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A description of Chern classes of semistable sheaves on a quadric surface

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Many conditions on rank r and Chern classes c_1, c_2 of a stable sheaf on an algebraic surface are known. One of them is Bogomolov's inequality

$$c_2 - \frac{r-1}{2r} \cdot c_1^2 > 0$$

that is valid for any surface. Another wonderful result is one of Drezet and LePotier. They gave a complete description of triples (r, c_1, c_2) that are possible for stable sheaves on $\mathbb{P}^2$. It is interesting that this set is not defined by a finite number of inequalities but has a fractal boundary ([D-L]).

The complete description of triples (r, c_1, c_2) for stable sheaves was known only for $\mathbb{P}^2$. In this paper we give such a description for semistable sheaves on a smooth quadric surface Q. It appears that there is quite a lot of similarities between the $\mathbb{P}^2$-case and the Q-case. In both cases the description depends on the Chern classes of exceptional sheaves. For Q these classes were studied in a separate paper ([R]). Our description is somewhat less constructive than in [D-L] and it is not clear at the moment how to convert it into an algorithm because the structure of the set of exceptional bundles is more complicated in our case. A formulation of the main theorem is in Section 3. Here I use the notion of Mukai lattice which is not really important for the paper but is – as I think – a natural frame to generalise the result.

The general lines of proving the theorem are the same ones as in [D-L]. The important new ingredients are $\bar{\gamma}$-stability (Section 2) and the way to prove the key lemma (Section 9) and maybe the way to formulate the theorem. The result of Drezet-LePotier about $\mathbb{P}^2$ can also be reformulated in a very similar style.

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1. Chern classes and the Mukai lattice

Let us begin with some notations and preliminaries.

Let X be a complete algebraic variety. For coherent sheaves E, F on X we define

$$\chi(E,F) = \sum_i (-1)^i \dim \operatorname{Ext}^i(E,F).$$

It is easy to see that $\chi(E,F)$ is a linear form for every argument and so is a bilinear function on a $\mathbb{Z}$-module $K_0(X)$. It is $\mathbb{Z}$-valued.

We will work most of the time with a smooth quadric surface Q. It is well known that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and we fix an isomorphism.

We choose the isomorphism $\operatorname{Pic} Q \simeq \mathbb{Z} \oplus \mathbb{Z}$ in such a manner that (m,n) corresponds to the line bundle $\mathcal{O}(m,n)$ equal to the tensor product of the preimage of $\mathcal{O}(m)$ from the first multiple and the preimage of $\mathcal{O}(n)$ from the second one. The line bundles $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ have as fibers of their linear systems $|\mathcal{O}(1,0)|$ and $|\mathcal{O}(0,1)|$ the vertical and horizontal lines in the decomposition $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Then the intersection number in $\operatorname{Pic} Q$ is written as

$$((a,b);(c,d)) = ad + bc.$$

We will write down explicitely the theorem of Riemann-Roch that will be used in Section 4 and especially throughout the computations in Section 9 and the Serre duality theorem.

The Riemann-Roch theorem. *Let r_E and r_F denote the rank of E and F, then*

(a) $$\chi(E,F) = r_E \cdot r_F + (r_E c_1(F) - r_F c_1(E);(1,1)) + r_F \cdot \left(\frac{1}{2}c_1^2 - c_2\right)(E) + r_E\left(\frac{1}{2}c_1^2 - c_2\right)(F) - c_1(E)\cdot c_1(F).$$

We fix the following notations for a sheaf E with $r_E \neq 0$:

$$v_E = (v_E', v_E'') = \frac{1}{r_E}\cdot c_1(E) \in \operatorname{Pic} Q \otimes \mathbb{Q} \simeq \mathbb{Q} \oplus \mathbb{Q},$$

$$\Delta_E = \frac{1}{r_E}\left(c_2(E) - \frac{r_E - 1}{2r_E}c_1(E)^2\right).$$

Then:

(b) $$\chi(E,F) = r_E r_F\big((v_F' - v_E' + 1)(v_F'' - v_E'' + 1) - \Delta_E - \Delta_F\big).$$

We will use the notation

$$E(m,n) = E \otimes \mathcal{O}(m,n).$$

The Serre duality theorem.

(a)
$$\operatorname{Ext}^i(E,F) \simeq \operatorname{Ext}^{2-i}(F,E(-2,-2))^*,$$
$$\chi(E,F) = \chi(F,E(-2,-2)).$$

Another standard notation that we will use throughout the paper is μ_E or $\mu(E)$:

$$\mu(E) = (v_E,(1,1)) = v'_E + v''_E .$$

One can easily check that

$$\mu(E(m,n)) = \mu(E) + m + n ,$$
$$v_{E(m,n)} = (v'_E + m, v''_E + n) ,$$
$$\Delta_{E(m,n)} = \Delta_E .$$

In order to define a Mukai lattice let us begin with some general considerations.

Let X be smooth, then from Serre duality we can see that

$$\chi(E,F) = (-1)^{\dim X} \chi(F, E \otimes K_X) .$$

So the left kernel of χ and the right one coincide and we can state the following

Definition. A $\mathbb{Z}$-module $\mathcal{M}u(X) = K_0(X)/\ker_\chi$ with a bilinear form induced by χ is called the *Mukai lattice* of X. We will call this form scalar product on $\mathcal{M}u(x)$ and use the notation $\chi(u,v)$ or just (u,v) for the value of this scalar product for $u, v \in \mathcal{M}u(X)$.

For $X = \mathbb{P}^2$ and for our case $X = Q$ there is even the equality $\mathcal{M}u(X) = K_0(X)$. But, for example, this is not so for a K3-surface.

For $X = Q$ we have also

$$\mathcal{M}u(Q) = K_0(Q) \simeq \mathbb{Z} \oplus \operatorname{Pic} Q \oplus \mathbb{Z} \simeq \mathbb{Z}^4 ,$$

where $[F] \mapsto (r(F), c_1(F), \frac{1}{2} c_1(F)^2 - c_2(F))$ and the scalar product for the right hand side is defined by the formula:

$$((r_1, a'_1, a''_1 d_1) \,|\, (r_2, a'_2, a''_2, d_2))$$
$$= r_1 r_2 + r_1(a'_2 + a''_2) - r_2(a'_1 + a''_1) + r_1 d_2 + r_2 d_1 - a'_1 a''_2 - a'_2 a''_1 .$$

One should mention that it is not symmetric. And, you see, this is a slightly different formulation of the Riemann-Roch theorem.

In the sequel we will use this isomorphism to identify left and right sides and will use the notation $[F]$ for the image in $\mathcal{M}u(X)$ of the class of the coherent sheaf F on X. If there is a stable (semistable) sheaf F for $u \in \mathcal{M}u(X)$ such that

$$[F] = u$$

then we call the element u stable (semistable).

Our main goal is to find the possible Chern classes for the stable sheaves, which we now can restate as follows *to find stable elements in* $\mathcal{M}u(X)$. A kind of answer for the case $X = Q$ will be given in Section 3. But we need to be more precise about stability first because there are different possibilities to choose a polarization on Q.

2. The stabilities

Let us recall the basic results about the Mumford-Takemoto stability (μ-stability) and the Gieseker stability (γ-stability).

Definition. One calls a sheaf E *μ-stable* if E has no torsion and for any subsheaf F such that both F and E/F have positive rank, one has

$$\mu(F) < \mu(E).$$

One calls a sheaf E *γ-stable* if E has no torsion and has positive rank and for any subsheaf F with positive rank one has

$$\frac{\chi(\mathcal{O}, F(n,n))}{r_F} < \frac{\chi(\mathcal{O}, E(n,n))}{r_E}$$

for sufficiently large n.

If we define *semistability* then we change $<$ to $\leqq$.

Using the Riemann-Roch theorem one can write

$$\frac{\chi(\mathcal{O}, E(n,n))}{r_E} = n^2 + (\mu_E + 2)n + (\nu'_E + 1)(\nu''_E + 1) - \Delta_E,$$

so if E is μ-stable then E is γ-stable.

Proposition 2.1. *Let E and F be μ-stable and F locally free and $E \neq F$; then*

$$\mu(F) - 4 < \mu(E) \leqq \mu(F) \Rightarrow \chi(F, E) \leqq 0.$$

Proof. First we derive that $\operatorname{Hom}(F, E) = 0$. Let $\varphi \in \operatorname{Hom}(F, E)$. Then we can put $\varphi = \alpha \circ \beta$, where β is surjective and α is injective

$$\varphi \colon F \xrightarrow{\beta} H \xrightarrow{\alpha} E.$$

Let $H \neq 0$. As E is without torsion, the same holds with H. Then either $\mu(F) < \mu(H)$ or β is an isomorphism. Also either cokernel α has positive rank and $\mu(H) < \mu(E)$ or the cokernel is a torsion sheaf and $\mu(H) = \mu(E)$. But $\mu(F) \geqq \mu(E)$ so the only possibility is $\mu(F) = \mu(N) = \mu(E)$ and then φ has finite cokernel. Then φ is an isomorphism because F is locally free and this is a contradiction.

Next we see that

$$\mathrm{Ext}^2(F, E) \simeq \mathrm{Hom}(E, F(-2, -2))^*$$

by Serre duality and this is equal to zero for the same reason. Then

$$\chi(F, E) = 0 - \dim \mathrm{Ext}^1(F, E) + 0 \leqq 0 .$$

Remark. If F is stable then $F(m, n)$ is also one. As a result of the proposition we can obtain from any stable F a condition of the type $\chi(F(m, n), e) \leqq 0$ for a stable sheaf E, $[E] = e$ to exist. But what we need is a refined version of the proposition with a new and more sophisticated notion of stability.

Definition. Given linear functions $\sigma_1, \ldots, \sigma_k$ on $K_0(X)$ let for a sheaf F with $\mathrm{rk}(F) = r \neq 0$

$$\gamma_i(F) = \frac{\sigma_i(F)}{r}$$

and form $\bar{\gamma}(F) = (\gamma_1(F); \ldots; \gamma_k(F))$. We call $\bar{\gamma}(F)$ a *vector slope* of F. We call a torsion free sheaf E *stable relative to* $\bar{\gamma}$ iff for any subsheaf F such that $0 < \mathrm{rk} F) \leqq \mathrm{rk}(E)$, $F \neq E$, there is $\bar{\gamma}(F) < \bar{\gamma}(E)$ (for the lexicographic order). If there is $\bar{\gamma}(F) \leqq \bar{\gamma}(E)$ under the same conditions then E is called $\bar{\gamma}$*-semistable.*

Remark. Gieseker stability is a special case of the definition because the condition

$$\frac{\chi(\mathcal{O}, F(n))}{\mathrm{rk} F} < \frac{\chi(\mathcal{O}, E(n))}{\mathrm{rk} E}$$

for large n is equivalent to lexicographic ordering of the coefficients of the polynomials.

Remark. As usual it follows from the linearity of functions σ_i that if there is an epimorphism $F \to H$ and $\mathrm{rk} H > 0$ then $\bar{\gamma}(F) < \bar{\gamma}(H)$ for a $\bar{\gamma}$-stable sheaf F and $\bar{\gamma}(F) \leqq \bar{\gamma}(H)$ for a $\bar{\gamma}$-semistable F.

Proposition 2.2. *If a-stability is defined by the functions $\sigma_1, \ldots, \sigma_k$ and b-stability is defined by $\sigma_1, \ldots, \sigma_m$, $m \geqq k$,* then

$$\begin{aligned} &E \text{ is a-stable} &&\Rightarrow E \text{ is b-stable},\\ &E \text{ is a-semistable} &&\Leftarrow E \text{ is b-semistable}. \end{aligned}$$

This follows just from the definitions.

Definition. Consider the functions

$$\gamma_1(F) = \mu(F),$$
$$\gamma_2(F) = \chi(\mathcal{O}, F)/\operatorname{rk} F,$$
$$\gamma_3(F) = \tilde{\mu}(F) \;= (c_1(F); (1,2))/\operatorname{rk} F$$

and let $\bar{\gamma}$ be a slope relative to these functions.

In this way we define a new stability for sheaves on Q.

Remark. If we take the two functions γ_1, γ_2 then the stability defined by this system is Gieseker stability for the polarisation $\mathcal{O}(1,1)$. If we take γ_3, γ_2 then also Gieseker stability arises for $\mathcal{O}(1,2)$. So our $\bar{\gamma}$-stability is "mixed" from two Gieseker stabilities.

Proposition 2.3. *Let F and E be $\bar{\gamma}$ semistable, then*

$$\mu(F) - 4 < \mu(E), \bar{\gamma}(E) < \bar{\gamma}(F) \Rightarrow \chi(F, E) \leqq 0 .$$

Proof. We will argue the same way as in the proof of Proposition 2.1. Let $\varphi \in \operatorname{Hom}(F, E)$. Then $\varphi = \alpha \circ \beta$ where β is surjective and α is injective

$$\varphi \colon F \xrightarrow{\beta} H \xrightarrow{\alpha} E .$$

Let $H \neq 0$. As E is without torsion, the same with H. Then $\operatorname{rk} H > 0$ and $\bar{\gamma}(F) \leqq \bar{\gamma}(H)$. Also $\bar{\gamma}(H) \leqq \bar{\gamma}(E)$. So $\bar{\gamma}(F) \leqq \bar{\gamma}(E)$ and this is impossible.

Consider now $\varphi \in \operatorname{Hom}(E, F(-2, -2))$. Since we have $\mu(E) < \mu(F(-2, -2))$, there is $\bar{\gamma}(E) < \bar{\gamma}(F(-2, -2))$. Then we have to check that $F(-2, -2)$ is $\bar{\gamma}$-semisimple and we leave this to the reader and by the previous part of the proof $\varphi = 0$. So we have the result.

Remark. Really F $\bar{\gamma}$-semistable $\Rightarrow$ $F(n, n)$ $\bar{\gamma}$-semistable.

3. Exceptional sheaves. The main theorem

In the sequel we will use the short notation

$${}^i\langle E \mid F \rangle := \operatorname{Ext}^i(E, F) .$$

Definition. We will call a coherent sheaf E over Q an exceptional sheaf iff

$${}^0\langle E \mid E \rangle = \mathbb{C}, \quad {}^1\langle E \mid E \rangle = 0, \quad {}^2\langle E \mid E \rangle = 0 .$$

Proposition 3.1. (1) *An exceptional sheaf is locally free and both γ-stable and $\bar{\gamma}$-stable.*

(2) *If E is a $\bar{\gamma}$- or γ-stable sheaf and $\chi(E,E) > 0$ then E is exceptional.*

(3) *For an exceptional sheaf E one has*

$$\chi(E,E) = 1, \quad \Delta_E = \frac{1}{2}\left(1 - \frac{1}{r^2}\right).$$

(4) *E is exceptional $\Leftrightarrow$ $E(m,n)$ is exceptional.*

(5) *All sheaves $\mathcal{O}(m,n)$ are exceptional.*

All statements except (2) are proven by Gorodentsev ([G]). For (2) he proved only a version with γ-stability. So let E be $\bar{\gamma}$-stable and $\varphi \in \mathrm{Hom}(E,E)$. Then we can put $\varphi = \alpha \circ \beta$ where β is surjective and α is injective,

$$\varphi\colon E \xrightarrow{\beta} H \xrightarrow{\alpha} E.$$

E has no torsion, so H also is without torsion. Let $H \neq 0$. As usual with stability there is $\bar{\gamma}(E) \leqq \bar{\gamma}(H) \leqq \bar{\gamma}(E)$ so $\bar{\gamma}(H) = \bar{\gamma}(E)$, then α and β are isomorphisms. Then φ is an isomorphism. If we choose $l \in \mathbb{C}$ such that $\varphi - l \cdot \mathrm{id}$ is not an isomorphism then it will be $\varphi = l \cdot \mathrm{id}$. So ${}^0\langle E|E\rangle = \mathbb{C}$. And a similar condition gives us that

$${}^0\langle E|E(-2,-2)\rangle = {}^2\langle E|E\rangle^* = 0.$$

Then from $\chi(E;E) > 0$ we derive that ${}^1\langle E|E\rangle = 0$ and that ends the proof.

Remark. By the same way we can prove that if $[E] = [F]$ for an exceptional E and $\bar{\gamma}$-semistable F then $E = F$.

Remark. So we see that Mukai classes of an exceptional bundle necessary lie on a "surface" $\chi(e,e) = 1$ and Mukai classes of other $\bar{\gamma}$-stable sheaves lie in the domain of $\chi(e,e) \leqq 0$. It is easy to prove that the condition $\chi(e,e) \leqq 1$ is equivalent to the Bogomolov inequality. So we have stronger inequalities for unexceptional stable sheaves. The main theorem gives us a more concrete description of the set of Mukai classes of $\bar{\gamma}$-stable sheaves on Q.

Theorem. *Let us denote by $\mathcal{E}xc$ the set of Mukai classes of exceptional sheaves on Q and let $e \in \mathcal{M}u(Q) - \mathcal{E}xc$. For a $\bar{\gamma}$-semistable sheaf E with $|E| = e$ to exist it is necessary and sufficient that for any $f \in \mathcal{E}xc$ with $\mathrm{rk}(f) \leqq \mathrm{rk}(e)$ the following condition is satisfied*:

(D-L) *If $\mu(f) - 2 < \mu(e)$, $\bar{\gamma}(e) < \bar{\gamma}(f)$ then $\chi(f,e) \leqq 0$*
and if $\bar{\gamma}(f) < \bar{\gamma}(e)$, $\mu(e) < \mu(f) + 2$ then $\chi(e,f) \leqq 0$.

If $\Delta(e) \neq \frac{1}{2}$ and $e \notin \mathbb{Z} \cdot \mathcal{E}xc$ then the condition (D-L) *is necessary and sufficient for the existence of a γ-stable sheaf E with $[E] = e$.*

Remarks. We know that (D-L) does not imply $\Delta(e) \neq \frac{1}{2}$. An example is $e \in \mathcal{M}u\, Q$ such that $\operatorname{rk}(e) = 2$, $c_1(e) = (1,0)$, $c_2(e) = -1$. But a description of the elements e in $\mathcal{M}u\, Q$ for which $\Delta(e) = \frac{1}{2}$ is an open question.

The notation (D-L) is chosen for Drezet-LePotier and their theorem for sheaves on $\mathbb{P}^2$ can be reformulated in a very similar manner.

To prove that (D-L) is necessary we need only compare Propositions 2.3 and 3.1. All the rest will be proved throughout the paper.

The plan of the proof is that in the beginning we make a family of sheaves $E(s)$ such that $[E(s)] = e$ for all $s \in S$. Then we prove that the subset of s, such that $E(s)$ is $\bar{\gamma}$-semistable, is not empty. Really a $\bar{\gamma}$-semistability of E is equivalent to the triviality of a canonical Harder-Narasimhan filtration. And we will go down on the length of the filtration in $E(s)$ in proving that a set of corresponding points is nonempty.

4. Making a family

Let $e \in \mathcal{M}u\, Q$ be under the conditions of the theorem.

Proposition 4.1. *There are a smooth variety S and a sheaf E on $\bar{Q} = Q \times S$ flat above S such that for any $s \in S$ the following conditions are satisfied:*

(1) *$E(s)$ has no torsion, $[E(s)] = e$.*

(2) *${}^2\langle E(s) | E(s)\rangle = 0$ and the Kodaira-Spencer morphism*

$$\omega \colon T_s S \to {}^1\langle E(s) | E(s)\rangle$$

is surjective.

(3) *A restriction $E(s)|_l$ on a general line l in a linear system $|\mathcal{O}(1,0)|$ or $|\mathcal{O}(0,1)|$ is rigid.*

Proof. Let us look at the polynomial $h(x,y)$ such that

$$h(m,n) = \chi(\mathcal{O}(m,n); e).$$

Then from the Riemann-Roch theorem one can see that h may be written in a form $h(x,y) = rxy + ax + by + c$ and $r > 0$. As the conditions of the theorem hold, so

$$h(m,n) \leqq 0 \quad \text{for} \quad \mu(e) < m + n < \mu(e) + 4$$

and we see that there exists a point $\bar{n}_0 = (m_0, n_0)$ such that

$$m_0 + n_0 \leqq \mu(e) \quad \text{and}$$

$$h(m_0, n_0) \geqq 0, \quad h(m_0 - 1, n_0 + 1) \leqq 0, \quad h(m_0, n_0 + 1) \leqq 0, \quad h(m_0 + 1, n_0) \leqq 0\,.$$

Let $\bar{n}_1 = (m_0 - 1, n_0)$, $\bar{n}_2 = (m_0 - 1, n_0 - 1)$, $\bar{n}_3 = (m_0 - 2, n_0 - 1)$ and

$$\begin{aligned} A &= h(m_0, n_0), \\ B &= -h(m_0 + 1, n_0), \\ C &= -h(m_0 - 1, n_0 + 1), \\ D &= -h(m_0, n_0 + 1). \end{aligned}$$

You see that A, B, C, D are nonnegative integers. Let

$$H = \mathrm{Hom}\,(\mathcal{O}(\bar{n}_3)^D;\ \mathcal{O}(\bar{n}_2)^C \oplus \mathcal{O}(\bar{n}_1)^B \oplus \mathcal{O}(\bar{n}_0)^A)\,.$$

Lemma 4.2. *The set $\mathcal{H}$ of monomorphisms is open in H and not empty.*

Proof. After elaborate but not difficult calculations one sees that

$$A + B + C - D = r = \mathrm{rk}\,(e)\,.$$

So the condition that $\alpha \in H$ is not a monomorphism on the fiber of $\mathcal{O}(\bar{n}_3)^D$ at $q \in Q$ cuts in H a subvariety of codimension $A + B + C - D + 1 = r + 1$. So non-monomorphisms form a subvariety H' of a codimension not less than $(r + 1) - 2 = r - 1$ and if $r > 1$ then H' is a proper closed set and its complement $\mathcal{H}$ is open and not empty. But $r \neq 1$ by $e \notin \mathcal{E}xc$, hence the lemma is proven.

Let us make a sheaf E on $Q \times \mathcal{H}$ taking $E(h) = \ker h$ for $h \in \mathcal{H}$, so that there is an exact sequence on $Q \times h$:

$$(*) \qquad 0 \to \mathcal{O}(\bar{n}_3)^D \to \mathcal{O}(\bar{n}_2)^C \oplus \mathcal{O}(\bar{n}_1)^B \oplus \mathcal{O}(\bar{n}_0)^A \to E(h) \to 0\,.$$

One can easily see that E is flat over $\mathcal{H}$. Let us prove that (1) and (2) are valid for E and for any restriction on a set $Q \times S$ where S is an open set in $\mathcal{H}$ and that (3) is valid for some open set S.

From $(*)$ one can calculate a Mukai vector for $E(h)$ and see that

$$[E(h)] = e\,.$$

To calculate cohomologies of a sheaf $E(h)$ you can use the fact that its image in the derived category of coherent sheaves on Q is equal to the image of the complex

$$K\colon 0 \to \mathcal{O}(\bar{n}_3)^D \to \mathcal{O}(\bar{n}_2)^C \oplus \mathcal{O}(\bar{n}_1)^B \oplus \mathcal{O}(\bar{n}_0)^A \to 0\,.$$

The sheaves $\mathcal{O}(\bar{n}_3)$, $\mathcal{O}(\bar{n}_2)$, $\mathcal{O}(\bar{n}_1)$, $\mathcal{O}(\bar{n}_0)$ are the base of a helix ([G], [R]) and that means here ${}^k\langle \mathcal{O}(\bar{n}_i) \,|\, \mathcal{O}(\bar{n}_j)\rangle = 0$ for either $i < j$ and $k \geqq 0$ or $i \geqq j$ and $k > 0$. So

$$^{i}\langle E(h)|E(h)\rangle = \mathrm{Ext}^{i}(K,K) = \mathbb{H}^{i}(\mathrm{Hom}^{\bullet}(K,K)).$$

Hence we derive that $^{2}\langle E(h)|E(h)\rangle = \mathrm{Ext}^{2}(K,K) = 0$ and that the natural morphism $\mathrm{Hom}^{1}(K,K) \to \mathrm{Ext}^{1}(K,K) = {}^{1}\langle E(h)|E(h)\rangle$ is an epimorphism. One sees that this epimorphism can be embedded in a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{O}(\bar{n}_3)^D; \mathcal{O}(\bar{n}_2)^C \oplus \mathcal{O}(\bar{n}_1)^B \oplus \mathcal{O}(\bar{n}_0)^A) \arrow[d] \arrow[dr] & \\ \mathrm{Hom}(\mathcal{O}(\bar{n}_3)^D; E(h)) \arrow[r] & {}^{1}\langle E(h)|E(h)\rangle \end{tikzcd}$$

and thus it is equal to the Kodaira-Spencer morphism

$$T_h\mathscr{H} \to {}^{1}\langle E(h)|E(h)\rangle$$

by Lemma 1.6 from [D-L]. So (1) and (2) are proven. By the same reasoning one can prove that

$$^{2}\langle E(h)|E(h) \oplus \mathcal{O}(-1,0)\rangle = {}^{2}\langle E(h)|E(h) \otimes \mathcal{O}(0,-1)\rangle = 0.$$

Then the Kodaira-Spencer morphism for a restriction $E(h)|_l$ can be put in a diagram

$$\begin{tikzcd} T_h\mathscr{H} \arrow[r] \arrow[dr] & {}^{1}\langle E(h)|E(h)\rangle \arrow[d] \\ & {}^{1}\langle E(h)|_l|E(h)|_l\rangle . \end{tikzcd}$$

Here the vertical morphism is a morphism from one of the next exact sequences

$$^{1}\langle E(h)|E(h)\rangle \to {}^{1}\langle E(h)|_l|E(h)|_l\rangle \to {}^{2}\langle E(h)|E(h) \otimes \mathcal{O}(-1,0)\rangle,$$

$$^{1}\langle E(h)|E(h)\rangle \to {}^{1}\langle E(h)|_l|E(h)|_l\rangle \to {}^{2}\langle E(h)|E(h) \otimes \mathcal{O}(0,-1)\rangle$$

which arise from the exact sequences of a restriction

$$0 \to E(h) \otimes \mathcal{O}(-1,0) \to E(h) \to E(h)|_l \to 0,$$

$$0 \to E(h) \otimes \mathcal{O}(0,-1) \to E(h) \to E(h)|_l \to 0.$$

Hence we see that the Kodaira-Spencer morphism is an epimorphism, so the set of $h \in \mathscr{H}$ such that $E(h)|_l$ is rigid is open. Denoting this by S proves the proposition.

5. Finiteness theorems and Harder-Narasimhan filtration

Having a good stability you can define in a torsion free sheaf a canonical filtration which becomes trivial for a semistable one. We want to do that for sheaves on Q.

Proposition 5.1. *Let E be a torsion free sheaf on Q. Then for the set of all subsheaves F in E one has the following properties*:

(1) *The values $\mu(F)$, $\tilde{\mu}(F)$ are upper bounded.*

(2) *If $\mu(F)$ is fixed then $\chi(\mathcal{O},F)$ is upper bounded.*

(3) *In the set of $\bar{\gamma}(F)$ there is a maximal element $\bar{\gamma}_{\max}$.*

(4) *There is a maximal subsheaf $F_{\max}$ in the set of subsheaves F, $\bar{\gamma}(F) = \bar{\gamma}_{\max}$.*

So as a result of the proposition we see that there is the subsheaf $F_{\max}$ in E such that for any other subsheaf F either $\bar{\gamma}(F) < \bar{\gamma}(F_{\max})$ or $F \subset F_{\max}$ and $\bar{\gamma}(F) = \bar{\gamma}(F_{\max})$. We will call $F_{\max}$ the maximal subsheaf in E.

Definition. A filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_p = E$ such that the sheaf $\mathrm{gr}_i = E_i/E_{i-1}$ is the maximal subsheaf in E/E_{i-1} is called *Harder-Narasimhan filtration* in E.

The existence and uniqueness (having a vector slope chosen) of the Harder-Narasimhan filtration in a torsion free sheaf E is a consequence of the proposition. Indeed, if $\tilde{F}$ is a preimage of a torsion sheaf in E/F then $\bar{\gamma}(\tilde{F}) \geqq \bar{\gamma}(F)$. Hence $E/F_{\max}$ is torsion free. So the maximal sheaf in $E/F_{\max}$ exists and so we make the filtration.

Proof of the Proposition. It is well known that if a factor sheaf F_2/F_1 has codimension 2 singularities then $\mu(F_1) = \mu(F_2)$ and $\tilde{\mu}(F_1) = \tilde{\mu}(F_2)$. This is the case for F^{**}/F, so proving (1) you can suppose that $E = E^{**}$ and $F = F^{**}$, hence that both E and F are locally free. The restriction $E|_l$ on a general line l from the linear system $|\mathcal{O}(0,1)|$ is a direct sum of $\mathcal{O}(n)$. Let n' be a maximal value of n in this sum and n'' be a similar maximum for a restriction on a line from $|\mathcal{O}(1,0)|$. One sees that for F it is

$$\mu(F) \leqq n' + n'', \quad \tilde{\mu}(F) \leqq 2n' + n''$$

and so is (1).

For (2) let us prove that if $d_1 \leqq \mu(F) \leqq d_2$ then $\sigma_2(F) = \chi(\mathcal{O}, F)$ is bounded. If a sheaf F_2/F_1 has only codimension 2 singularities then

$$\chi(F_1) \leqq \chi(F_2)$$

so we only need a proof for the case $E = E^{**}$, $F = F^{**}$.

Let us use an induction by the rank of E. For $\mathrm{rk}\, E = 1$ the statement is obvious. Then if the sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

is exact and $F \subset E$ then sheaves $F_1 \subset E_1$, $F_2 \subset E_2$ exist such that the restriction of the above sequence

$$0 \to F_1 \to F \to F_2 \to 0$$

is exact. Using (1) we see that the values of $\mu(F_1)$, $\mu(F_2)$ are bounded, so by induction $\chi(\mathcal{O}, F_1)$, $\chi(\mathcal{O}, F_2)$ are bounded and their sum

$$\chi(\mathcal{O}, F) = \chi(\mathcal{O}, F_1) + \chi(\mathcal{O}, F_2)$$

is also bounded. The statement (3) obviously follows from (1) and (2).

To prove (4) let us note that if there are two subsheaves F_1, F_2 with

$$\bar{\gamma}(F_1) = \bar{\gamma}(F_2) = \gamma_{\max}$$

and $F = F_1 + F_2$ then from an exact sequence

$$0 \to F_1 \cap F_2 \to F_1 \oplus F_2 \to F \to 0$$

and inequalities $\bar{\gamma}(F_1 \cap F_2) \leqq \bar{\gamma}_{\max}$, $\bar{\gamma}(F) \leqq \bar{\gamma}_{\max}$, $\bar{\gamma}(F_1 \oplus F_2) = \gamma_{\max}$ and the definition of $\bar{\gamma}$ one derives that $\bar{\gamma}(F) = \bar{\gamma}(F_1 \cap F_2) = \bar{\gamma}_{\max}$. Of course this implies (4).

6. Comparison of filtrations and a maximal filtration

The following definitions are really quite general but we will look only on our case.

We will associate with a filtration in a sheaf E, $\operatorname{rk}(E) = r$ a piecewise linear mapping of a segment $[0, r]$ to $\mathbb{R}^3$. We will call this mapping a weight of the filtration as in [D-L].

Definition. Given a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_m = E$ in a torsion free sheaf E, take points

$$(\operatorname{rk} F_i; \sigma_1(F_i); \sigma_2(F_i); \sigma_3(F_i))$$

in $[0, r] \times \mathbb{R}^3$ as vertices of a graph of a piecewise linear mapping $\bar{\sigma}_{\{F_i\}} : [0, r] \to \mathbb{R}^3$. This mapping will be called a *weight* of the filtration.

A mapping $\bar{n}$: $[0, r] \to \mathbb{R}^3$ will be called *convex* iff for any $a, b \in [0, r]$

$$\bar{n}\left(\frac{a+b}{2}\right) \geqq \frac{\bar{n}(a) + \bar{n}(b)}{2}$$

for the lexicographic order.

If a weight of a filtration is convÖex then one calls the filtration convex.

Remark. A Harder-Narasimhan filtration is convex.

Let us make an order on the mappings defining $\bar{u} \leqq \bar{v}$ iff for any $x \in [0, r]$ there is $\bar{u}(x) \leqq \bar{v}(x)$ lexicographically.

Proposition 6.1. *For a torsion free sheaf E of rank r*

(a) *a set of weights of convex filtrations in E is finite,*

(b) *the weight of the Harden-Narasimhan filtration dominates a weight of any other filtration in E.*

Proof. Let $0 = F_0 \subset F_1 \subset \cdots \subset F_m = E$ be a convex filtration in E. Then its associated grading satisfies

$$\bar{\gamma}(\mathrm{gr}_1) \geqq \bar{\gamma}(\mathrm{gr}_2) \geqq \cdots \geqq \bar{\gamma}(\mathrm{gr}_m) .$$

This implies that $\bar{\gamma}(F_1) \geqq \bar{\gamma}(F_2) \geqq \cdots \geqq \bar{\gamma}(F_m) = \bar{\gamma}(E)$, hence $\mu(F_i) \geqq \mu(E)$. Then from Proposition 5.1 one derives that there is only a finite quantity of possibilities for $\mu(F_i)$, $\chi(\mathcal{O}, F_i)$ and $\tilde{\mu}(F_i)$ and so for $\bar{\gamma}(F_i)$.

Now let $0 = F_0 \subset F_1 \subset \cdots \subset F_m = E$ be a maximal filtration. If E/F_1 is not torsion free then taking F_i' such that

$$F_i'/F_1 = F_i/F_1 + \operatorname{tors} E/F_1$$

one has a filtration $0 = F_0' \subset F_1' \subset \cdots \subset F_m' = E$ which dominates the previous one. Thus E/F_1 is torsion free and E/F_i also by the same reason. The maximality of $\{F_i\}$ implies $\bar{\gamma}$-semistability of F_1. Really if $F' \subset F_1$ and $\bar{\gamma}(F') > \bar{\gamma}(F_1)$ then the filtration

$$0 = F_0 \subset F' \subset F_1 \subset \cdots \subset F_m = E$$

is bigger $\{F_i\}$. Hence $\bar{\gamma}(F_1) \leqq \bar{\gamma}_{\max}$, and $F_1 \subset E_{\max}$. Using induction by rank one can easily derive (b) from this.

Proposition 6.2. *Let T be an algebraic variety and E be a T-flat coherent sheaf on $Q \times T$ such that $r(E(t))$, $\bar{\gamma}(E(t))$ are independent of $t \in T$. Then the weights of Harden-Narasimhan filtrations in $E(t)$ belong to a finite set.*

Proof. Denote by $\mathrm{gr}_i(E(t))$ the factors of a Harder-Narasimhan filtration in $E(t)$. General theorems about cohomologies of flat families imply that there is only a finite set of possibilities for $E(t)_l$ where l is a generic line from $|\mathcal{O}(1,0)|$ or $|\mathcal{O}(0,1)|$. Hence we see that values $\mu(\mathrm{gr}_i E(t))$ and $\tilde{\mu}(\mathrm{gr}_i E(t))$ are bounded and so lie in a finite set. Then from the Riemann-Roch theorem for $\chi(\mathcal{O}, E(t))$ one sees that $\sum r_i \Delta(\mathrm{gr}_i E(t))$ has a finite set of values. Integers r_i are taken from a finite set, so to prove the proposition it is sufficient to prove that for a $\bar{\gamma}$-semistable torsion free sheaf F there is $\Delta(F) \geqq 0$. There are numbers m', m'' such that

$$0 \leqq v_F' - m' < 1, \quad 0 \leqq v_F'' - m'' < 1 .$$

Then either $\bar{\gamma}(F) = \bar{\gamma}(\mathcal{O}(m', m''))$ and $\Delta(F) = \Delta(\mathcal{O}(m', m'')) = 0$ or $\bar{\gamma}(F) \neq \bar{\gamma}(\mathcal{O}(m', m''))$. In the last case also either $\bar{\gamma}(F) > \bar{\gamma}(\mathcal{O}(m', m''))$ or $\bar{\gamma}(\mathcal{O}(m', m'')) > \bar{\gamma}(F)$ and $v(F) = (m', m'')$. Now we can apply Proposition 2.3 either for $\mathcal{O}(m', m'')$, F or for $F, \mathcal{O}(m', m'')$. So one sees that either

$$\chi(F, \mathcal{O}(m', m'')) = r_F((m' - v_F' + 1)(m'' - v_F'' + 1) - \Delta(F)) \leqq 0$$

or

$$\chi(\mathcal{O}(m', m''), F) = r_F((0+1)(0+1) - \Delta(F)) \leqq 0 .$$

Then in both cases $\Delta(F) \geqq 0$.

7. Filtrations and cohomologies

Here we recall some properties of cohomologies $\mathrm{Ext}^i_{F,+}$ and $\mathrm{Ext}^i_{F,-}$ for sheaves or complexes with a filtration from [D-L]. Here if K is a sheaf or a complex of sheaves then we use the notation $F_i K$ for members of the filtration F in K.

(1) There is an exact sequence

$$\to \mathrm{Ext}^i_{F,-}(K,K) \to \mathrm{Ext}^i(K,K) \to \mathrm{Ext}^i_{F,+}(K,K) \to \mathrm{Ext}^{i+1}_{F,-}(K,K) \to .$$

(2) If K is left bounded then there is a spectral sequence with limit $\mathrm{Ext}^\bullet_{F,+}(K,K)$ where

$$E_1^{p,q} = \begin{cases} \prod_i \mathrm{Ext}^{p+q}(\mathrm{gr}_i K, \mathrm{gr}_{i-p} K) & \text{for } p < 0 , \\ 0 & \text{for } p \geqq 0 . \end{cases}$$

(3) Under the same conditions there is a spectral sequence with limit $\mathrm{Ext}^\bullet_{F,-}(K,K)$ where

$$E_1^{p,q} = \begin{cases} \prod_i \mathrm{Ext}^{p+q}(\mathrm{gr}_i K, \mathrm{gr}_{i-p} K) & \text{for } p \geqq 0 , \\ 0 & \text{for } p < 0 . \end{cases}$$

(4) Let $\tilde{K} = K/F_1 K$ and a filtration in $\tilde{K}$ be induced from K. Then there is an exact sequence

$$\to \mathrm{Ext}^i_{F/F_1,+}(\tilde{K}, \tilde{K}) \to \mathrm{Ext}^i_{F,+}(K,K) \to \mathrm{Ext}^i(F_1, \tilde{K}) \to \mathrm{Ext}^{i+1}_{F/F_1}(\tilde{K}, \tilde{K}) \to .$$

Proposition 7.1. *Let E be a torsion free sheaf on Q with its Harder-Narasimhan filtration F. Then*

$$\mathrm{Ext}^0_{F,+}(E,E) = 0 \quad \textit{and} \quad \mathrm{Ext}^2_{F,-}(E,E) = 0 .$$

This follows immediately from the above spectral sequences and the definitions of semistability and of the Harder-Narasimhan filtration.

8. Filtrations with fixed weight

Our reasoning here is a slight generalisation of the similar one in [D-L]. Given a torsion free sheaf E of a rank r and a piecewise linear mapping $\bar{n}$: $[0,r] \to \mathbb{R}^3$ let us look at the functor Drap: $\mathcal{S}chm \to \mathcal{S}et$ such that Drap(S) is a set of filtrations

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = \varrho^* E$$

where ϱ is a projection $Q \times S \to Q$ and F_i are sheaves on $Q \times S$ for which

(a) factors $\mathrm{gr}_i = F_i/F_{i-1}$ are S-flat,

(b) the weight of the induced filtration $0 = F_0(s) \subset F_1(s) \subset \cdots \subset F_m(s) = E$ is equal to $\bar{n}$ for any $s \in S$.

Proposition 8.1. *The described functor* Drap *is represented by a projective variety* $\mathrm{Drap}^{\bar{n}}(E)$. *Points in* $\mathrm{Drap}^{\bar{n}}(E)$ *are corresponding to filtrations in* E *with the weight* $\bar{n}$ *bijectively. If* F *is such a filtration then the Zariski tangent space for* $\mathrm{Drap}^{\bar{n}}(E)$ *at* F *is* $\mathrm{Ext}^0_{F,+}(E, E)$ *and the condition* $\mathrm{Ext}^1_{F,+}(E, E) = 0$ *is sufficient for* F *to be a nonsingular point in* $\mathrm{Drap}^{\bar{n}}(E)$.

Proof. Let n be a mapping $[0, r] \to \mathbb{R}^2$ made out of $\bar{n}$ just by dropping out the last coordinate in $\mathbb{R}^3$. Then n is equal to the weight of the filtration in E in the sense of Drezet and LePotier ([D-L]) or say a γ-weight. Having a γ-weight fixed you have a finite quantity of possible weights. One can prove this from Proposition 5.1 (1) for E and i^*E where $i: Q \to Q$ is an involution such that $i^*\mathcal{O}(1,0) = \mathcal{O}(0,1)$. Thus our flag variety $\mathrm{Drap}^{\bar{n}}(E)$ is a component in the Drezet-LePotier flag variety, so the formula for the tangent space and the nonsingularity condition are the same.

Let $\tilde{Q} = Q \times S$ and $\mathcal{E}$ be a coherent S-flat sheaf on $\tilde{Q}$. For an S-scheme $f: S' \to S$ look at a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = (\mathrm{id} \times f)^*\mathcal{E}$$

such that

(a) F_i/F_{i-1} are S'-flat sheaves on $Q \times S'$,

(b) for any $s \in S'$ the induced filtration in $(\mathrm{id} \times f)^*\mathcal{E}(s)$ is of weight $\bar{n}$.

So we can make a functor Drap: S-$\mathcal{S}chm \to \mathcal{S}et$, defining Drap S equal to the set of filtrations of the above type.

Proposition 8.2. (1) *The functor* Drap *is represented by a projective S-scheme*

$$\pi: \mathrm{Drap}^{\bar{n}}(\mathcal{E}) \to S$$

and the fiber of π *over* s *is* $\mathrm{Drap}^{\bar{n}}(\mathcal{E}(s))$.

(2) *Let* $s \in S$ *and* $F \in \mathrm{Drap}^{\bar{n}}(\mathcal{E}(s)) \subset \mathrm{Drap}^{\bar{n}}(\mathcal{E})$. *Then there is an exact sequence*

$$0 \to \mathrm{Ext}^0_{F,+}(\mathcal{E}(s), \mathcal{E}(s)) \to T_F\,\mathrm{Drap}^{\bar{n}}(\mathcal{E}) \to T_S S \xrightarrow{\omega_+} \mathrm{Ext}^1_{F,+}(\mathcal{E}(s), \mathcal{E}(s))$$

where ω_+ *is a composition of the Kodaira-Spencer morphism* ω *and a cohomology morphism from the exact sequence in* 7.1:

$$\omega_+: T_s S \to \mathrm{Ext}^1(\mathcal{E}(s), \mathcal{E}(s)) \to \mathrm{Ext}^1_{F,+}(\mathcal{E}(s), \mathcal{E}(s)).$$

(3) *Let S be smooth in s and* $\operatorname{Ext}^2(\mathscr{E}(s), \mathscr{E}(s)) = 0$ *and* ω_+ *surjective. Then* $\operatorname{Drap}^{\bar{n}}(\mathscr{E})$ *is smooth at F.*

One can derive that in a similar way from [D-L].

The most important conclusion of the previous consideration is the following:

Suppose that S is a smooth variety and $\mathscr{E}$ is a coherent S-flat sheaf on $\tilde{Q} = Q \times S$ such that for any $s \in S$ the sheaf $\mathscr{E}(s)$ is torsion free of rank r and $\bar{\gamma}(\mathscr{E}(s)) = \bar{\alpha}$ is independent of s.

Denote by $\bar{n}\colon [0, r] \to \mathbb{R}^3$ a piecewise linear mapping with $\bar{n}(0) = 0$, $\bar{n}(r) = \bar{\alpha}$ and by H the Harder-Narasimhan filtration of $\mathscr{E}(s)$.

Proposition 8.3. *Let for any $s \in S$*

(1) $\operatorname{Ext}^2(\mathscr{E}(s), \mathscr{E}(s)) = 0$,

(2) *the Kodaira-Spencer morphism* $\omega\colon T_s S \to \operatorname{Ext}^1(\mathscr{E}(s), \mathscr{E}(s))$ *is surjective.*

Then:

(a) *The set* $\Omega(\bar{n}) = \{s \in S \mid \bar{\sigma}_{H(\mathscr{E}(s))} \leqq \bar{n}\}$ *is open in S.*

(b) *The points* $s \in \Omega(\bar{n})$, *for which* $\bar{n} = \bar{\sigma}_{H(\mathscr{E}(s))}$ *holds, constitute a closed smooth subvariety in* $\Omega(\bar{n})$ *and its normal space at s is* $\operatorname{Ext}^1_{H,+}(\mathscr{E}(s), \mathscr{E}(s))$.

Proof. Let $X(\bar{n}) = \{s \in S \mid \bar{\sigma}_{H(\mathscr{E}(s))} = \bar{n}\}$; $s \in X(\bar{n})$ implies $s \in \operatorname{Im}(\pi\colon \operatorname{Drap}^{\bar{n}}(\mathscr{E}) \to S)$ in notations Proposition 8.2. Let $Y(\bar{n}) = \operatorname{Im}(\pi\colon \operatorname{Drap}^{\bar{n}}(\mathscr{E}) \to S)$, then $Y(\bar{n})$ is closed because of projectivity of $\operatorname{Drap}^{\bar{n}}(\mathscr{E})$. From Proposition 6.1 (b) follows

$$X(\bar{n}) \subset Y(\bar{n}) \subset \bigcup_{\bar{v} \geqq \bar{n}} X(\bar{v})$$

and Proposition 6.2 implies that $X(\bar{n}) \neq \emptyset$ only for a finite number of $\bar{v}$.

Thus we can conclude that

$$\bigcup_{\bar{v} > \bar{n}} X(\bar{v}) = \bigcup_{\bar{v} > \bar{n}} Y(\bar{v})$$

and so we have (a).

Look now at a restriction of a structure morphism π'

$$\pi'\colon \operatorname{Drap}^{\bar{n}}(\mathscr{E}|_{\Omega(\bar{n})}) \to \Omega(\bar{n})\,.$$

If $s \in \Omega(\bar{n})$ and $F \in \pi'^{-1}(s)$ then the weight of F is $\bar{n}$ and F is a Harder-Narasimhan filtration because of Proposition 6.1 and the definition of $\Omega(\bar{n})$. So the fiber of π' consists of not more than one point. And from Proposition 7.1 it follows that

$$\mathrm{Ext}^0_{F,+}\left(\mathscr{E}(s),\mathscr{E}(s)\right)=\mathrm{Ext}^2_{F,-}\left(\mathscr{E}(s),\mathscr{E}(s)\right)=0\,,$$

so the standard morphism

$$\mathrm{Ext}^1\left(\mathscr{E}(s),\mathscr{E}(s)\right)\to\mathrm{Ext}^1_{F,+}\left(\mathscr{E}(s),\mathscr{E}(s)\right)$$

is surjective. Hence ω_+ is surjective and we can apply Proposition 8.2 to $\mathscr{E}|_{\Omega(\bar{n})}$. We see that $d\pi'$ is an imbedding and from the diagram

$$\begin{array}{ccccccccc}
\mathrm{Ext}^0_{F,+}\left(\mathscr{E}(s),\mathscr{E}(s)\right) & \to & T_F\,\mathrm{Drap}^{\bar{n}}(\mathscr{E}|_{\Omega(\bar{n})}) & \to & T_sS & \to & \mathrm{Ext}^1_{F,+}\left(\mathscr{E}(s),\mathscr{E}(s)\right) & \to & 0\\
\| & & {\scriptstyle d\pi'}\downarrow & & \| & & \| & & \\
0 & \to & T_sX(\bar{n}) & \to & T_sS & \to & \mathrm{Ext}^1_{F,+}\left(\mathscr{E}(s),\mathscr{E}(s)\right) & \to & 0
\end{array}$$

we conclude the formula for a normal space.

Corollary. *If $s\in S$ and $\mathrm{Ext}^1_{H,+}\left(\mathscr{E}(s),\mathscr{E}(s)\right)\neq 0$ then there is $s'\in S$ such that the weight of the Harder-Narasimhan filtration for $\mathscr{E}(s')$ is strictly less than the one for $\mathscr{E}(s)$.*

Indeed such a point s is contained in a closure of the set $\Omega(\bar{n})-X(\bar{n})$ by the proposition, so there is some $\bar{w}$ such that s belongs to a closure of $X(\bar{w})$. Then $\bar{n}>\bar{w}$ and by definition of $X(\bar{w})$ any $s'\in X(\bar{w})$ fulfills the corollary.

9. The key lemma

Proposition 9.1. *Let e satisfy the conditions of the theorem and E be a locally free sheaf on Q such that*

(i) *the restrictions $E|_{l_1}$ and $E|_{l_2}$ on a general line of linear systems $|\mathcal{O}(1,0)|$ and $|\mathcal{O}(0,1)|$ are rigid,*

(ii) *the Harder-Narasimhan filtration in E is nontrivial,*

(iii) $[E]=e$.

Then $\mathrm{Ext}^1_{F,+}(E,E)\neq 0$.

Proof. Let $0=F_0\subset F_1\subset\cdots\subset F_k=E$ be the Harder-Narasimhan filtration in E and $k>1$ by assumption. Let gr_i, $i=1,\dots,k$, be factors of this filtration.

We can rewrite the condition (i) as follows

$$E|_{l_i}=\mathcal{O}(m_i)^{s_i}\oplus\mathcal{O}(m_i+1)^{t_i}\,,$$

where $s_i+t_i=\mathrm{rk}\,E$.

As before we will use the notation

$$v' = (c_1; (0,1)),$$
$$v'' = (c_1; (1,0)).$$

There is a monomorphism

$$0 \to \mathrm{Hom}(\mathcal{O}(n), F_1|_{l_i}) \to \mathrm{Hom}(\mathcal{O}(n), E|_{l_i})$$

and $F_1 \simeq \mathrm{gr}_1$, so this implies

$$v'(\mathrm{gr}_1) \leqq m_1 + 1, \quad v''(\mathrm{gr}_1) \leqq m_2 + 1.$$

Also for gr_k there are exact sequences (k is the last index in the filtration):

$$\mathrm{Ext}^1(\mathcal{O}(n), \mathcal{E}|_{l_i}) \to \mathrm{Ext}^1(\mathcal{O}(n), \mathrm{gr}_k|_{l_i}) \to 0.$$

They show us that for a direct summand of the type $\mathcal{O}(m)$ in $\mathrm{gr}_k|_{l_i}$ there is $m \geqq m_i$, so

$$v'(\mathrm{gr}_k) \geqq m_1, \quad v''(\mathrm{gr}_k) \geqq m_2.$$

From these four inequalities we derive that

$$\mu(\mathrm{gr}_1) - \mu(\mathrm{gr}_k) \leqq 2$$

and the definition of a Harder-Narasimhan filtration gives us

$$0 \leqq \mu(\mathrm{gr}_1) - \mu(\mathrm{gr}_k).$$

So there is

$$0 \leqq \mu(\mathrm{gr}_1) - \mu(\mathrm{gr}_k) \leqq 2. \tag{1}$$

To calculate $\mathrm{Ext}^i_{F,+}(E, E)$ one can use the spectral sequence from Section 7 with the first terms

$$E_1^{p,q} = \bigoplus \mathrm{Ext}^{p+q}(\mathrm{gr}_j, \mathrm{gr}_{j-p}) \quad \text{for } p < 0,$$
$$E_1^{p,q} = 0 \quad \text{for } p \geqq 0.$$

In our situation for $p > 0$, $\bar{\gamma}(\mathrm{gr}_j) > \bar{\gamma}(\mathrm{gr}_{j+p})$ so $\mathrm{Ext}^0(\mathrm{gr}_j, \mathrm{gr}_{j+p}) = 0$ and we have

$$\mathrm{Ext}^2(\mathrm{gr}_j, \mathrm{gr}_{j+p}) = \mathrm{Ext}^0(\mathrm{gr}_{j+p}, \mathrm{gr}_j(-2,-2))^*.$$

But from (1) we conclude

$$\mu(\mathrm{gr}_{j+p}) \geqq \mu(\mathrm{gr}_k) \geqq \mu(\mathrm{gr}_1) - 2 \geqq \mu(\mathrm{gr}_j) - 2$$

so $\bar{\gamma}(\mathrm{gr}_{j+p}) > \bar{\gamma}(\mathrm{gr}_j(-2,-2))$, hence

$$\mathrm{Ext}^2(\mathrm{gr}_j, \mathrm{gr}_{j+p}) = 0.$$

Thus $E_1^{p,q} = 0$ for $p + q \neq 1$.

As a result we see that all the differentials in the spectral sequence are trivial and then

$$\dim \operatorname{Ext}^1_{F,+}(E, E) = \sum_{j,p>0} \dim \operatorname{Ext}^1(\mathrm{gr}_j, \mathrm{gr}_{j+p}) .$$

Let us suppose that the conclusion is false, so

$$\operatorname{Ext}^1(\mathrm{gr}_j, \mathrm{gr}_{j+p}) = 0 \quad \text{for all } j, p > 0 .$$

This gives us $\chi(\mathrm{gr}_j, \mathrm{gr}_{j+p}) = 0$ for $j, p > 0$. Then from the bilinearity of χ it follows that

$$\chi(\mathrm{gr}_1, E) = \chi(\mathrm{gr}_1, \mathrm{gr}_1) ,$$
$$\chi(E, \mathrm{gr}_k) = \chi(\mathrm{gr}_k, \mathrm{gr}_k) ,$$
$$\chi(\mathrm{gr}_1, \mathrm{gr}_k) = 0 .$$

We want to show that this system of equations is selfcontradictory. Let us use the Riemann-Roch theorem for the explicit computation of χ. Then

$$\chi(E, F) = r_E r_F \big((v'_F - v'_E + 1)(v''_F - v''_E + 1) - 1 + \delta_E + \delta_F\big) ,$$

where $\delta = \dfrac{1}{2} - \Delta$, $\Delta = \dfrac{1}{r}\left(c_2 - \dfrac{r-1}{2r} c_1^2\right)$. And from this formula we also have

$$\delta_F = \frac{\chi(F, F)}{2r^2} .$$

Let $v'_i = v'(\mathrm{gr}_i)$, $v''_i = v''(\mathrm{gr}_i)$, $\delta_i = \delta_{\mathrm{gr}_i}$.

Lemma 9.2. *Either $\delta_1 = \delta_k = 0$ or one of these numbers is strictly positive.*

Proof. As $\chi(\mathrm{gr}_1, \mathrm{gr}_k) = 0$, so

$$(v'_k - v'_1 + 1)(v''_k - v''_1 + 1) - 1 + \delta_1 + \delta_k = 0 .$$

Setting $a = v'_k - v'_1$, $b = v''_k - v''_1$ we have proved that

$$a + b \leqq 0, \quad a \geqq -1, \quad b \geqq -1 .$$

Then the maximum value of $(a+1)(b+1)$ is equal to 1 for $a = b = 0$. Hence

$$-(\delta_1 + \delta_k) = (a+1)(b+1) - 1 \leqq 0 ,$$

thus $\delta_1 + \delta_k \geqq 0$ and the lemma is proved.

In the following we will consider several cases for δ_1, δ_k.

Case 1: $\delta_1 = \delta_k = 0$. In this situation follows from the proof of the lemma that $v_1' = v_k'$, $v_1'' = v_k''$, so

$$\mu(\mathrm{gr}_1) = \mu(\mathrm{gr}_2) = \cdots = \mu(\mathrm{gr}_k).$$

Also from $\delta_1 = \delta_k$ follows that $\sigma_2(\mathrm{gr}_1) = \sigma_2(\mathrm{gr}_k)$ so

$$\gamma_2(\mathrm{gr}_1) = \gamma_2(\mathrm{gr}_2) = \cdots = \gamma_2(\mathrm{gr}_k).$$

Then from the definition of the filtration $\gamma_3(\mathrm{gr}_1) > \gamma_3(\mathrm{gr}_k)$ but this gives us a contradiction as $\gamma_3(\mathrm{gr}_i) = 2v_i' + v_i''$.

Lemma. *If $\delta > 0$ for a semistable sheaf G then there is a stable sheaf G_1 such that*

$$[G] = m[G_1],$$

where m is a positive integer.

Proof. As always there is a Jordan-Hölder filtration

$$0 \subset G_1 \subset G_2 \subset \cdots \subset G_m = G,$$

where each factor is a stable sheaf with the same slope as G. The condition $\delta_G > 0$ implies $\chi(G, G) > 0$ so for any two factors G', G'' of the filtration $\chi(G', G'') > 0$ since the sign of χ depends only on the slopes of the sheaves. So we have $\mathrm{Hom}(G', G'') \neq 0$ and from their stability it follows that $G' \simeq G''$. Then $[G] = m[G_1]$ as needed.

Case 2: $\delta_1 > 0$. For a semistable class $[\mathrm{gr}_1]$ there is a stable element g such that

$$[\mathrm{gr}_1] = mg.$$

Then $\chi(g, g) > 0$ and by Proposition 3.1, $g \in \mathscr{E}xc$. Also we have either

$$\mu(\mathrm{gr}_1) > \mu(E) > \mu(\mathrm{gr}_k) \geqq \mu(\mathrm{gr}_1) - 2$$

or

$$\mu(\mathrm{gr}_1) = \mu(E) = \mu(\mathrm{gr}_k) > \mu(\mathrm{gr}_1) - 2$$

and $\bar{\gamma}(g) = \bar{\gamma}(\mathrm{gr}_1)$. Thus

$$\bar{\gamma}(g) > \bar{\gamma}(E), \quad \mu(E) > \mu(g) - 2$$

and

$$m\chi(g, E) = \chi(\mathrm{gr}_1, E) = \chi(\mathrm{gr}_1, \mathrm{gr}_1) > 0.$$

But this is impossible by the condition (D-L) of the theorem.

Case 3: $\delta_k > 0$. Here we can write

$$[\mathrm{gr}_k] = mg$$

and $g \in \mathscr{E}xc$, $\bar{\gamma}(g) = \bar{\gamma}(gr_k)$. It is

$$\bar{\gamma}(\mathrm{gr}_k) < \bar{\gamma}(E), \quad \mu(E) < \mu(\mathrm{gr}_k) + 2$$

and

$$m\chi(g, E) = \chi(\mathrm{gr}_k, E) = \chi(E, \mathrm{gr}_k) = \chi(\mathrm{gr}_k, \mathrm{gr}_k) > 0\,,$$

but this contradicts the condition (D-L) and the proposition is proved.

10. Proof of the theorem

Our first step is to prove the existence of a $\bar{\gamma}$-semistable sheaf in the class e. Such a sheaf will be also γ-semistable and μ-semistable by Proposition 2.2.

Consider the sheaf $\mathscr{E}$ on $\tilde{Q} = Q \times S$ defined in Proposition 4.1. Then $\mathscr{E}$ is a family of sheaves on Q with base S. Proposition 8.3 gives us the stratification of S and by Proposition 6.2 there is a finite number of strata. From Propositions 9.1 and 8.3 we conclude that a stratum with a nontrivial Harder-Narasimhan filtration has a nonzero codimension. Thus a restriction $\mathscr{E}'$ of $\mathscr{E}$ on the open stratum S' is a family of $\bar{\gamma}$-semistable sheaves. It is important to mention that the conditions of Proposition 4.1 are also valid for $\mathscr{E}'$.

Our next step is to prove that if $\Delta(e) \neq \frac{1}{2}$ then for some $s \in S'$ the sheaf $\mathscr{E}'(s)$ is γ-stable. This will complete the proof.

Here we can use some results from [D-L]. Let

$$0 \subset F_0 \subset F_1 \subset \cdots \subset F_k = \mathscr{E}'(s)$$

for some $s \in S'$ be a filtration with γ-stable factors gr_i without torsion and $\mu(\mathrm{gr}_i) = \mu(e)$, $\Delta(\mathrm{gr}_i) = \Delta(e)$. One calls such a filtration a Jordan-Hölder filtration.

Lemma 10.1. *For such a filtration*

$$\mathrm{Ext}^2_{F,-}(\mathscr{E}'(s), \mathscr{E}'(s)) = 0\,.$$

Proof. We see from 7(3) that it is sufficient to prove that $\mathrm{Ext}^2(\mathrm{gr}_i, \mathrm{gr}_{i-p}) = 0$ for $p \geqq 0$, $i = 1, \ldots, k$. But $\mathrm{Ext}^2(\mathrm{gr}_i, \mathrm{gr}_{i-p}) = \mathrm{Hom}(\mathrm{gr}_{i-p}, \mathrm{gr}_i(2,2))^*$ by Serre duality, so it is equal to zero because

$$\mu(\mathrm{gr}_{i-p}) = \mu(\mathrm{gr}_i) > \mu(\mathrm{gr}_i(-2,-2))$$

as a result of γ-stability.

Denote by H_i the Hilbert polynomial for gr_i with respect to the polarisation $\mathcal{O}(1,1)$. Propositions (1.5) and (1.7) from [D-L] state that a subset of points s in S' for which in

$\mathscr{E}'(s)$ there is a Jordan-Hölder filtration with Hilbert polynomials $(H_1, \dots, H_k)$ is equal to the image of a canonical mapping

$$\pi\colon \mathrm{Drap}^{H_1, \dots, H_k} \to S'.$$

And if s is a regular point in the image then the codimension of the image is equal to

$$\dim \mathrm{Ext}^1_{F,+}(\mathscr{E}'(s), \mathscr{E}'(s)).$$

Lemma 10.2. *If the number k of factors in the filtration F is more than one then*

$$\dim \mathrm{Ext}^1_{F,+}(\mathscr{E}'(s), \mathscr{E}'(s)) \neq 0.$$

Proof. It is sufficient to prove that

$$c = \sum (-1)^i \dim \mathrm{Ext}^i_{F,+}(\mathscr{E}'(s), \mathscr{E}'(s))$$

is negative. By the spectral sequence from 7(2) and the Riemann-Roch theorem

$$c = \sum_{1 \leqq i < j \leqq k} \chi(\mathrm{gr}_i, \mathrm{gr}_j) = \sum r_i r_j (1 - 2\Delta(e))$$

because all the factors have the same slope.

Suppose c is non-negative then $\Delta(e) = \Delta(\mathrm{gr}_i) < \frac{1}{2}$ and by Proposition 3.1 the sheaves gr_i are exceptional. They all have the same rank r because

$$\Delta(\mathrm{gr}_i) = \Delta(e) = \frac{1}{2}\left(1 - \frac{1}{r^2}\right).$$

But since $\Delta(\mathscr{E}'(s)) = \Delta(e)$ one can rewrite this as

$$\frac{1}{2}\left(1 - \frac{\chi(\mathscr{E}'(s), \mathscr{E}'(s))}{(kr)^2}\right) = \frac{1}{2}\left(1 - \frac{1}{r^2}\right).$$

So $\chi(\mathscr{E}'(s), \mathscr{E}'(s)) = k^2$.

But on the other hand we have

$$\chi(\mathscr{E}'(s), \mathscr{E}'(s)) = k + \sum_{i \neq j} \chi(\mathrm{gr}_i, \mathrm{gr}_j).$$

Sublemma. *If F_1, F_2 are exceptional and $\mu(F_1) = \mu(F_2)$, $F_1 \neq F_2$, then $\chi(F_1, F_2) \leqq 0$.*

This follows from Proposition 2.1 and 3.1.

So the only possibility for us is that all gr_i are isomorphic. Then

$$\chi(\mathrm{gr}_i, \mathscr{E}(s)) = k\chi(\mathrm{gr}_1, \mathrm{gr}_1) = k > 0$$

and this contradicts the condition (D-L) for $\mathscr{E}'(s)$. Thus the Lemma 10.2 is proven.

Now we see from Proposition (1.7) of [D-L] and from the finiteness of systems of Hilbert polynomials that there is an open subset in S' where $k = 1$ and thus $\mathscr{E}'(s)$ is γ-stable.

The theorem is proved.

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