Lipschitzian Operators of Substitution in the Algebra $\Lambda\mathrm{BV}$[1]

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Dedicated to Professor A. N. Sharkovsky on the occasion of his 65th birthday

Abstract We present necessary and sufficient conditions on the generating functions of operators of substitution (= Nemytskii superposition operators), which map the Waterman's space $\Lambda\mathrm{BV}$ of functions of Λ-bounded variation on the interval into another space of this type and satisfy the Lipschitz condition.

Keywords Functions of Λ-bounded variation, Operators of substitution, Lipschitz condition

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1 Introduction

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty} \subset \mathbb{R}$ be a sequence of real numbers such that

$$0 < \lambda_i \le \lambda_{i+1} \quad \text{for all} \quad i \in \mathbb{N} \quad \text{and} \quad \sum_{i=1}^{\infty} 1/\lambda_i = \infty. \tag{1}$$

A function $f : I = [a, b] \to \mathbb{R}$ is said to be of Λ-*bounded variation* (in the sense of Waterman [22], [23]), in which case we write $f \in \Lambda\mathrm{BV}$, if the following quantity, called the Λ-*variation of* f *on the interval* I, is finite:

$$V_\Lambda(f) \equiv V_\Lambda(f, I) = \sup \sum_{i=1}^{m} |f(b_i) - f(a_i)|/\lambda_i\,; \tag{2}$$

the supremum being taken over all $m \in \mathbb{N}$ and all (non-ordered) collections of non-overlapping intervals $[a_i, b_i] \subset [a, b]$, $i = 1, \ldots, m$.

It is easily seen that $\Lambda\mathrm{BV} = \mathrm{BV}$, the space of functions of ordinary Jordan bounded variation on I, if and only if Λ is a bounded sequence. Consequently, if we suppose that $\sup_{i\in\mathbb{N}} \lambda_i = \infty$, then BV is a proper subspace of $\Lambda\mathrm{BV}$.

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The classes ΛBV with different sequences Λ have proven to be of interest in the study of Fourier series (e.g., [8], [9], [20], [22], [23], and references therein). Our aim in this note is to present a different type of application of the class ΛBV connected with the characterization of superposition operators on ΛBV satisfying the global Lipschitz condition.

Let $\mathbb{R}^I$ stand for the set of all functions mapping I into $\mathbb{R}$. Given a function of two variables $h : I \times \mathbb{R} \to \mathbb{R}$, the *operator of substitution* (also called the *Nemytskii* or *superposition operator*) $H = H_h : \mathbb{R}^I \to \mathbb{R}^I$, *generated by* h, is defined for $f : I \to \mathbb{R}$ by

$$(Hf)(x) \equiv H(f)(x) = h(x, f(x)), \qquad x \in I. \tag{3}$$

Let $\mathrm{B} \subset \mathbb{R}^I$ be a Banach function space with norm $\|\cdot\|$. We are interested in characterization of those functions h, for which the corresponding operator of substitution H maps the space B into itself and is *Lipschitzian* in the sense that there exists a number $\mu > 0$ such that

$$\|H(f_1) - H(f_2)\| \le \mu \, \|f_1 - f_2\| \quad \text{for all} \quad f_1,\, f_2 \in \mathrm{B}. \tag{4}$$

The study of operators of substitution H satisfying (4) is important in connection with systems with hysteresis [10] and difference equations [21].

Matkowski [12] proved that if B = Lip, the space of Lipschitz functions on I with the usual Lipschitzian norm, then condition (4) is equivalent to the existence of two functions $h_0, h_1 \in \mathrm{Lip}$ such that $h(x,\xi) = h_0(x) + h_1(x)\xi$ for all $x \in I$ and $\xi \in \mathbb{R}$. Matkowski and Miś [16] showed that if B = BV and condition (4) is satisfied, then there exist two functions $h_0, h_1 \in \mathrm{BV}$, which are continuous from the left on $(a,b]$, such that $h^*(x,\xi) = h_0(x) + h_1(x)\xi$, $(x,\xi) \in (a,b] \times \mathbb{R}$, where $h^*(x,\xi) = \lim_{y \to x-0} h(y,\xi)$ is the left regularization of h in the first variable. We note (cf. [12, p. 131]) that the spaces Lip and BV above cannot be replaced by the space C of continuous functions on I with the uniform norm or by the space L^p $(p > 0)$ of Lebesgue p-summable functions on I with the standard norm (e.g., $h(x,\xi) = \sin\xi$).

The above two results have been further extended to various spaces of functions and mappings of generalized bounded variation of one variable ([1]–[5], [13]–[15], [17], [18]) and two real variables ([6], [7]).

The main result of this paper (Theorem 2.1) asserts that the generating functions h of Lipschitzian operators of substitution H on the space $\mathrm{B} = \Lambda\mathrm{BV}$ admit the representation of Matkowski and Miś. The general idea of this representation belongs to Matkowski, and the test functions f_j used in the proof of Theorem 2.1 are those from [2], [4], [5] and [7]. The preliminaries and main results of this paper are presented in Section 2, while their proofs are given in Section 3. Section 4 contains some generalizations when functions under consideration take their values in normed linear spaces (Theorem 4.1).

2 Main Results

It is known (cf. [23, Sec. 3]) that $\Lambda\mathrm{BV}$ is a Banach space with respect to the norm

$$\|f\|_\Lambda = |f(a)| + V_\Lambda(f), \qquad f \in \Lambda\mathrm{BV}. \tag{5}$$

We note, moreover, that it was shown in [11, Theorem 4] that $\Lambda\mathrm{BV}$ is a normed Banach algebra (for details see step 4 in the proof of Theorem 2.1).

Any function $f \in \Lambda\mathrm{BV}$ has the limit from the left $f(x-0) = \lim_{y\to x-0} f(y)$ at each point $x \in (a, b]$ and the limit from the right $f(x+0) = \lim_{y\to x+0} f(y)$ at each point $x \in [a, b)$, and the set of discontinuities of f is at most countable (cf. [23, Theorems 2, 3]).

Given $f \in \Lambda\mathrm{BV}$, we define its *left regularization* $f^* : I \to \mathbb{R}$ by

$$f^*(x) = f(x-0) \quad \text{if} \quad a < x \le b, \quad \text{and} \quad f^*(a) = \lim_{x\to a+0} f^*(x). \tag{6}$$

The existence of the second limit in (6) will be proved in Section 3 (see Lemma 3.1). We set

$$\Lambda\mathrm{BV}^* = \{f \in \Lambda\mathrm{BV} \mid f \ \text{ is continuous from the left on } (a, b]\}.$$

Let $\Lambda' = \{\lambda'_i\}_{i=1}^\infty \subset \mathbb{R}$ be a sequence satisfying conditions of the form (1) and let $\Lambda'\mathrm{BV}$ be the associated space of functions of Λ'-bounded variation.

The main result of the present paper is the following

Theorem 2.1 *Suppose that $H : \mathbb{R}^I \to \mathbb{R}^I$ is an operator of substitution generated by the function $h : I \times \mathbb{R} \to \mathbb{R}$ according to formula* (3).

If H maps $\Lambda'\mathrm{BV}$ into $\Lambda\mathrm{BV}$ and is Lipschitzian in the sense that

$$\exists\, \mu > 0 \text{ such that } \|H(f_1) - H(f_2)\|_\Lambda \le \mu\, \|f_1 - f_2\|_{\Lambda'} \quad \forall\, f_1, f_2 \in \Lambda'\mathrm{BV}, \tag{7}$$

then there exists a constant $\mu_0 > 0$, depending on μ, λ_1 and λ'_1, such that

$$|h(x, \xi_1) - h(x, \xi_2)| \le \mu_0 |\xi_1 - \xi_2|, \qquad x \in I, \quad \xi_1, \xi_2 \in \mathbb{R}, \tag{8}$$

and there exist two functions $h_0, h_1 \in \Lambda\mathrm{BV}^*$ such that

$$h^*(x, \xi) = h_0(x) + h_1(x)\xi, \qquad x \in I, \quad \xi \in \mathbb{R}, \tag{9}$$

where $h^*(x, \xi)$ is the left regularization of $x \mapsto h(x, \xi)$ for each fixed $\xi \in \mathbb{R}$.

Conversely, if $h_0, h_1 \in \Lambda\mathrm{BV}$, $h(x, \xi) = h_0(x) + h_1(x)\xi$, $x \in I$, $\xi \in \mathbb{R}$, and the following condition is satisfied:

$$\exists\, C > 0 \quad \text{such that} \quad \sum_{i=1}^n 1/\lambda_i \le C \sum_{i=1}^n 1/\lambda'_i \quad \text{for all} \quad n \in \mathbb{N}, \tag{10}$$

then H maps $\Lambda'\mathrm{BV}$ into $\Lambda\mathrm{BV}$ and is Lipschitzian in the sense of (7).

This theorem will be proved in Section 3. Now we present its corollary.

Corollary 2.2 *Let $h : I \times \mathbb{R} \to \mathbb{R}$ be a function such that h^* exists and $h^* = h$ on $I \times \mathbb{R}$, and let H be the operator of substitution generated by h. Then the following two conditions are equivalent*:

(a) *H maps the algebra $\Lambda\mathrm{BV}$ into itself and is Lipschitzian*;

(b) *there exist two functions $h_0, h_1 \in \Lambda\mathrm{BV}^*$ such that $h(x,\xi) = h_0(x) + h_1(x)\xi$ for all $x \in I$ and $\xi \in \mathbb{R}$.*

3 Proofs

In order to prove Theorem 2.1 we need three lemmas.

Lemma 3.1 *The second limit in* (6) *exists for any $f \in \Lambda\mathrm{BV}$.*

Proof. Let $\varepsilon > 0$. From the definition of $f(a+0)$ we find a $\delta = \delta(\varepsilon) > 0$ such that $|f(x) - f(a+0)| \le \varepsilon$ for all $x \in (a, a+\delta]$. Let $x_1, x_2 \in (a, a+\delta]$. For small $\sigma > 0$ (such that $\sigma < x_j - a$, $j = 1,2$) we have:

$$|f^*(x_1) - f^*(x_2)| \le |f^*(x_1) - f(x_1 - \sigma)| + |f(x_1 - \sigma) - f(x_2 - \sigma)| + \\ + |f(x_2 - \sigma) - f^*(x_2)|,$$

where the expression in the middle is estimated by

$$|f(x_1 - \sigma) - f(x_2 - \sigma)| \le |f(x_1 - \sigma) - f(a+0)| + |f(a+0) - f(x_2 - \sigma)| \le 2\varepsilon.$$

From the existence of $f^*(x_j)$ we can choose a (smaller) $\sigma > 0$ such that $|f^*(x_j) - f(x_j - \sigma)| \le \varepsilon$, $j = 1,2$. It follows that $|f^*(x_1) - f^*(x_2)| \le 4\varepsilon$ for all $x_1, x_2 \in (a, a+\delta]$, and it remains to apply the Cauchy criterion for the existence of a limit. □

Lemma 3.2 *If $f \in \Lambda\mathrm{BV}$, then $f^* \in \Lambda\mathrm{BV}^*$ and $V_\Lambda(f^*) \le V_\Lambda(f)$.*

Proof. Clearly, f^* is continuous from the left on $(a, b]$, so we prove that $f^* \in \Lambda\mathrm{BV}$. Since $f^* = f$ at the points of continuity of f and, by virtue of (6) and Lemma 3.1, f^* has an internal saltus at each of its points of discontinuity (i.e., if $x \in I$ is a point of discontinuity of f^*, then $\liminf_{y\to x} f^*(y) \le f^*(x) \le \limsup_{y\to x} f^*(y)$), then, applying Theorem 2 from [19], we conclude that $f^* \in \Lambda\mathrm{BV}$ and $V_\Lambda(f^*) \le V_\Lambda(f)$. □

The next lemma easily follows from definition (2).

Lemma 3.3 *If $f : I \to \mathbb{R}$ is monotone, then $V_\Lambda(f) = |f(b) - f(a)|/\lambda_1$.*

Proof of Theorem 2.1. For clarity we divide the proof into four steps.

1. *General part.* Given $f_1, f_2 \in \Lambda'\mathrm{BV}$, by (5) and (7), we have, in particular, $V_\Lambda(Hf_1 - Hf_2) \le \mu \|f_1 - f_2\|_{\Lambda'}$, so that if $m \in \mathbb{N}$ and $[a_i, b_i] \subset [a, b]$, $i = 1, \ldots, m$, are non-overlapping intervals, then, by virtue of (2),

$$\sum_{i=1}^{m} \frac{|(Hf_1 - Hf_2)(b_i) - (Hf_1 - Hf_2)(a_i)|}{\lambda_i} \le \mu \|f_1 - f_2\|_{\Lambda'},$$

or, according to (3),

$$\sum_{i=1}^{m} \frac{|h(b_i, f_1(b_i)) - h(b_i, f_2(b_i)) - h(a_i, f_1(a_i)) + h(a_i, f_2(a_i))|}{\lambda_i} \le$$
$$\le \mu \|f_1 - f_2\|_{\Lambda'}. \tag{11}$$

If $\alpha,\ \beta \in \mathbb{R}$, $\alpha < \beta$, we define auxiliary Lipschitz functions $\eta_{\alpha,\beta} : \mathbb{R} \to [0, 1]$ by

$$\eta_{\alpha,\beta}(y) = \begin{cases} 0 & \text{if} \quad y \le \alpha, \\ (y - \alpha)/(\beta - \alpha) & \text{if} \quad \alpha \le y \le \beta, \\ 1 & \text{if} \quad y \ge \beta. \end{cases}$$

2. *Proof of* (8). Let $\xi_1, \xi_2 \in \mathbb{R}$. Suppose first that $x \in (a, b]$. Setting $m = 1$, $b_1 = x$ and $a_1 = a$ in (11), we substitute into (11) two functions defined by

$$f_j(y) = \eta_{a,x}(y)\xi_j, \qquad y \in I, \quad j = 1, 2.$$

Since $f_j(a) = 0$, $j = 1, 2$, and $V_{\Lambda'}(f_1 - f_2) = |\xi_1 - \xi_2|/\lambda'_1$ by Lemma 3.3, we have $\|f_1 - f_2\|_{\Lambda'} = |\xi_1 - \xi_2|/\lambda'_1$. It follows from (11) now that

$$|h(x, \xi_1) - h(x, \xi_2)|/\lambda_1 \le \mu\, |\xi_1 - \xi_2|/\lambda'_1,$$

which is (8) with $\mu_0 = \mu\, \lambda_1/\lambda'_1$. If $x = a$, we set $m = 1$, $b_1 = b$ and $a_1 = a$ in (11) and substitute into (11) two functions defined by

$$f_j(y) = (1 - \eta_{a,b}(y))\xi_j, \qquad y \in I, \quad j = 1, 2.$$

Now $f_j(a) = \xi_j$, $j = 1, 2$, and so, as above, $\|f_1 - f_2\|_{\Lambda'} = (1 + 1/\lambda'_1)|\xi_1 - \xi_2|$. Inequality (11) gives:

$$|h(a, \xi_1) - h(a, \xi_2)|/\lambda_1 \le \mu\, (1 + 1/\lambda'_1)|\xi_1 - \xi_2|,$$

which is (8) with the desired constant $\mu_0 = \mu\, \lambda_1(1 + 1/\lambda'_1)$.

By the definition of $h^*(\cdot, \xi)$, we have from (8):

$$|h^*(x, \xi_1) - h^*(x, \xi_2)| \le \mu_0|\xi_1 - \xi_2|, \qquad x \in I, \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{12}$$

3. *Proof of* (9). Let $\xi_1, \xi_2 \in \mathbb{R}$ and $x \in (a, b]$. Suppose that $m \in \mathbb{N}$ and $a < a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_m < b_m < x$ in (11). We substitute into (11) the following two functions:

$$f_j(y) = \eta_m(y)\xi_1 + (2 - j)\xi_2, \qquad y \in I, \quad j = 1, 2,$$

where the Lipschitz function $\eta_m : I \to [0,1]$ is defined as follows:

$$\eta_m(y) = \begin{cases} 0 & \text{if} \quad a \le y \le a_1, \\ \eta_{a_i,b_i}(y) & \text{if} \quad a_i \le y \le b_i, \quad i = 1,\dots,m, \\ 1 - \eta_{b_i,a_{i+1}}(y) & \text{if} \quad b_i \le y \le a_{i+1}, \quad i = 1,\dots,m-1, \\ 1 & \text{if} \quad b_m \le y \le b. \end{cases} \tag{13}$$

We note that $(f_1 - f_2)(y) = \xi_2$ for all $y \in I$, and so, $\|f_1 - f_2\|_{\Lambda'} = |\xi_2|$. Since $f_1(b_i) = \xi_1 + \xi_2$, $f_2(b_i) = \xi_1$, $f_1(a_i) = \xi_2$ and $f_2(a_i) = 0$, (11) yields

$$\sum_{i=1}^{m} \frac{|h(b_i, \xi_1 + \xi_2) - h(b_i, \xi_1) - h(a_i, \xi_2) + h(a_i, 0)|}{\lambda_i} \le \mu\, |\xi_2|. \tag{14}$$

Because constant functions belong to $\Lambda'\mathrm{BV}$ and H takes its values in $\Lambda\mathrm{BV}$, we infer that $h(\cdot, \xi) = H(\xi) \in \Lambda\mathrm{BV}$ for all $\xi \in \mathbb{R}$. Taking into account the definition of h^* and passing to the limit as $a_1 \to x - 0$ in (14), we find that for each $x \in (a, b]$ the following inequality holds:

$$|h^*(x, \xi_1 + \xi_2) - h^*(x, \xi_1) - h^*(x, \xi_2) + h^*(x, 0)| \le \mu\, |\xi_2| \big/ \textstyle\sum_{i=1}^{m} (1/\lambda_i).$$

Therefore, the last inequality holds also at $x = a$. Passing to the limit as $m \to \infty$ and taking into account (1) we arrive at the equality

$$h^*(x, \xi_1 + \xi_2) - h^*(x, \xi_1) - h^*(x, \xi_2) + h^*(x, 0) = 0, \quad x \in I, \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{15}$$

The remaining part of the proof of (9) is the same as in [12] (for the reader's convenience we present it here). For a fixed $x \in I$ we define $S_x : \mathbb{R} \to \mathbb{R}$ by $S_x(\xi) = h^*(x, \xi) - h^*(x, 0)$, $\xi \in \mathbb{R}$, so that (15) can be rewritten in the form $S_x(\xi_1 + \xi_2) = S_x(\xi_1) + S_x(\xi_2)$ for all $\xi_1, \xi_2 \in \mathbb{R}$, showing that S_x is an additive function. On the other hand, (12) implies $|S_x(\xi_1) - S_x(\xi_2)| \le \mu_0 |\xi_1 - \xi_2|$, and so, S_x is continuous on $\mathbb{R}$. Thus, there exists $h_1 : I \to \mathbb{R}$ such that $S_x(\xi) = h_1(x)\xi$ for all $x \in I$ and $\xi \in \mathbb{R}$. Setting $h_0(x) = h^*(x, 0)$, $x \in I$, we arrive at (9). Noting that $h_0(\cdot) = h^*(\cdot, 0)$ and $h_1(\cdot) = h^*(\cdot, 1) - h^*(\cdot, 0)$ and applying Lemma 3.2, we conclude that $h_0, h_1 \in \Lambda\mathrm{BV}^*$.

4. In order to prove the converse assertion, we note that the operator of substitution H is given by

$$(Hf)(x) = h_0(x) + h_1(x) f(x), \qquad f \in \Lambda'\mathrm{BV}, \quad x \in I. \tag{16}$$

By virtue of [11, Theorem 4] the following inequality holds:

$$V_\Lambda(fg) \le \|f\|_u V_\Lambda(g) + V_\Lambda(f) \|g\|_u, \qquad f,\, g \in \Lambda\mathrm{BV}, \tag{17}$$

where $\|f\|_u = \sup_{x \in I} |f(x)|$. It is clear from (1) and (2) that

$$\|f\|_u \le |f(a)| + \lambda_1 V_\Lambda(f), \qquad f \in \Lambda\mathrm{BV}. \tag{18}$$

Making use of (5), (17) and (18), we find that

$$\|fg\|_\Lambda \le \max\{1, 2\lambda_1\}\|f\|_\Lambda\|g\|_\Lambda, \qquad f,\, g \in \Lambda\mathrm{BV}. \tag{19}$$

According to [19, Theorem 3], condition (10) is equivalent to the inclusion $\Lambda'\mathrm{BV} \subset \Lambda\mathrm{BV}$. Let us show that this implies the existence of a positive constant $\kappa = \kappa(\Lambda', \Lambda)$ such that

$$\|f\|_\Lambda \le \kappa\, \|f\|_{\Lambda'} \quad \text{for all} \quad f \in \Lambda'\mathrm{BV}. \tag{20}$$

In fact, the identity operator Id, defined by $\mathrm{Id}(f) = f$, maps the Banach space $\Lambda'\mathrm{BV}$ into the Banach space $\Lambda\mathrm{BV}$ and is closed by virtue of (5) and (18), and so, by the closed graph theorem, it is continuous. It sufficies to define the constant $\kappa(\Lambda', \Lambda)$ to be equal to the operator norm of Id.

Now, (10), (16) and (19) imply that H maps $\Lambda'\mathrm{BV}$ into $\Lambda\mathrm{BV}$. Finally, if $f_1, f_2 \in \Lambda'\mathrm{BV}$, it follows from (19) and (20) that

$$\begin{aligned}\|H(f_1) - H(f_2)\|_\Lambda = \|h_1(f_1 - f_2)\|_\Lambda \le \max\{1, 2\lambda_1\}\|h_1\|_\Lambda\|f_1 - f_2\|_\Lambda \le \\ \le \max\{1, 2\lambda_1\}\kappa(\Lambda', \Lambda)\|h_1\|_\Lambda\|f_1 - f_2\|_{\Lambda'},\end{aligned} \tag{21}$$

and so, H satisfies condition (7). This completes the proof. □

Remark 3.4 *A theorem similar to Theorem* 2.1 *holds for the right regularization of $h(\cdot, \xi)$. However, the function h^* in* (9) *in general cannot be replaced by h* (*see Example on p.* 157 *in* [16]). *When Λ and Λ' are constant or bounded sequences, Theorem* 2.1 *gives the results of* [16].

Remark 3.5 *If $h_0, h_1 \in \Lambda\mathrm{BV}$ and $\|h_1\|_\Lambda < 1/\max\{1, 2\lambda_1\}$, then, by Banach's contraction principle and* (21) (*with $\Lambda' = \Lambda$, in which case $\kappa(\Lambda', \Lambda) = 1$*), *there exists a unique function $f \in \Lambda\mathrm{BV}$ such that $f(x) = h_0(x) + h_1(x)f(x)$ for all $x \in I$.*

4 Generalizations

Here we extend the result of Theorem 2.1 to the case when functions from $\Lambda\mathrm{BV}$ take their values in normed or Banach spaces.

Let X be a Banach space with norm $|\cdot|$. For a function $f : I \to X$ we write $f \in \Lambda\mathrm{BV}(I; X)$ provided the value (2) is finite.

(i) Each function $f \in \Lambda\mathrm{BV}(I; X)$ has the limit from the left $f(x - 0) \in X$ at any point $x \in (a, b]$ and the limit from the right $f(x + 0) \in X$ at any point $x \in [a, b)$, and the set of discontinuities of f is at most countable. This follows from [23], since Theorems 2 and 3 there carry over to the present case with the same proofs.

(ii) $\Lambda\mathrm{BV}(I; X)$ is a Banach space with respect to norm (5). To see the completeness of this space, we first note that the functional $V_\Lambda(\cdot)$ is lower semicontinuous, i.e., if $f, f_n : I \to X$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for all $x \in I$ (pointwise convergence on I), then

$$V_\Lambda(f) \le \liminf_{n\to\infty} V_\Lambda(f_n). \tag{22}$$

In fact, let $m \in \mathbb{N}$ and $[a_i, b_i] \subset [a, b]$, $i = 1, \dots, m$, be arbitrary non-overlapping intervals. From the definition of $V_\Lambda(f_n)$ we have:

$$\sum_{i=1}^{m} |f_n(b_i) - f_n(a_i)|/\lambda_i \le V_\Lambda(f_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

Taking the limit inferior as $n \to \infty$ in both sides of this inequality and taking into account the pointwise convergence of f_n to f, we get:

$$\sum_{i=1}^{m} |f(b_i) - f(a_i)|/\lambda_i \le \liminf_{n\to\infty} V_\Lambda(f_n),$$

and (22) follows from the definition of $V_\Lambda(f)$.

Now, suppose that $\{f_n\}_{n=1}^{\infty} \subset \Lambda\mathrm{BV}(I; X)$ is a Cauchy sequence:

$$\|f_n - f_m\|_\Lambda = |f_n(a) - f_m(a)| + V_\Lambda(f_n - f_m) \to 0 \quad \text{as} \quad n, m \to \infty. \tag{23}$$

From the estimate (18) (which is obvious for $f \in \Lambda\mathrm{BV}(I; X)$ as well) we find that, for each $x \in I$, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in X and, by virtue of completeness of X, we may denote its limit by $f(x) \in X$. Since $|V_\Lambda(f_n) - V_\Lambda(f_m)| \le V_\Lambda(f_n - f_m)$, the sequence $\{V_\Lambda(f_n)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$, and, hence, it is bounded and convergent. By (22), $f \in \Lambda\mathrm{BV}(I; X)$. Once again, (22) and the pointwise convergence of $\{f_m\}_{m=1}^{\infty}$ give: $\|f_n - f\|_\Lambda \le \lim_{m\to\infty} \|f_n - f_m\|_\Lambda$ for all $n \in \mathbb{N}$, and so, due to (23),

$$\limsup_{n\to\infty} \|f_n - f\|_\Lambda \le \lim_{n\to\infty} \lim_{m\to\infty} \|f_n - f_m\|_\Lambda = 0,$$

which means that f_n converges to f as $n \to \infty$ in the norm of $\Lambda\mathrm{BV}(I; X)$.

(iii) We define the left regularization f^* of f according to (6). Then Lemma 3.1 holds for $f \in \Lambda\mathrm{BV}(I; X)$ with the same proof. Lemma 3.2 is valid for $f \in \Lambda\mathrm{BV}(I; X)$ as well, but we have to change from the previous (real valued case) proof to a more direct one, which we present now.

Let us show that f^* is continuous from the left at $x \in (a, b]$. By (i), there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of continuity of f lying strictly at the left of x such that $x_n \to x$ as $n \to \infty$. Thus, we have in X:

$$\lim_{y\to x-0} f^*(y) = \lim_{n\to\infty} f^*(x_n) = \lim_{n\to\infty} f(x_n) = \lim_{y\to x-0} f(y) = f^*(x).$$

Let us prove that $f^* \in \Lambda\mathrm{BV}(I; X)$ and $V_\Lambda(f^*) \le V_\Lambda(f)$. Let $Q = \{1, 2, 3, \dots\}$ be a finite or countable set and $\{x_n\}_{n\in Q} \subset (a, b]$ be the set of points of discontinuity from the left of f. Let us define $f_1 : I \to X$ by

$f_1(x) = f(x)$ if $x \neq x_1$ and $f_1(x_1) = f(x_1 - 0)$, so that f_1 and f differ only at x_1, and let us show that $V_\Lambda(f_1) \leq V_\Lambda(f)$. Given $m \in \mathbb{N}$ and non-overlapping intervals $[a_i, b_i] \subset I$, $i = 1, \ldots, m$, we have the following three possibilities: a) $x_1 \neq a_i$ and $x_1 \neq b_i$ for all $i \in \{1, \ldots, m\}$; b) there exists $i_0 \in \{1, \ldots, m\}$ such that $x_1 = b_{i_0}$; c) there exists $i_0 \in \{1, \ldots, m\}$ such that $x_1 = a_{i_0}$. In case a) we have:

$$\sum_{i=1}^{m} |f_1(b_i) - f_1(a_i)|/\lambda_i = \sum_{i=1}^{m} |f(b_i) - f(a_i)|/\lambda_i \leq V_\Lambda(f).$$

If case b) holds, then

$$\sum_{i=1}^{m} \frac{|f_1(b_i) - f_1(a_i)|}{\lambda_i} = \sum_{i=1}^{i_0-1} \frac{|f(b_i) - f(a_i)|}{\lambda_i} + \frac{|f(x_1 - 0) - f(a_{i_0})|}{\lambda_{i_0}} + \\ + \sum_{i=i_0+1}^{m} \frac{|f(b_i) - f(a_i)|}{\lambda_i},$$

where the first or the last sum on the right hand side should be omitted depending on whether $i_0 = 1$ or $i_0 = m$ in the case $m \geq 2$, or both these sums should be omitted if $m = 1$. Let $\varepsilon > 0$. By the definition of $f(x_1 - 0)$, there exists a $y \in (a_{i_0}, x_1)$ such that $|f(x_1 - 0) - f(y)| \leq \varepsilon\, \lambda_1$, and so,

$$\frac{|f(x_1 - 0) - f(a_{i_0})|}{\lambda_{i_0}} \leq \frac{|f(y) - f(a_{i_0})|}{\lambda_{i_0}} + \varepsilon.$$

Since the intervals $[a_1, b_1], \ldots, [a_{i_0-1}, b_{i_0-1}]$, $[a_{i_0}, y]$, $[a_{i_0+1}, b_{i_0+1}], \ldots, [a_m, b_m]$ are still non-overlapping, we find from the above that

$$\sum_{i=1}^{m} |f_1(b_i) - f_1(a_i)|/\lambda_i \leq V_\Lambda(f) + \varepsilon.$$

In a similar manner we treat case c). Thus, we have proved that $V_\Lambda(f_1) \leq V_\Lambda(f) + \varepsilon$ for all $\varepsilon > 0$.

If functions $f_1, \ldots, f_{n-1}$ are already constructed and $x \in I$, we set $f_n(x) = f_{n-1}(x)$ for $x \neq x_n$ and $f_n(x_n) = f_{n-1}(x_n - 0) = f(x_n - 0)$, $n = 2, 3, \ldots$. By induction,

$$V_\Lambda(f_n) \leq V_\Lambda(f_{n-1}) \leq \ldots \leq V_\Lambda(f_1) \leq V_\Lambda(f), \qquad n \in Q.$$

If Q is finite, we are through, so let Q be infinite. Define $f_* : I \to X$ by $f_*(x) = f(x)$ if $x \notin \{x_n\}_{n=1}^{\infty}$ and $f_*(x_n) = f(x_n - 0)$ for $n \in Q$, and note that f_n converges in X pointwise on I to f_* as $n \to \infty$, so that, by (22), $V_\Lambda(f_*) \leq V_\Lambda(f)$. Finally, since $f^*(x) = f_*(x)$ if $x \neq a$, and $f^*(a) = f_*(a + 0)$, so that f^* and f_* differ only at a, by the above argument, we conclude that $V_\Lambda(f^*) \leq V_\Lambda(f_*) \leq V_\Lambda(f)$, which was to be proved.

(iv) The Banach algebra property of $\Lambda\mathrm{BV}$ is extended in the following way. Let X, Y and Z be normed spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ with norms $|\cdot|$ (the same symbol $|\cdot|$ for norms won't lead to ambiguities). Let $M : X\times Y \to Z$ be a bilinear map (called a multiplication) such that $|M(\xi,\eta)| \le |\xi|\cdot|\eta|$ for all $\xi \in X$ and $\eta \in Y$. We have: if $f \in \Lambda\mathrm{BV}(I;X)$ and $g \in \Lambda\mathrm{BV}(I;Y)$, then the product $fg : I \to Z$ defined by $(fg)(x) = M(f(x), g(x))$, $x \in I$, is in $\Lambda\mathrm{BV}(I;Z)$ and inequality (19) holds. This is a consequence of (17) and (5), which are valid in this more general case.

(v) Denote by $L(X;Y)$ the normed space of all bounded linear operators from X into Y. Given $h : I \times X \to Y$, we define the *operator of substitution* $H : X^I \to Y^I$ by (3) provided $f \in X^I$ (i.e. $f : I \to X$). Let $P(I;X) \subset X^I$ be a family of functions with the following property: for all $\xi_1, \xi_2 \in X$, $m \in \mathbb{N}$ and $a < a_1 < b_1 < \ldots < a_m < b_m < b$ the polygonal function defined by $I \ni x \mapsto \eta_m(x)\xi_1 + \xi_2 \in X$ belongs to $P(I;X)$, where η_m is defined in (13). Clearly, $P(I;X) \subset \Lambda\mathrm{BV}(I;X)$.

The analysis of the proof of Theorem 2.1 shows that the following counterpart and generalization of this theorem holds:

Theorem 4.1 *If X is a real normed space, Y is a Banach space and H maps $P(I;X)$ into $\Lambda\mathrm{BV}(I;Y)$ and is Lipschitzian* (*in the sense of the norms in these spaces*)*, then inequality* (8) *holds for all $x \in I$ and $\xi_1, \xi_2 \in X$, and there exist two functions $h_0 \in \Lambda\mathrm{BV}^*(I;Y)$ and $h_1 : I \to L(X;Y)$ with the property that $h_1(\cdot)\xi \in \Lambda\mathrm{BV}^*(I;Y)$ for all $\xi \in X$ such that* (9) *holds for all $x \in I$ and $\xi \in X$.*

Conversely, if X and Y are normed spaces, $h(x,\xi) = h_0(x) + h_1(x)\xi$, $x \in I$, $\xi \in X$, where $h_0 \in \Lambda\mathrm{BV}(I;Y)$ and $h_1 \in \Lambda\mathrm{BV}(I;L(X;Y))$, then the operator of substitution H maps $\Lambda\mathrm{BV}(I;X)$ into $\Lambda\mathrm{BV}(I;Y)$ and is Lipschitzian.

Proof of Theorem 4.1 is the same as that of Theorem 2.1; however, two remarks are in order. In step 3 of the proof we have: since X is real, the additivity and continuity of S_x imply $S_x \in L(X;Y)$ for all $x \in I$, and so, setting $h_1(x)\xi = S_x(\xi)$, $x \in I$, $\xi \in X$, we find that $h_1 : I \to L(X;Y)$ and $h_0, h_1(\cdot)\xi \in \Lambda\mathrm{BV}^*(I;Y)$ for all $\xi \in X$. In step 4 we note that, by virtue of (iv), applied with X there replaced by $L(X;Y)$, Y — by X and Z — by Y, we get: $h_1 f \in \Lambda\mathrm{BV}(I;Y)$.

(vi) At the end of this paper we present an extension ℓ_Λ of the space of summable sequences ℓ_1 in the spirit of Waterman and show that the counterpart of Theorem 2.1 is wrong in it. Let Λ satisfy conditions (1). A sequence of real numbers $x = \{x_i\}_{i=1}^\infty$ is said to be Λ-*summable* (in symbols, $x \in \ell_\Lambda$) if the following quantity is finite:

$$\|x\|_\Lambda = \sup\left\{\sum_{i=1}^\infty |x_{j(i)}|/\lambda_i \;\middle|\; j : \mathbb{N} \to \mathbb{N} \text{ is bijective}\right\}.$$

One can easily check that $\|\cdot\|_\Lambda$ is a norm in ℓ_Λ, and that, given $x, y \in \ell_\Lambda$, we have: $\sup_{i\in\mathbb{N}} |x_i| \le \lambda_1 \|x\|_\Lambda$ and $\|xy\|_\Lambda \le \lambda_1 \|x\|_\Lambda \|y\|_\Lambda$, where $xy = \{x_i y_i\}_{i=1}^\infty$.

If $h : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, the operator of substitution $H : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is defined by the formula: $(Hx)(i) = h(i, x_i)$, $i \in \mathbb{N}$, $x = \{x_i\}_{i=1}^{\infty}$. Let, in particular, $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying: $\exists\, \mu > 0$ such that $|h(\xi) - h(\eta)| \leq \mu\, |\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}$ (e.g. $h(\xi) = \sin \xi$). Then for $x, y \in \ell_\Lambda$ we have: $\|Hx\|_\Lambda \leq \mu\, \|x\|_\Lambda$ and $\|Hx - Hy\|_\Lambda \leq \mu\, \|x - y\|_\Lambda$.

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