same, and the resulting statement is that the matrix polynomial $\triangle(L)$ identically vanishes. ∎

Here are few final comments.

1. The characteristic identity provides us with the compact expression for the inverse matrix of L:

$$L^{-1} = \frac{1}{\sigma_q(N)}\sum_{i=0}^{N-1}(-L)^i\sigma_q(N-i-1)\,.$$

2. Multiplying the characteristic identity by L^p, and taking the q-trace one gets the expressions of higher symmetric polynomials $s_q(N+p)$ in terms of the basic ones $\sigma_q(i)$.

3. On passing to concrete REA representations the order of the characteristic identity may decrease due to the basic symmetric polynomials $\sigma_q(i)$ become dependent. This is illustrated in a recent paper [20] where the L-matrices were realized as pseudo-differential operators acting on the quantum plane and they were found to possess the characteristic identity of the second order.

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