Rationality of quotients by linear actions of affine groups

We feel honored to dedicate this article to our friend, colleague and teacher
Fabrizio Catanese on the occasion of his 60th birthday

BOGOMOLOV Fedor[1,*], BÖHNING Christian[2],
& GRAF VON BOTHMER Hans-Christian[2]

[1]*Department of Mathematics, Courant Institute of Mathematical Sciences, New York University,*
New York, NY 10012, USA;
[2]*Mathematisches Institut, University of Gottingen, Gottingen D-37073, Germany*
Email: bogomolo@cims.nyu.edu, boehning@uni-math.gwdg.de, bothmer@uni-math.gwdg.de

Abstract Let $G = \mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ be the (special) affine group. In this paper we study the representation theory of G and in particular the question of rationality for V/G, where V is a generically free G-representation. We show that the answer to this question is positive (Theorem 6.1) if the dimension of V is sufficiently large and V is indecomposable. We explicitly characterize two-step extensions $0 \to S \to V \to Q \to 0$, with completely reducible S and Q, whose rationality cannot be obtained by the methods presented here (Theorem 5.3).

Keywords rationality, linear group quotients, affine groups

MSC(2000): 14E08, 14M20, 14L24

1 Introduction

The well-known rationality problem in invariant theory asks whether V/G is always rational if V is a linear representation of a connected linear algebraic group G over $\mathbb{C}$. This seems to be extremely difficult, in general. However, it becomes a little more accessible if the unipotent radical of G is large in a certain sense, of which Miyata's theorem is the first example: if the action of G on V can be made triangular, then V/G is rational. We will give further evidence for the previous viewpoint in this paper by studying generically free quotients V/G, where $G = \mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ is the special affine group. In fact, if V is indecomposable and of sufficiently large dimension, these quotients are always rational, cf. Theorem 6.1 and Theorem 5.3 below. Some indecomposability assumption is really needed as there are families of decomposable generically free and arbitrarily large G-representations for which a proof of rationality amounts to proving stable rationality of level 1 for all generically free $\mathrm{SL}_n(\mathbb{C})$-representations, which is expected to be a hard problem, cf. Remark 5.4. One should also note that many rationality questions for reductive groups reduce to parabolic subgroups by the method of taking sections for the action. We remark that the methods of this paper apply, in principle, more generally to the affine groups

---

*Corresponding author

$\mathrm{GL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$, $\mathrm{Sp}_n(\mathbb{C}) \ltimes \mathbb{C}^n$, $\mathrm{SO}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ and other affine extensions of semisimple groups, or even to other nilpotent extensions of reductive groups where one knows stable rationality of some level for the reductive part such as jet groups. However, treating all cases uniformly might have rendered the presentation less transparent, and we decided to focus on a sample case to illustrate the methods.

2 Preliminaries

We begin by recalling some standard facts and conventions. Throughout this paper, we work over the complex numbers.

(A) Let G be a connected linear algebraic group; it is an extension

$$1 \to U \to G \to R \to 1,$$

where U is the unipotent radical of G, and R is the reductive part (representations of it are completely reducible). Thus U is nilpotent as a group (the descending central series terminates in the trivial group), and all the elements u of U are unipotent, i.e., $n = 1 - u$ is nilpotent. Then G is a semidirect product $G = R \ltimes U$ (Levi decomposition) and the reductive part may be written as

$$R = (T \times S)/C,$$

where T is a torus, S is semisimple, and C is a finite central subgroup. A (finite-dimensional) G-representation V has a Jordan-Hölder filtration

$$(0) \subset V_0 \subset V_1 \subset \cdots \subset V_{l-1} \subset V_l = V \tag{1}$$

by G-invariant subspaces such that the quotient V_{i+1}/V_i is a completely reducible G-representation (so U acts trivially on V_{i+1}/V_i) and maximal with that property.

(B) If Γ is any linear algebraic group $(G, U, \ldots)$, and W is a Γ-representation, then W/Γ will always denote the quotient in the sense of Rosenlicht, i.e., a birational model of $\mathbb{C}(W)^{\Gamma}$.

(C) The groups $G = \mathrm{SL}_n(\mathbb{C})$ and $G = \mathrm{SAff}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ are special (every étale locally trivial principal G-bundle is Zariski locally trivial), so for a generically free G-representation V, V/G is stably rational of level $\dim G$ ($= n^2 - 1$ resp. $n^2 - 1 + n$).

(D) Let $U = \mathbb{C}^n$ be the n-dimensional additive group (which occurs as unipotent radical, e.g., in the affine group $\mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$).

Lemma 2.1.   *Let $0 \to A \to B \to C \to 0$ be an exact sequence of U-representations such that the U action on A and C is trivial. Then $B/U \to C/U = C$ is birationally a vector bundle over C.*

Proof.   After choosing a section σ_0 of the projection $B \to C$, B becomes a trivial vector bundle $A \oplus C$ over C with zero section σ_0. The U-orbit of σ_0 inside B is then generically a vector subbundle of B: an element $t \in \mathbb{C}^n$ acts on the fibre $A \times \{c_0\}$ as $(a, c_0) \mapsto (a + t(c_0), c_0)$ via translations. Thus $U \cdot \sigma_0$ is a family of vector subspaces in each fibre, trivialized by the sections $e_1 \cdot \sigma_0, \ldots, e_n \cdot \sigma_0$, where $e_1, \ldots, e_n$ is a basis of $\mathbb{C}^n$, over some open set in C (where the dimension of the space of translations they span is the generic one). Then B/U may be identified with the quotient bundle $B/U \cdot \sigma_0$. $\qquad \square$

3 Representations of affine groups

In this section we review the representation theory of the affine group. See also [9] for more details. Let $\mathrm{SAff}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ be the n-dimensional special affine group. Sometimes we will write $U = \mathbb{C}^n$ to avoid confusion, when we consider it as a subgroup of $\mathrm{SAff}_n(\mathbb{C})$. Elements of $\mathrm{SAff}_n(\mathbb{C})$ can be written in matrix form as

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix},$$

where $A \in \mathrm{SL}_n(\mathbb{C}) \subset \mathbb{C}^{n \times n}$, v is a vector in $\mathbb{C}^{n \times 1}$, and $0 \in \mathbb{C}^{1 \times n}$. Thus we can write elements $g \in \mathrm{SAff}_n(\mathbb{C})$ as $g = (A, v)$, and matrix multiplication yields

$$(A, 0) \cdot (\mathrm{id}, v) = (\mathrm{id}, Av) \cdot (A, 0). \tag{2}$$

Let V be an N-dimensional representation of $\mathrm{SAff}_n(\mathbb{C})$. In a suitable basis, the image of the additive subgroup $\mathbb{C}^n$ (of pairs (id, v)) under $\varrho : \mathrm{SAff}_n(\mathbb{C}) \to \mathrm{Aut}(V) = \mathrm{GL}_N(\mathbb{C})$ is contained in the unipotent subgroup U_N of upper triangular $N \times N$ matrices with ones on the diagonal, which is an affine space. Thus

$$\varrho(v) = \sum_{|\alpha| \leqslant d} F_\alpha v^\alpha, \tag{3}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $v^\alpha = v_1^{\alpha_1} \cdots v_n^{\alpha_n}$ as usual, and F_α is in $\mathrm{Mat}_{N \times N}(\mathbb{C})$. The representation V has two natural filtrations:

$$0 \subset V_0 \subset V_1 \subset \cdots \subset V_{l-1} \subset V_l = V, \tag{4}$$

and

$$0 \subset V_0' \subset V_1' \subset \cdots \subset V_{l-1}' \subset V_l' = V, \tag{5}$$

which are defined inductively as follows: $Q_i := V_i/V_{i-1}$, for $i = 0, \ldots, l$ (we put $V_{-1} = 0$), is the maximal completely reducible subrepresentation of V/V_{i-1}; and $Q_{l-j}' := V_{l-j}'/V_{l-j-1}'$, for $j = 0, \ldots, l$, is the maximal completely reducible quotient representation of V_{l-j}'. Thus

$$V = Q_0 \oplus Q_1 \oplus \cdots \oplus Q_l = Q_0' \oplus \cdots \oplus Q_l'$$

as $\mathrm{SL}_n(\mathbb{C})$-representations (this is sometimes called the semisimplification of V).

Remark 3.1.   The two methods of filtering a representation of the affine group are related to duality as follows: if V has a filtration of type (4) with quotients $Q_i = V_i/V_{i-1}$, then the dual $W := V^\vee$ has a filtration

$$0 \subset W_0' \subset W_1' \subset \cdots \subset W_{l-1}' \subset W_l' = W$$

of type (5) with $W_{l-j}'/W_{l-j-1}' =: Q_{l-j}' = Q_j^\vee$.

First we consider filtrations of type (4), and unless explicitly stated otherwise, the term filtration will mean filtration of type (4). Since $\rho(\exp(u)) = \exp(d\varrho_e(u))$ for the linearization $d\varrho_e : \mathbb{C}^n \to \mathrm{End}(V)$, we see that d in formula (3) can be chosen equal to l. More precisely, $\varrho(v)$ is represented by some $N \times N$ block matrix

$$\begin{pmatrix} \mathrm{Id}_{q_0} & N_{01} & N_{02} & \ldots & N_{0l} \\ 0 & \mathrm{Id}_{q_1} & N_{12} & \ldots & N_{1l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathrm{Id}_{q_{l-1}} & N_{l-1,l} \\ 0 & 0 & 0 & \ldots & \mathrm{Id}_{q_l} \end{pmatrix}, \tag{6}$$

where $q_i = \dim Q_i$, N_{ij} is a $q_i \times q_j$-matrix depending on v, $N_{ij} = N_{ij}(v)$, and $N_{ij}(v)$ is a polynomial in v of total degree $\leqslant j - i$. Clearly $\mathrm{SAff}_n(\mathbb{C})$ is a subgroup of $\mathrm{SL}_{n+1}(\mathbb{C})$ in the natural way, and every $\mathrm{SL}_{n+1}(\mathbb{C})$-representation yields an $\mathrm{SAff}_n(\mathbb{C})$-representation by restriction. In particular, $\mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee$ yields an $\mathrm{SAff}_n(\mathbb{C})$-representation with a filtration

$$0 \subset S_0 \subset \cdots \subset S_l,$$

such that $S_i/S_{i-1} = \mathrm{Sym}^i(\mathbb{C}^n)^\vee$ as $\mathrm{SL}_n(\mathbb{C})$-representations (the action of $\mathrm{SAff}_n(\mathbb{C})$ on affine functions of degree less than or equal to l).

Proposition 3.2.   *Every representation V of the group $\mathrm{SAff}_n(\mathbb{C})$ with a filtration as in (4) is a subrepresentation of*

$$V_0 \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee$$

with filtration induced from

$$0 \subset V_0 \subset V_0 \otimes S_1 \subset \cdots \subset V_0 \otimes S_{l-1} \subset V_0 \otimes S_l = V_0 \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee .$$

Proof.   Let $v, w \in U = \mathbb{C}^n$, $A \in \mathrm{SL}_n(\mathbb{C})$ and $x \in V$. Then

$$\varrho(v)(\varrho(A)x) = \varrho(A)(\varrho(A^{-1}v)x) = \sum_{|\alpha| \leqslant l} \varrho(A) F_\alpha(x)(A^{-1}v)^\alpha$$

and

$$\varrho(v)(\varrho(w)x) = \sum_{|\alpha| \leqslant l} F_\alpha(x)(v + w)^\alpha,$$

which means that there is an $\mathrm{SAff}_n(\mathbb{C})$-equivariant map

$$V \to (V_0 \oplus V_1/V_0 \oplus \cdots \oplus V_l/V_{l-1}) \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee,$$
$$x \mapsto f_x,$$

where f_x is the (affine) polynomial function on $U = \mathbb{C}^n$ (with coefficients in V viewed as $Q_0 \oplus \cdots \oplus Q_l$ now as $\mathrm{SAff}_n(\mathbb{C})$-module!) given by

$$f_x(v) = \varrho(v)(x).$$

There is an $\mathrm{SAff}_n(\mathbb{C})$-equivariant projection

$$(V_0 \oplus V_1/V_0 \oplus \cdots \oplus V_l/V_{l-1}) \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee \to V_0 \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee.$$

It gives us an $\mathrm{SAff}_n(\mathbb{C})$-equivariant map

$$\iota : V \to V_0 \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee,$$

and it remains to check injectivity for this map. Injectivity follows from the assumption that the filtration (4) is such that the Q_{i+1} are maximal completely reducible subrepresentations of V/V_i in each step (we will actually only be using that Q_0 is the maximal completely reducible submodule of V in the proof of the proposition and the full assertion in the proof of the following corollary): for assuming to the contrary that injectivity fails. By $\mathrm{SL}_n(\mathbb{C})$-equivariance this is equivalent to saying that there exists an $\mathrm{SL}_n(\mathbb{C})$-irreducible summand S of $V = Q_0 \oplus \cdots \oplus Q_l$ which is mapped to 0 under ι. Then the $\mathrm{SAff}_n(\mathbb{C})$-span $\bar{S}$ of S in V is contained entirely in $Q_I \oplus \cdots \oplus Q_l$ for some $I \geqslant 1$. But $\bar{S}$ contains a completely reducible submodule (a minimal $\mathrm{SAff}_n(\mathbb{C})$-submodule) which intersects Q_0 trivially. This contradicts the maximality of Q_0.   □

Corollary 3.3.   *For a representation V of $\mathrm{SAff}_n(\mathbb{C})$ as above we have for $j \geqslant i$*

$$Q_j \subset Q_i \otimes \mathrm{Sym}^{j-i}(\mathbb{C}^n)^\vee$$

as $\mathrm{SL}_n(\mathbb{C})$-representations.

Proof.   It suffices to show that $Q_i \subset Q_0 \otimes \mathrm{Sym}^i(\mathbb{C}^n)^\vee$ follows from the previous Proposition 3.2. The general case follows by replacing V by V/V_{i-1}. We have the $\mathrm{SAff}_n(\mathbb{C})$-equivariant map

$$\iota : V \to V_0 \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee,$$

and, in particular, an induced $\mathrm{SL}_n(\mathbb{C})$-equivariant map

$$Q_i \to Q_0 \otimes \mathrm{Sym}^i(\mathbb{C}^n)^\vee,$$

which is the restriction of the previous map to Q_i composed with the $\mathrm{SL}_n(\mathbb{C})$-equivariant projection. We just have to prove it is nonzero on every irreducible summand S of Q_i. If it were zero on S then $Q_0 \oplus Q_1 \oplus \cdots \oplus Q_{i-1} \oplus S$, and also the $\mathrm{SAff}_n(\mathbb{C})$-submodule $\bar{S}$ generated by it, would be mapped under ι to the $\mathrm{SAff}_n(\mathbb{C})$-submodule $V_0 \otimes \mathrm{Sym}^{i-1}(\mathbb{C}^{n+1})^\vee$ of $V_0 \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee$. This would mean that $\bar{S}$ has a filtration

$$0 \subset \bar{S}_0 \subset \cdots \subset \bar{S}_{i-1} = \bar{S}$$

with completely reducible quotients, which contradicts the fact that the Q_{i+1} are the maximal completely reducible subrepresentations of V/V_i in each step (here we are using this fact in its full strength). $\qquad\square$

Let us now consider filtrations of type (5) and dualize the statements in Proposition 3.2 and Corollary 3.3.

Proposition 3.4.   *Let V be a representation of $\mathrm{SAff}_n(\mathbb{C})$ with a filtration of type (5):*

$$0 \subset V_0' \subset V_1' \subset \cdots \subset V_{l-1}' \subset V_l' = V$$

with $Q_{l-j}' := V_{l-j}'/V_{l-j-1}'$, for $j = 0, \ldots, l$, the maximal completely reducible quotient representation of V_{l-j}'. Then V is a quotient of $Q_l' \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})$ and for $i \leqslant j$,

$$Q_i' \subset Q_j' \otimes \mathrm{Sym}^{j-i}(\mathbb{C}^n)\,.$$

Proof.   The maximal completely reducible subrepresentation of $V^\vee$ is $(Q_l')^\vee$, so by Proposition 3.2, $V^\vee$ is a subrepresentation of $(Q_l')^\vee \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee$ from which the first assertion follows. By Remark 3.1 and Corollary 3.3, one has for $t \geqslant s$,

$$(Q_{l-t}')^\vee \subset (Q_{l-s}')^\vee \otimes \mathrm{Sym}^{t-s}(\mathbb{C}^n)^\vee,$$

from which the second assertion follows putting $i = l - t$, $j = l - s$ and dualizing. $\qquad\square$

4   Minimal subvarieties of Severi-Brauer varieties

References for the theory of Severi-Brauer varieties are [1, 5, 8]. Recall that a Severi-Brauer variety P over a field K is a K-form of the projective space, i.e., $P_{\bar{K}} \simeq \mathbb{P}_{\bar{K}}$, over an algebraic closure of K. Thus a fibration $X \to Y$ which is generically a projective bundle in the étale topology over Y gives rise to a Severi-Brauer variety over the function field $K = \mathbb{C}(Y)$. If A is an Azumaya (central simple) algebra of degree n over K (i.e., $A \otimes_K \bar{K} \simeq \mathrm{Mat}_{n \times n}(\bar{K})$), then the set of all minimal (i.e., dimension n) right ideals I of A is a closed subvariety $\mathbb{P}_A$ of $\mathrm{Grass}(n, A)$ defined by the conditions that I is a right ideal. This is a Severi-Brauer variety as can be seen from the fact that for $A \simeq \mathrm{Mat}_{n \times n}(K) = \mathrm{End}(V)$ the right ideals of dimension n are in bijective correspondence with $\mathbb{P}(V)$ by associating with a one-dimensional subspace l in the n-dimensional K-vector space V those $f \in \mathrm{End}(V)$ with image contained in l. Conversely, any Severi-Brauer variety arises in this way since both isomorphism classes of Severi-Brauer varieties of dimension $n-1$ over K and isomorphism classes of degree n Azumaya algebras over K are classified by the nonabelian Galois cohomology set $H^1(\mathrm{Gal}(\bar{K}/K), \mathrm{PGL}_n(\bar{K})) = H^1(K, \mathrm{PGL}_n(\bar{K}))$ (noting $\mathrm{PGL}_n(\bar{K}) = \mathrm{Aut}(\mathrm{Mat}_{n \times n}(\bar{K}))$, so the automorphism groups over $\bar{K}$ of K-forms of projective space and K-forms of matrices are the same). The inductive limit $H^1(K, \mathrm{PGL}_\infty)$ of the sets $H^1(K, \mathrm{PGL}_n)$ via the maps $H^1(K, \mathrm{PGL}_n) \to H^1(K, \mathrm{PGL}_{mn})$ (diagonal embedding) carries a natural group structure induced by the tensor product $\mathrm{PGL}_n \times \mathrm{PGL}_m \to \mathrm{PGL}_{m \times n}$. Then there is the isomorphism

$$H^1(K, \mathrm{PGL}_\infty) \simeq \mathrm{Br}(K)$$

with the Brauer group $\mathrm{Br}(K)$ of K, and each Severi-Brauer variety P has its class $[P] \in \mathrm{Br}(K)$. We need the following two lemmas linking the birational geometry and algebra of Severi-Brauer varieties.

Lemma 4.1.   *If D is a division algebra over K and $A = \mathrm{Mat}_{r \times r}(D)$ then the associated Severi-Brauer variety P_A over K is birational to the projectivisation of a vector bundle on P_D.*

Proof.   Let e be an idempotent with $eAe = D$ and let $P_A \to P_D$ be the map that sends a right ideal I in A to eIe. After base change to $\kappa = K(P_D)$, the generic point of P_D, this map can be identified with the projection

$$\pi : \mathbb{P}(V_\kappa^{\oplus r}) = P_A \otimes \kappa \to \mathbb{P}(V_\kappa) = P_D \otimes \kappa$$

onto a summand. Note that κ is a splitting field for both D and A. The generic fibre of $P_A \to P_D$ is the preimage of the point defined by the generic point in $P_D \otimes \kappa$ under π.   □

Lemma 4.2.   *If two Severi-Brauer varieties P_1 and P_2 over K are stably birationally isomorphic over K, then $[P_1]$ and $[P_2]$ generate the same subgroup of $\mathrm{Br}(K)$ and conversely. More precisely, if P_1 and P_2 are both of rank r, then*

$$P_1 \times_K \mathbb{P}^r \simeq P_2 \times_K \mathbb{P}^r,$$

where $\simeq$ denotes birational equivalence.

Proof.   This follows from Amitsur's theorem (Theorem 5.4.1 of [5]) that the kernel of $\mathrm{Br}(K) \to \mathrm{Br}(K(P))$ for a Severi-Brauer variety P over the function field K is generated by the class of $[P]$, and from the fact that $\mathrm{Br}(K(P)) \to \mathrm{Br}(K(P)(t))$ is injective for an indeterminate t. See [5, Remark 5.4.3] for the converse.   □

Lemma 4.3.   *Suppose S_1 is some rank r Severi-Brauer variety over K. Then the class $n[S_1] \in \mathrm{Br}(K)$ is also representable by a Severi-Brauer variety of rank r.*

Proof.   By hypothesis, S_1 corresponds to an Azumaya algebra A of degree $r + 1$ over K. The index of $A^{\otimes n}$ divides the index of A (see [5, Proposition 4.5.8]). Recall that the index of an Azumaya algebra is the degree of the unique division algebra in its Brauer equivalence class. In other words, there is an Azumaya algebra of the same degree as A which represents the class of $n[S_1]$.   □

Proposition 4.4.   *Let*

$$0 \to V_0 \to V \to Q_1 \to 0$$

be a two-step filtration of a generically free G-representation V, where G is the special affine group $\mathrm{SAff}_n(\mathbb{C})$ (this means here simply that V_0 is some completely reducible subrepresentation, and Q_1 is a completely reducible quotient). Assume that $\mathrm{SL}_n(\mathbb{C})/(\mathbb{Z}/m\mathbb{Z})$ acts generically freely on the quotient Q_1 for some $m \mid n$. Suppose moreover that $\dim V_0 \geqslant n^2 + 2n$. Then V/G is rational.

Proof.   Look at the fibre product diagram

$$
\begin{array}{ccc}
X & \longrightarrow & V'/\mathrm{SL}_n(\mathbb{C}) \\
\downarrow & & \downarrow \\
(Q_1 \oplus \mathbb{C}^n)/\mathrm{SL}_n(\mathbb{C}) & \longrightarrow & Q_1/\mathrm{SL}_n(\mathbb{C}),
\end{array}
$$

where $V' = V/U$. Here X is a vector bundle over both $V'/\mathrm{SL}_n(\mathbb{C})$ and $(Q_1 \oplus \mathbb{C}^n)/\mathrm{SL}_n(\mathbb{C})$. This follows from the no-name lemma of [4] because the action of $\mathrm{SL}_n(\mathbb{C})$ on both V' and $Q_1 \oplus \mathbb{C}^n$ is generically free. This means that if we divide out homotheties in the fibres and consider the Severi-Brauer varieties

$$S_1 : \mathbb{P}(V')/\mathrm{SL}_n(\mathbb{C}) \to Q_1/\mathrm{SL}_n(\mathbb{C}),$$
$$S_2 : (Q_1 \oplus \mathbb{P}(\mathbb{C}^n))/\mathrm{SL}_n(\mathbb{C}) \to Q_1/\mathrm{SL}_n(\mathbb{C}),$$

then S_1 and S_2 are stably equivalent Severi-Brauer varieties. By Lemma 4.2, S_1 and S_2 generate the same subgroup of $\mathrm{Br}(K)$, where $K = \mathbb{C}(Q_1/\mathrm{SL}_n(\mathbb{C}))$. By Lemma 4.3, the class of S_1 is also represented by some $\mathbb{P}^{n-1}$-bundle S'. By Lemma 4.2, S' and S_2 are stably equivalent of level $n - 1$. Moreover, S_2 is stably rational of level n^2. Furthermore, by Lemma 4.1, S_1 is birational to a vector bundle over S' provided its rank is bigger than $n - 1$. If the rank of S_1 is bigger than $n^2 + n - 1$, we consequently get rationality of S_1 from the stable rationality of level n^2 of S'. The latter follows because S_2 is stably rational of level n^2 and S' and S_2 are stably equivalent of level $n - 1$, and $n^2 \geqslant n - 1$. Together with rationality for S_1 we obtain of course also rationality of V/G. Finally the rank of S_1 is bigger than or equal to $\dim V_0 - n - 1$ (the subgroup $\mathbb{C}^n$ still acts via translations). Hence the assertion holds.   □

5 Rationality for 2-step extensions

Definition 5.1. *We call an $\mathrm{SL}_n(\mathbb{C})$-representation W bad if no group isogeneous to $\mathrm{SL}_n(\mathbb{C})$ (i.e., obtained from $\mathrm{SL}_n(\mathbb{C})$ by quotienting by a finite subgroup of its centre) acts generically freely on W. If $n > 9$ the irreducible bad representations are*

$$\Lambda^2(\mathbb{C}^n), \quad S^2(\mathbb{C}^n), \quad \mathbb{C}^n, \quad \mathbb{C}, \quad \mathrm{Ad}_0 = \Sigma^{(2,1,\ldots,1,\,0)}(\mathbb{C}^n)$$

or one of the duals (where Ad_0 is the trace zero part of the adjoint representation and Σ denotes the Schur functor). Every bad representation is a direct sum of bad irreducible representations. If W is not bad we will also say that W is good.

Remark 5.2. For given n there are — up to addition of trivial summands $\mathbb{C}$ — only finitely many bad representations. Indeed, every bad $\mathrm{SL}_n(\mathbb{C})$-representation is a direct sum of irreducible bad ones, and for each of the irreducible bad representations R other than $\mathbb{C}$ (they are all listed in an appendix table of [6]) it is true that there is a t such that $R^t = R \oplus \cdots \oplus R$ (t-times) is good.

Theorem 5.3. *If V is a generically free G-representation with a filtration of type (5) of length $l = 2$, then we can write V as $V = V_1 \oplus W$, where V_1 has a length 2 filtration*

$$0 \to S \to V_1 \to Q \to 0$$

with $S \subset Q \otimes \mathbb{C}^n$, $Q \subset S \otimes (\mathbb{C}^n)^\vee$, W is an $\mathrm{SL}_n(\mathbb{C})$-representation, and V_1 does not split off another $\mathrm{SL}_n(\mathbb{C})$-representation. Then V/G is rational if

(A) there exists an $\mathrm{SL}_n(\mathbb{C})$-equivariant decomposition $W = W_1 \oplus W_2$ such that $Q \oplus W_2$ is good and $\dim(S \oplus W_1) \geqslant n^2 + 2n$; or

(B) Q contains $\geqslant n^2 - 1$ copies of $\mathbb{C}$.

Proof. Assertion (A) of the theorem follows from Proposition 4.4. For Part (B), remark that if Q does contain more than $n^2 - 1$ copies of $\mathbb{C}$ then there is a natural $(G, \mathrm{SL}_n(\mathbb{C}))$-section for the action of G on V. Namely, writing $Q := Q' \oplus \mathbb{C}$, the action of U on V gives an $\mathrm{SL}_n(\mathbb{C})$-equivariant map

$$U \simeq \mathbb{C}^n \to \mathrm{Hom}(Q' \oplus \mathbb{C}, S) \to \mathrm{Hom}(\mathbb{C},\, S) \simeq S,$$

and the map of $\mathbb{C}^n$ to S cannot be zero, hence must be an inclusion, since otherwise the action of U on the summand $\mathbb{C}$ in Q would be trivial and hence S would not be maximal with the property of being a completely reducible subrepresentation of V_1. So $S = S' \oplus \mathbb{C}^n$, and as $(G, \mathrm{SL}_n(\mathbb{C}))$-section we take $S' \oplus Q \oplus W$. Now $Q = Q'' \oplus m\mathbb{C}$ with $m \geqslant n^2 - 1$. By assumption, $\mathrm{SL}_n(\mathbb{C})$ operates generically freely on $S' \oplus Q \oplus W$ and therefore also on $S' \oplus Q'' \oplus W$. But then $(S' \oplus Q \oplus W)/\mathrm{SL}_n(\mathbb{C})$ is generically a rank m vector bundle over $(S' \oplus Q'' \oplus W)/\mathrm{SL}_n(\mathbb{C})$ which is in turn stably rational of level $n^2 - 1$. It follows that V is also rational in this case. $\square$

Remark 5.4. There are cases of generically free G-representations V which do not satisfy the hypotheses of Theorem 5.3 and for which a proof of rationality for V/G seems to be a difficult problem: take $V = \mathbb{C}^{n+1} \oplus W$, where W is an irreducible generically free $\mathrm{SL}_n(\mathbb{C})$-representation. Then, as in the preceding proof, V has a $(G, \mathrm{SL}_n(\mathbb{C}))$-section $\mathbb{C} \oplus W$. Thus, if we could prove rationality for V/G, we would obtain stable rationality of level 1 for $W/\mathrm{SL}_n(\mathbb{C})$, which is expected to be a hard problem in general.

Definition 5.5. *We call a G-representation V with a length two filtration exceptional if it does not satisfy the hypotheses of Theorem 5.3. Hence we cannot conclude that V/G is rational immediately by the methods here. More precisely, if we write $V = V_1 \oplus W$ as above, this means that either V is not generically free or it is so but*

(1) for all decompositions $W = W_1 \oplus W_2$, $Q \oplus W_2$ is bad or $\dim S \oplus W_1 < n^2 + 2n$; and

(2) Q contains $< n^2 - 1$ copies of $\mathbb{C}$.

For later use, and to characterize exceptional two-step representations, we prove the following technical

Lemma 5.6.   *Let*

$$0 \to S \to V \to Q \to 0$$

be an exact sequence of G-representations with S the maximal completely reducible submodule of V. Assume that

(a) *the maximal completely reducible quotient of Q is good as $\mathrm{SL}_n(\mathbb{C})$-representation;*

(b) *the maximal completely reducible subrepresentation of Q is not one of the following finitely many* (*bad*) $\mathrm{SL}_n(\mathbb{C})$*-representations:*

$$R_1 = \mathbb{C}^n \ or \ R_2 = \Lambda^2(\mathbb{C}^n)^\vee \ or \ R_3 = \mathbb{C} \oplus \cdots \oplus \mathbb{C} \oplus (\mathbb{C}^n)^\vee \oplus \cdots \oplus (\mathbb{C}^n)^\vee$$

(*at most $(n-1)$-summands in total in R_3*).

Then V is generically free as G-representation.

Proof.   From (a) one gets that V is generically free for $\mathrm{SL}_n(\mathbb{C})$. We argue by contradiction and suppose that V is not generically G-free. This means that for general $v \in V$ there exists an element $(A, t) \in G = \mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$ with $t \neq 0$ such that

$$\begin{pmatrix} \varphi_1(A) & \psi(A, t) \\ 0 & \varphi_2(A, t) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where v_1 and v_2 are the components of v w.r.t. the $\mathrm{SL}_n(\mathbb{C})$-equivariant splitting $V = S \oplus Q$ and $\varphi_1(A)$, $\varphi_2(A, t)$ and $\psi(A, t)$ denote the components of (A, t) relative to the representation V in $\mathrm{End}(S)$, $\mathrm{End}(Q)$ and $\mathrm{Hom}(Q, S)$, respectively. Since a group isogeneous to $\mathrm{SL}_n(\mathbb{C})$ acts generically freely on the maximal completely reducible quotient of Q, we obtain that A is multiplication by some root of unity ζ_A. Thus there exists an element $v_1^0 \in S$ such that for almost all $v_2 \in Q$ we get the equation

$$\psi(A, t)v_2 = (1 - \varphi_1(A))v_1^0.$$

We consider the variety

$$M := \{(A, t, v_2) \in \mathbb{C}^{n \times n} \times (\mathbb{C}^n \backslash \{0\}) \times Q \mid \psi(A, t)v_2 = (1 - \varphi_1(A))v_1^0\}.$$

By the discussion above, the projection to Q of M is dominant, and since M is the union of components M_ζ corresponding to different ζ's, one of them will dominate Q, so that we can consider $\zeta = \zeta_0$ and also $A = A_0$ as fixed. Now consider the projection

$$p : M_{\zeta_0} \to \mathbb{P}(\mathbb{C}^n) = \mathbb{P}(U),$$
$$(A_0, t, v_2) \mapsto [t].$$

For a $[t_0]$ in the image of p with maximal fibre dimension, we get

$$\dim p^{-1}(t_0) \geqslant \dim M_{\zeta_0} - (n-1) \geqslant \dim Q - (n-1).$$

And since for $w \in p^{-1}(t_0)$ fixed and general $v_2 \in p^{-1}(t_0)$ we have $\psi(A_0, t_0)(v_2 - w) = 0$, we get

$$\dim \ker(\psi(A_0, t_0)) \geqslant \dim p^{-1}(t_0) \geqslant \dim Q - (n-1).$$

We may view $\psi(A_0, \cdot)$ as a family of $\mathrm{SL}_n(\mathbb{C})$-equivariant homomorphisms

$$\psi_i(A_0, \cdot) : \mathrm{Sym}^i(\mathbb{C}^n) \to \mathrm{Hom}(Q_i, S)$$

for i between 1 and the filtration length L of V (thus there is a natural grading). By $\mathrm{SL}_n(\mathbb{C})$-equivariance

$$\dim \ker(\psi(A_0, t)) \geqslant \dim Q - (n-1)$$

for all t. We now work on $\mathbb{P}^{n-1} = \mathbb{P}(U)$, and, since $H^0(\mathbb{P}(U), \mathcal{O}(1)) = U^\vee$, we may view ψ as giving rise to maps of vector bundles

$$0 \longrightarrow \ker(\psi)(A_0, \cdot) \longrightarrow \bigoplus_{i=1}^L Q_i \otimes \mathcal{O}(-i+1) \xrightarrow{\psi(A_0, \cdot)} S \otimes \mathcal{O}(1),$$

where $\ker(\psi)(A_0, \cdot)$ is a vector bundle by $\mathrm{SL}_n(\mathbb{C})$-equivariance. We will suppress A_0 from the notation in the sequel and restate our basic inequality in the form

$$\mathrm{rk}(\ker(\psi)) + \dim \mathbb{P}(U) \geqslant \dim Q. \tag{7}$$

Now factor ψ:

$$\bigoplus_{i=1}^L Q_i \otimes \mathcal{O}(-i+1) \xrightarrow{\alpha} \mathrm{im}(\psi) \xrightarrow{\beta} S \otimes \mathcal{O}(1).$$

Since by (7) we have $n - 1 \geqslant \mathrm{rk}(\mathrm{im}(\psi))$ one can only have

$$\mathrm{im}(\psi) = \mathcal{T}_{\mathbb{P}^{n-1}}(k) \quad \text{or} \quad \mathrm{im}(\psi) = \Omega^1_{\mathbb{P}^{n-1}}(k) \quad \text{or} \quad \bigoplus_{i=1}^l \mathcal{O}(k_i), \quad l \leqslant n - 1.$$

We will now narrow down the number of possibilities for $\mathrm{im}(\psi)$ even further. The map induced by the bundle map ψ on the H^0-level corresponds to the map $\mathbb{C}^n \to \mathrm{Hom}(Q_1, S)$, where Q_1 is the maximal completely reducible subrepresentation of Q, thus is nonzero and hence

$$H^0(\mathrm{im}(\psi)) \neq 0 \quad \text{and} \quad H^0((\mathrm{im}(\psi))^\vee(1)) \neq 0. \tag{8}$$

Using (8), we find that $\mathrm{im}(\psi)$ can only be one of the following:

$$\mathcal{T}(-1), \quad \Omega^1(2), \quad (X_0 \otimes \mathcal{O}) \oplus (X_1 \otimes \mathcal{O}(1)) \oplus \cdots \oplus (X_{L-1} \otimes \mathcal{O}(-(L-1))).$$

Here $X_0, \ldots, X_{L-1}$ are some $\mathrm{SL}_n(\mathbb{C})$-representations since we are looking at homogeneous vector bundles. We have $H^0(\Omega(2)) = \Lambda^2(\mathbb{C}^n)^\vee$, $H^0(\mathcal{T}(-1)) = \mathbb{C}^n$, and all X's must be direct sums of trivial representations $\mathbb{C}$ since $\mathrm{rk}(\mathrm{im}(\psi)) \leqslant n - 1$. We will argue that none of these cases can actually occur under the hypotheses of the lemma. For in the induced sequence

$$Q_1 \otimes \mathcal{O} \xrightarrow{\alpha} \mathrm{im}(\psi) \xrightarrow{\beta} S \otimes \mathcal{O}(1),$$

the arrows α and β are equivariant. Hence it follows in each of the three cases above that Q_1 contains $R_1 = \mathbb{C}^n$ or $R_2 = \Lambda^2(\mathbb{C}^n)^\vee$ or a direct sum

$$R_3 = \mathbb{C} \oplus \ldots \mathbb{C} \oplus (\mathbb{C}^n)^\vee \oplus \cdots \oplus (\mathbb{C}^n)^\vee$$

(at most $(n-1)$-copies in total since $\mathrm{rk}(\mathrm{im}(\psi)) \leqslant n - 1$) and the arrow α is induced by the arrows in the Euler sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathbb{C}^n \otimes \mathcal{O} \longrightarrow \mathcal{T}_{\mathbb{P}^{n-1}}(-1) \longrightarrow 0,$$

and the identity map $\mathcal{O} \to \mathcal{O}$. Hence there are two possibilities:

• Q_1 is equal to R_1, R_2 or R_3 which is impossible by Assumption (b).

• $Q_1 \otimes \mathcal{O}$ splits as $(R_j \oplus Q_1') \otimes \mathcal{O}$, and Q_1' is a nonzero subrepresentation of Q_1 which is in the kernel of ψ. This contradicts the hypothesis that S is maximal with the property of being completely reducible inside V. □

Corollary 5.7. *Let $V = V_1 \oplus W$, $0 \to S \to V_1 \to Q \to 0$, be an exceptional two step representation (notation as in Theorem 5.3). If Q contains $< n^2 - 1$ summands of $\mathbb{C}$, then there are only finitely many possibilities for Q and S for any fixed n.*

Proof. Suppose V is exceptional because the action of G on it, hence on V_1, is not generically free. Then by Lemma 5.6, Q must be bad and by Remark 5.2, since it contains $< n^2 - 1$ summands of $\mathbb{C}$, there are only finitely many possibilities for Q, hence since $S \subset Q \otimes \mathbb{C}^n$ also finitely many possibilities for S. If V is generically free, but (1) of Definition 5.5 is satisfied, then the dimension of S must be $< n^2 + 2n$

or Q must be bad, which in view of $S \subset Q \otimes \mathbb{C}^n$ and $Q \subset S \otimes (\mathbb{C}^n)^\vee$ again limits both S and Q to finitely many possibilities.   □

Corollary 5.8.   *If V is a G-representation of large enough filtration length l (with respect to filtration types* (4) *or* (5)*), then V is generically free.*

Proof.   There are two cases.

(1) Q_l in the type (4)-filtration for V is good. Then $Q_{l-1} \to V/V_{l-2} \to Q_l$ is a quotient of V which is generically free by Lemma 5.6, so V is generically free.

(2) Q_l is bad, a sum of irreducible bad representations. The G-span of Q_l is contained in $Q_l \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})$. We use the following immediate consequence of the Littlewood-Richardson rule:

For an irreducible $\mathrm{SL}_n(\mathbb{C})$-representation $W = \Sigma^{(\lambda_1,\ldots,\lambda_n)}(\mathbb{C}^n)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ a non-increasing sequence of non-negative integers, put $\lambda(W) := \lambda_1(W) - \lambda_2(W)$. Then, if U is an irreducible summand of $W \otimes \mathrm{Sym}^k(\mathbb{C}^n)$, we have $\lambda(U) \geqslant k - \lambda_1(W)$.

Since λ and λ_1 are bounded on irreducible bad representations, we see that if l is sufficiently large, then we can assume the following properties for V:

(a) V is an extension $V_1 \to V \to V_2$ of G-representations with V_1 generically free for the action of $\mathrm{SL}_n(\mathbb{C})$ (we can also take V_1 as a length two G-representation).

(b) the quotient V_2 of V has a length two subrepresentation which is generically free for G (use Lemma 5.6).

By upper-semicontinuity of the dimension of the stabilizers, the generic stabilizer in V_2 is finite. Suppose that the generic stabilizer in V are nontrivial. Then, denoting by $v = (v_1, v_2)$ the decomposition of a vector $v \in V$ with respect to the $\mathrm{SL}_n(\mathbb{C})$-equivariant splitting $V = V_1 \oplus V_2$, we get: there exists a (fixed) $v_2 \in V_2$ such that for general $v_1 \in V_1$ the stabilizer G_v of $v = (v_1, v_2)$ inside G is finite and nontrivial, and is contained in the (fixed) finite group $G_{v_2} \subset G$. Since $\mathrm{SL}_n(\mathbb{C})$ is a reductive Levi subgroup in G, hence maximal reductive, and G_{v_2} is finite, there is an element $t \in U = \mathbb{C}^n$ (the unipotent radical) such that $t G_{v_2} t^{-1} \subset \mathrm{SL}_n(\mathbb{C})$. Hence we get, replacing v by $t \cdot v = (v_1', v_2')$, that there is a $v_2' \in V_2$ such that for general $v_1' \in V_1$ the stabilizer of (v_1', v_2') in G is a nontrivial finite subgroup of $\mathrm{SL}_n(\mathbb{C})$ which contradicts the property of V_1 in (a) (being generically free for $\mathrm{SL}_n(\mathbb{C})$).   □

6   Rationality if the representation dimension is large

The aim of this section is to prove

Theorem 6.1.   *Let V be a generically free indecomposable representation of the special affine group $G = \mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$. Then there is a constant $k = k(n)$ depending on n such that if $\dim V \geqslant k(n)$, then V/G is rational.*

We consider exclusively filtrations of type (5) for V in this section: recall that this is a filtration

$$0 \subset V_0' \subset V_1' \subset \cdots \subset V_{l-1}' \subset V_l' = V,$$

defined inductively as follows: $Q_{l-j}' := V_{l-j}'/V_{l-j-1}'$, for $j = 0,\ldots,l$, is the maximal completely reducible quotient representation of V_{l-j}'. The indecomposability assumption on V cannot be dropped due to Remark 5.4. The proof of Theorem 6.1 will be preceded by some lemmas.

Lemma 6.2.   *Fix n and the filtration length l of V. Suppose that $\tilde{V} = V/V_{l-2}'$ is an exceptional two-step extension $\tilde{V} = V_1 \oplus W$, $0 \to S \to V_1 \to Q \to 0$ (notations similar to Theorem 5.3, so W is an $\mathrm{SL}_n(\mathbb{C})$-representation and $S \subset Q \otimes \mathbb{C}^n$, $Q \subset S \otimes (\mathbb{C}^n)^\vee$) and $Q \oplus W$ contains $< n^2 - 1$ copies of $\mathbb{C}$. Suppose that V is indecomposable, and write $W = W_1 \oplus \cdots \oplus W_k$ where the W_i's are defined inductively as follows*:

• W_1 *contains all irreducible summands W' of W such that for the G-spans inside V we have $\langle G \cdot W' \rangle \cap \langle G \cdot Q \rangle \neq 0$.*

• W_{j+1} *contains all irreducible summands W'' of W which are not already in $W_1,\ldots,W_j$ and satisfy*

$$\langle G \cdot W'' \rangle \cap \langle G \cdot (Q + W_1 + \cdots + W_j) \rangle \neq 0$$

(so in fact $\langle G \cdot W'' \rangle \cap \langle G \cdot W_j \rangle \neq 0$, and then, by the indecomposability of V, we have $W = W_1 \oplus \cdots \oplus W_k$).

Then for $s \in \mathbb{N}$ there is a constant $c(s)$ such that if $\dim V \geqslant c(s)$ then $k \geqslant s$, and for $\dim V$ sufficiently large, V has a generically free G-quotient $\hat{V}$ such that $V \to \hat{V}$ has fibre dimension larger than $n^2 + n - 1$, so that V/G is rational.

Proof.   By Corollary 5.7 we know already that S and Q are limited to finitely many possibilities, so we have to show the same for the W's. By Propositions 3.2 and 3.4 we obtain inclusions of $\mathrm{SL}_n(\mathbb{C})$-representations,

$$\langle G \cdot Q \rangle \subset Q \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1}),$$
$$W_1 \subset Q \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1}) \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee,$$
$$W_2 \subset Q \otimes (\mathrm{Sym}^l(\mathbb{C}^{n+1}) \otimes \mathrm{Sym}^l(\mathbb{C}^{n+1})^\vee)^{\otimes 2},$$

and so forth, so that together with the possibilities for Q, also those for the W's are limited. Thus for $\dim V$ to become arbitrarily large, we need k to become very large, i.e., W is highly decomposable. Here one should note that this does not necessarily imply that eventually $\tilde{V}$ is no longer exceptional: the action of G on $\tilde{V}$ cannot be generically free and remain so after adding arbitrary nontrivial $\mathrm{SL}_n(\mathbb{C})$-summands to $\tilde{V}$, e.g., if $\tilde{V} = (\mathbb{C}^{n+1})^\vee$ or, more generally, there are some nontrivial translations in the stabilizer in general position. So we have to resort to some other type of argument here, namely we show directly that if k becomes very large, then V has a generically free quotient $\hat{V}$ with large fibre dimension as claimed in the statement of the lemma. Note that there can only be finitely many trivial summands $\mathbb{C}$ by hypothesis among the W's, so that if k becomes large, $W = W_1 \oplus \cdots \oplus W_k$, will eventually be a good $\mathrm{SL}_n(\mathbb{C})$-representation. Together with W, the maximal completely reducible subrepresentation V_0 of V becomes large and highly reducible by the construction of the W_i's. An $\mathrm{SL}_n(\mathbb{C})$-subrepresentation R_0 of V_0 gives rise to a quotient $\hat{V}_{R_0}$ of V as follows: take the G-span of $R_0^\vee \subset V^\vee$ (inclusion as $\mathrm{SL}_n(\mathbb{C})$-representation) and take the dual of this span. For k large, we may choose R_0 in such a way that $W_1 \oplus \cdots \oplus W_\kappa$ is contained (as $\mathrm{SL}_n(\mathbb{C})$-representation) in $\hat{V}_{R_0}$ modulo its maximal completely reducible subrepresentation, and $W_1 \oplus \cdots \oplus W_\kappa$ is good, so that $\hat{V}_{R_0}$ is G-generically free by Lemma 5.6 (note that Condition (b) of this lemma will be automatically satisfied if the dimension of $W_1 \oplus \cdots \oplus W_\kappa$ is sufficiently large, using Proposition 3.2, since the filtration length of $\hat{V}_{R_0}$ modulo its maximal completely reducible subrepresentation is bounded with l). For large k, we can then also arrange that the fibre dimension of $V \to \hat{V}_{R_0}$ becomes large, since $\dim V_0$ grows with k, and V_0 becomes highly reducible. $\qquad\square$

Remark 6.3.   The complicated procedure used in the proof of the previous Lemma 6.2 is justified by the complexity of G-representations for which we want to give some examples:

• The examples of $(\mathbb{C}^n)^\vee \otimes (\mathbb{C}^{n+1})^\vee$ and its dual show that V can have reducible V_0 or Q_l without being decomposable.

• Consider the subrepresentation $V \subset (\mathbb{C}^n)^\vee \otimes \mathrm{Sym}^2(\mathbb{C}^{n+1})^\vee$ with

$$Q_0 = (\mathbb{C}^n)^\vee, \quad Q_1 = \mathrm{Sym}^2(\mathbb{C}^n)^\vee \oplus \Lambda^2(\mathbb{C}^n)^\vee, \quad Q_2 = \mathrm{Sym}^3(\mathbb{C}^n)^\vee.$$

Here $\Lambda^2(\mathbb{C}^n)^\vee$ is not in the G-span of $\mathrm{Sym}^3(\mathbb{C}^n)^\vee$ and

$$Q_0' = (\mathbb{C}^n)^\vee, \quad Q_1' = \mathrm{Sym}^2(\mathbb{C}^n)^\vee, \quad Q_2' = \mathrm{Sym}^3(\mathbb{C}^n)^\vee \oplus \Lambda^2(\mathbb{C}^n)^\vee.$$
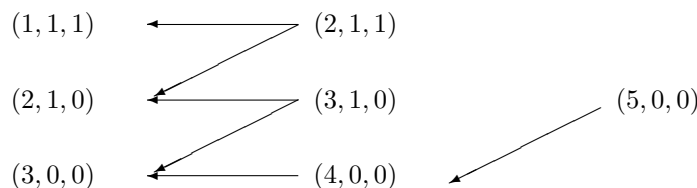
• Consider the subrepresentation V of

$$(\mathrm{Sym}^3(\mathbb{C}^n)^\vee \oplus \Sigma^{2,1}(\mathbb{C}^n)^\vee \oplus \Lambda^3(\mathbb{C}^n)^\vee) \otimes \mathrm{Sym}^2(\mathbb{C}^{n+1})^\vee$$

with type (4) filtration such that

$$Q_0 = \mathrm{Sym}^3(\mathbb{C}^n)^\vee \oplus \Sigma^{2,1}(\mathbb{C}^n)^\vee \oplus \Lambda^3(\mathbb{C}^n)^\vee,$$
$$Q_1 = \mathrm{Sym}^4(\mathbb{C}^n)^\vee \oplus \Sigma^{3,1}(\mathbb{C}^n)^\vee \oplus \Sigma^{2,1,1}(\mathbb{C}^n)^\vee,$$
$$Q_2 = \mathrm{Sym}^5(\mathbb{C}^n)^\vee.$$

Diagrammatically, we can picture which of these summands map to which ones under the translations in $\mathbb{C}^n$ as follows:

$$(1,1,1) \longleftarrow (2,1,1)$$
$$(2,1,0) \longleftarrow (3,1,0) \qquad (5,0,0)$$
$$(3,0,0) \longleftarrow (4,0,0)$$

Thus the G-span S_1 of $\mathrm{Sym}^5(\mathbb{C}^n)^\vee$ intersects the span S_2 of $\Sigma^{3,1}(\mathbb{C}^n)^\vee$ nontrivially, and S_2 intersects the G-span S_3 of $\Sigma^{2,1,1}(\mathbb{C}^n)^\vee$ nontrivially, but $S_1 \cap S_3 = 0$. Moreover, the filtration of V of type (5) has

$$Q_0' = \mathrm{Sym}^3(\mathbb{C}^n)^\vee,$$
$$Q_1' = \mathrm{Sym}^4(\mathbb{C}^n)^\vee \oplus \Sigma^{2,1}(\mathbb{C}^n)^\vee \oplus \Lambda^3(\mathbb{C}^n)^\vee,$$
$$Q_2' = \mathrm{Sym}^5(\mathbb{C}^n)^\vee \oplus \Sigma^{3,1}(\mathbb{C}^n)^\vee \oplus \Sigma^{2,1,1}(\mathbb{C}^n)^\vee.$$

We now turn to the proof of Theorem 6.1.

Proof.   Recall that we use filtrations of type (5) throughout this proof. The theorem is true for two-step filtrations by Theorem 5.3, so we can suppose that the filtration length l of V satisfies $l \geqslant 3$. V is supposed to be generically free and indecomposable, and we will distinguish cases according to the type of two-step extension $\tilde{V} = V/V_{l-2}'$, $\tilde{V} = V_1 \oplus W$ (as in Lemma 6.2) which V has as quotient.

1) For $0 \to S \to V_1 \to Q \to 0$, we have that $Q \oplus W$ contains $\geqslant n^2 - 1$ copies of $\mathbb{C}$. Then we obtain rationality for V/G by taking a $(G, \mathrm{SL}_n(\mathbb{C}))$-section as in the proof of Theorem 5.3. So we suppose $Q \oplus W$ contains $< n^2 - 1$ copies of $\mathbb{C}$ in the following.

2) $\tilde{V}$ is not exceptional. Then $\tilde{V}/G$ is rational by Theorem 5.3, and V/G is generically a vector bundle over it (as G acts generically freely on $\tilde{V}$), so V/G is rational.

3) $\tilde{V}$ is exceptional (and $Q \oplus W$ is assumed to contain $< n^2 - 1$ copies of $\mathbb{C}$ by Step 1). Then by Lemma 6.2, it only remains to consider the case where the filtration length l becomes large. In this case we get rationality of V/G from the stable rationality of generically free G-representations and by Corollary 5.8. $\square$

References

1   Artin M. Brauer Severi Varieties (notes by Verschoren A.). In: Brauer Groups in Ring Theory and Algebraic Geometry. Lect Notes in Math, vol. 917. Berlin: Springer-Verlag, 1982, 194–210

2   Bogomolov F. Holomorphic tensors and vector bundles on projective varieties. Math USSR Izvestija, 1979, 13: 499–555

3   Bogomolov F. Stable rationality of quotient varieties by simply connected groups. Mat Sbornik, 1986, 130: 3–17

4   Bogomolov F, Katsylo P. Rationality of some quotient varieties. Mat Sbornik, 1985, 126: 584–589

5   Gille P, Szamuely T. Central Simple Algebras and Galois Cohomology. Cambridge Studies in Advanced Mathematics, vol. 101. Cambridge: Cambridge University Press, 2006

6   Popov V L, Vinberg E B. Invariant Theory. In: Parshin A N, Shafarevich I R, eds. Algebraic Geometry IV. Encyclopedia of Mathematical Sciences, vol. 55. Berlin: Springer-Verlag, 1994

7   Rosenlicht M. Some basic theorems on algebraic groups. Amer J Math, 1956, 78: 401–443

8   Saltman D. Lectures on Division Algebras. CBMS Regional Conference Series in Mathematics, vol. 94. Providence, RI: Amer Math Soc, 2009

9   Specht W. Darstellungstheorie der affinen Gruppe. Math Z, 1938, 43: 120–160