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The master T-operator for vertex models with trigonometric R-matrices as classical tau-function

A. Zabrodin *

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Abstract

The construction of the master T-operator recently suggested in [1] is applied to integrable vertex models and associated quantum spin chains with trigonometric R-matrices. The master T-operator is a generating function for commuting transfer matrices of integrable vertex models depending on infinitely many parameters. At the same time it turns out to be the tau-function of an integrable hierarchy of classical soliton equations in the sense that it satisfies the the same bilinear Hirota equations. The class of solutions of the Hirota equations that correspond to eigenvalues of the master T-operator is characterized and its relation to the classical Ruijsenaars-Schneider system of particles is discussed.

1 Introduction

The master T-operator was recently introduced in [1]. It is a generating function for commuting transfer matrices of integrable vertex models and associated quantum spin chains which unifies the transfer matrices on all levels of the nested Bethe ansatz and Baxter's Q-operators in one commuting family. It was also proven in [1] that the master T-operator, as a function of infinitely many auxiliary parameters (one of which being the usual spectral parameter), satisfies the same hierarchy of bilinear Hirota equations as the classical τ-function does. Since the operator-valued generating functions commute for all values of the auxiliary parameters, there is no problem with their ordering in the bilinear equations.

A similarity between quantum transfer matrices and classical τ-functions was first pointed out in [2] (see also [3]), where a discrete integrable dynamics in the space of commuting integrals of motion of a quantum integrable model was introduced. This classical dynamics was identified with the discrete 3-term Hirota equation with special boundary conditions. The diagonalization of transfer matrices by means of the nested Bethe ansatz technique was shown to be equivalent to an "undressing" chain of Bäcklund

*Institute of Biochemical Physics, 4 Kosygina st., 119334, Moscow, Russia and ITEP, 25 B.Cheremushkinskaya, 117218, Moscow, Russia and NRU-HSE, Vavilova str. 7, 117312 Moscow, Russia

transformations for the discrete Hirota equation. Later this approach was extended to supersymmetric integrable models [4]. An essential further step was made in the important paper [5], where an operator realization of the Bäcklund flow describing the "undressing" process was constructed for generalized quantum spin chains with rational $GL(N)$-invariant R-matrices. In fact the master T-operator was already used implicitly in that construction. A more explicit and more general definition was given in [1].

In this paper we review the construction of [1] trying to avoid technical details. Here we deal with the class of integrable lattice vertex models of statistical mechanics with trigonometric R-matrices. The main claim is that the master T-operator for these models is a τ-function of the classical MKP hierarchy.

We also characterize the class of solutions of the Hirota equations that correspond to eigenvalues of the master T-operator and make explicit the close connection with the classical Ruijsenaars-Schneider system of particles [6] which emerges as the dynamical system for zeros of the (eigenvalues of) the master T-operator. In an equivalent way, the connection emerges from the Baker-Akhiezer function for the Ruijsenaars-Schneider system which generates the algebra of commuting operators (transfer matrices) for the vertex model (the Bethe algebra). As is well known, the Ruijsenaars-Schneider model is a limiting case of the Calogero-Moser system of particles. In this connection let us note that a similar relation between the quantum Gaudin model (which can be regarded as a degeneration of quantum spin chains or vertex models with rational R-matrices) and classical Calogero-Moser system was found in [7] from a different reasoning.

2 The transfer matrices

We consider generalized quantum integrable vertex models with trigonometric R-matrix. The simplest R-matrix is the operator in $\mathbb{C}^N \otimes \mathbb{C}^N$ of the form

$$\begin{aligned}\mathsf{R}(u) = (e^{\gamma(u+1)} - e^{-\gamma(u+1)}) \sum_{a=1}^{N} e_{aa} \otimes e_{bb} + (e^{\gamma u} - e^{-\gamma u}) \sum_{1\le a\neq b\le N} e_{aa} \otimes e_{bb} \\ + (e^{\gamma} - e^{-\gamma}) \sum_{1\le a\neq b\le N} e^{\mathrm{sign}(b-a)\gamma u} e_{ab} \otimes e_{ba}.\end{aligned} \tag{2.1}$$

Here $u \in \mathbb{C}$ is the spectral parameter and e_{ab} denotes the $N \times N$ matrix with 1 in position (a, b) and 0 elesewhere. The deformation (anisotropy) parameter γ is assumed to be such that $q = e^{\gamma}$ is not a root of unity. Following the tradition, we call this R-matrix *trigonometric* although the coefficients are hyperbolic functions of u like $\sinh \gamma u$. Let $V_i = \mathbb{C}^N$ be copies of the space $\mathbb{C}$, then by $\mathsf{R}_{ij}(u)$ denote the R-matrix acting in $V_i \otimes V_j$. The R-matrix (2.1) satisfies the Yang-Baxter equation

$$\mathsf{R}_{12}(u_1 - u_2)\mathsf{R}_{13}(u_1 - u_3)\mathsf{R}_{23}(u_2 - u_3) = \mathsf{R}_{23}(u_2 - u_3)\mathsf{R}_{13}(u_1 - u_3)\mathsf{R}_{12}(u_1 - u_2), \tag{2.2}$$

where the both sides are operators in $V_1 \otimes V_2 \otimes V_3$. For any diagonal $N \times N$ matrix g set $g_1 = g \otimes \mathbb{I}$, $g_2 = \mathbb{I} \otimes g$, then the R-matrix commutes with $g_1 g_2$:

$$\mathsf{R}_{12}(u)\, g_1 g_2 = g_1 g_2\, \mathsf{R}_{12}(u). \tag{2.3}$$

This property will be referred to as g-invariance of the R-matrix.

Fix a diagonal matrix $g = \mathrm{diag}\,(p_1, p_2, \ldots, p_N)$. We call it the twist matrix. Below we assume that all $p_i \in \mathbb{C}$ are in general position, i.e., $p_i/p_j \neq e^{2\gamma n}$ for any $i \neq j$ and any integer n. The transfer matrix for the vertex model with twisted boundary conditions and with inhomogeneity parameters u_i at each site is defined as

$$T(u) = \mathrm{tr}_0 \Big(\mathsf{R}_{10}(u - u_1)\mathsf{R}_{20}(u - u_2) \ldots \mathsf{R}_{L0}(u - u_L)\, g \Big). \tag{2.4}$$

The R-matrices and g are mulitiplied as matrices in the common space V_0 (the auxiliary space). Trace tr_0 is taken in the auxiliary space. The result is an operator acting in the tensor product of vector representations $\mathcal{H} = \otimes_{j=1}^{L} V_j = \mathbb{C}^{\otimes L}$ (the quantum space). Formally, our setting includes also models with higher representations at the sites because they can be obtained by "fusing" several vector representations with properly chosen parameters u_i. By construction, the operator (2.4) is a Laurent polynomial in $e^{\gamma u}$.

It follows from the Yang-Baxter equation and from the g-invariance of the R-matrix that the transfer matrices for models with the same γ and g commute for all u and can be diagonalized simultaneously. Their diagonalization is the basic problem of the theory of vertex models. The standard method is the nested Bethe ansatz technique.

The full commutative family of operators in the quantum space is in general larger than the one generated by coefficients of $T(u)$. The algebraic construction of higher commuting transfer matrices essentially relies on representation theory of the q-deformed algebras $U_q(\widehat{gl}(N))$ and $U_q(gl(N))$ (see, e.g., [10, 11, 12, 13]).

The algebra $U_q(gl(N))$ has generators L^+_{ab} with $1 \leq a \leq b \leq N$ and L^-_{ab} with $1 \leq b \leq a \leq N$ such that $L^+_{aa}L^-_{aa} = L^-_{aa}L^+_{aa} = 1$. Combining them into matrices

$$\mathsf{L}^+ = \sum_{a \leq b} e_{ab} \otimes L^+_{ab}, \quad \mathsf{L}^- = \sum_{a \geq b} e_{ab} \otimes L^-_{ab}$$

with $U_q(gl(N))$-valued matrix elements, one can represent the defining relations of the algebra in the form [14]

$$\mathsf{R}_{12}\mathsf{L}^\pm_1\mathsf{L}^\pm_2 = \mathsf{L}^\pm_2\mathsf{L}^\pm_1\mathsf{R}_{12}, \quad \mathsf{R}_{12}\mathsf{L}^+_1\mathsf{L}^-_2 = \mathsf{L}^-_2\mathsf{L}^+_1\mathsf{R}_{12}$$

with the u-independent R-matrix

$$\mathsf{R}_{12} = \lim_{e^{\gamma u} \to \infty} (e^{-\gamma u}\mathsf{R}(u)) = q\sum_{a=1}^{N} e_{aa} \otimes e_{bb} + \sum_{1 \leq a \neq b \leq N} e_{aa} \otimes e_{bb} + (q - q^{-1}) \sum_{1 \leq a < b \leq N} e_{ab} \otimes e_{ba}.$$

The diagonal elements $L^\pm_{aa}$ can be understood as exponents of the commuting Cartan generators h_a:

$$L^\pm_{aa} = q^{\pm \mathsf{h}_a}.$$

Let π_λ be the irreducible finite-dimensional representation of $U_q(gl(N))$ with the highest weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ such that $\lambda_i \in \mathbb{Z}_+$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. The highest weight vector v obeys

$$L^-_{ab}\mathsf{v} = 0, \quad a > b, \qquad L^-_{aa}\mathsf{v} = q^{-\lambda_a}\mathsf{v}.$$

The representation space $V^{(\lambda)}$ is generated by repeated action of the generators L^+_{ab} on the highest weight vector and subsequent factorizing (see [11] for details). These representations are q-deformations of the highest weight finite-dimensional representations of $U(gl(N))$. The highest weights are naturally identified with Young diagrams (partitions) λ.

The representation $\pi_{(1)}$ corresponding to the one-box diagram is the vector representation in $\mathbb{C}^N$. On the Cartan generators h_a introduced above it looks exactly like for the usual non-deformed algebra $gl(N)$: $\pi_{(1)}(\mathsf{h}_a) = e_{aa}$. Given the diagonal matrix $g = \mathrm{diag}\,(p_1, p_2, \dots, p_N)$, we set

$$\mathsf{g} = p_1^{\mathsf{h}_1} p_2^{\mathsf{h}_2} \dots p_N^{\mathsf{h}_N}, \tag{2.5}$$

then $g = \pi_{(1)}(\mathsf{g})$.

The R-matrix $\mathsf{R}^\lambda(u)$ acting in $\mathbb{C}^N \otimes V^{(\lambda)}$ is

$$\mathsf{R}^\lambda(u) = e^{\gamma u} \sum_{a \le b} e_{ab} \otimes \pi_\lambda(L^+_{ab}) - e^{-\gamma u} \sum_{a \ge b} e_{ab} \otimes \pi_\lambda(L^-_{ab}). \tag{2.6}$$

The R-matrices $\mathsf{R}^\lambda(u)$, $\mathsf{R}^\mu(u)$ are intertwined by a more general R-matrix $R^{\lambda\mu}(u)$ which acts in $V^{(\lambda)} \otimes V^{(\mu)}$:

$$\mathsf{R}^{\lambda\mu}_{12}(u_1 - u_2)\mathsf{R}^\lambda_{13}(u_1 - u_3)\mathsf{R}^\mu_{23}(u_2 - u_3) = \mathsf{R}^\mu_{23}(u_2 - u_3)\mathsf{R}^\lambda_{13}(u_1 - u_3)\mathsf{R}^{\lambda\mu}_{12}(u_1 - u_2) \tag{2.7}$$

This Yang-Baxter relation generalizes (2.2). Here space 1 is $V^{(\lambda)}$, space 2 is $V^{(\mu)}$ and space 3 is $\mathbb{C}^N$. The explicit form of $R^{\lambda\mu}(u)$ is much more complicated than (2.6). It can be obtained from the universal R-matrix for the quantum affine algebra $U_q(\widehat{gl}(N))$ [15] by specifying it to finite-dimensional evaluation representations or by the fusion procedure [16, 17, 13] applied to the fundamental R-matrix $\mathsf{R}(u)$. The g-invariance (2.3) is extended to $R^{\lambda\mu}(u)$ as follows:

$$\mathsf{R}^{\lambda\mu}_{12}(u)\, \pi_\lambda(\mathsf{g})_1\, \pi_\mu(\mathsf{g})_2 = \pi_\lambda(\mathsf{g})_1\, \pi_\mu(\mathsf{g})_2\, \mathsf{R}^{\lambda\mu}_{12}(u). \tag{2.8}$$

The higher transfer-matrices, or T-operators, are constructed in a similar way to (2.4) by taking trace of the product of R-matrices $\mathsf{R}^\lambda(u - u_i)$ in the auxiliary space $V^{(\lambda)}$:

$$T^\lambda(u) = \mathrm{tr}_{V^{(\lambda)}} \Big(\mathsf{R}^\lambda_{10}(u - u_1)\mathsf{R}^\lambda_{20}(u - u_2) \dots \mathsf{R}^\lambda_{L0}(u - u_L)\, \pi_\lambda(\mathsf{g}) \Big). \tag{2.9}$$

Here the space with index 0 is the auxiliary space $V^{(\lambda)}$. These T-operators act in the same quantum space $\mathcal{H} = \mathbb{C}^{\otimes L}$. If $\lambda = (1)$ is the 1-box diagram, then definition (2.9) coincides with (2.7). By analogy, we will call g of the form (2.5) the twist element. The Yang-Baxter equation (2.7) and the g-invariance (2.8) imply that the T-operators with the same g commute for all u and λ: $[T^\lambda(u), T^\mu(v)] = 0$, and can be diagonalized simultaneously.

An important property of the T-operators defined by (2.9) is that they vanish identically if the first column of λ is longer than N.

Set

$$H_a = \sum_{l=1}^{L} e^{(l)}_{aa}, \qquad e^{(l)}_{aa} = \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{l-1} \otimes e_{aa} \otimes \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{L-l},$$

then the g-invariance implies that $[T^\lambda(u),\, H_a] = 0$. Therefore, the eigenstates of the transfer matrices can be classified according to eigenvalues of the operators H_a. Let

$$\mathcal{H} = \bigoplus_{M_1,\ldots,M_N} \mathcal{H}(\{M_a\})$$

be the decomposition of the quantum space $\mathcal{H}$ into the direct sum of eigenspaces for the operators H_a with the eigenvalues $M_a \in \mathbb{Z}_+$, $a = 1, 2, \ldots, N$, then eigenstates of $T^\lambda(u)$ lie in the spaces $\mathcal{H}(\{M_a\})$. Since $\sum_a e_{aa} = \mathbb{I}$ is the unit matrix, $\sum_a H_a = L\,\mathbb{I}^{\otimes L}$ and thus

$$\sum_{i=1}^{N} M_i = L. \tag{2.10}$$

For the trivial representation (corresponding to the empty Young diagram $\emptyset$) $\pi_\emptyset(L^{\pm}_{ab}) = 1$ if $a = b$ and 0 otherwise, $\pi_\emptyset(\mathsf{g}) = 1$. Formula (2.6) yields $R^\emptyset(u) = e^{\gamma u} - e^{-\gamma u}$, where multiplication by the unity matrix $\mathbb{I}$ is implied. Therefore, we can define the T-operator for the trivial representation as follows:

$$T^\emptyset(u) = 2^L \prod_{i=1}^{L} \sinh(\gamma(u - u_i)). \tag{2.11}$$

For the one-dimensional representation $\pi_{(1^N)}$ (corresponding to the Young diagram (1^N) with one column of height N) $\pi_{(1^N)}(q^{\mathsf{h}_a}) = q$ for diagonal generators and 0 otherwise, $\pi_\emptyset(\mathsf{g}) = \det g$. Formula (2.6) yields $R^{(1^N)}(u) = e^{\gamma(u+1)} - e^{-\gamma(u+1)}$, where multiplication by the unity matrix $\mathbb{I}$ is implied. Therefore, the T-operator for the representation with the highest weight (1^N) (the quantum determinant of the quantum monodromy matrix) is given by:

$$T^{(1^N)}(u) = 2^L \det g \prod_{i=1}^{L} \sinh(\gamma(u + 1 - u_i)) = \det g\, T^\emptyset(u+1). \tag{2.12}$$

For general λ the T-operator is the Laurent polynomial in $e^{\gamma u}$ of the similar form:

$$T^\lambda(u) = \sum_{k=-L/2}^{L/2} G_k^\lambda\, e^{2k\gamma u}. \tag{2.13}$$

The coefficients G_k^λ of the T-operators with fixed g generate the full family of commuting operators (the Bethe algebra of the vertex model).

The operators $T^\lambda(u)$ appear to be functionally dependent. They are known to obey some functional relations which are given by the Cherednik-Bazhanov-Reshetikhin (CBR) determinant formulas [8, 9]. These formulas express $T^\lambda(u)$ for arbitrary λ through the transfer matrices $T_s(u) := T^{(s)}(u)$ corresponding to 1-row diagrams of length s or through the transfer matrices $T^a(u) := T^{(1^a)}(u)$ corresponding to 1-column diagrams of height a:

$$T^\lambda(u) = \Big(\prod_{k=1}^{\lambda_1'-1} T^\emptyset(u-k)\Big)^{-1} \det_{i,j=1,\ldots,\lambda_1'} T_{\lambda_i - i + j}(u-j+1), \tag{2.14}$$

$$T^\lambda(u) = \Big(\prod_{k=1}^{\lambda_1-1} T^\emptyset(u+k)\Big)^{-1} \det_{i,j=1,\ldots,\lambda_1} T^{\lambda_i' - i + j}(u+j-1). \tag{2.15}$$

Hereafter λ' denotes the transposed diagram (with respect to the main diagonal), so that λ_1' is the height of the first column, and $\emptyset$ is the empty diagram. One can show that formulas (2.15) follow from (2.14) and vice versa.

3 The master T-operator

Let $\mathbf{t} = \{t_1, t_2, t_3, \dots\}$ be an infinite set of parameters which we call times because they will have the meaning of hierarchical times in the MKP hierarchy. The Schur polynomials $s_\lambda(\mathbf{t})$ labeled by Young diagrams λ can be defined by the determinant formula

$$s_\lambda(\mathbf{t}) = \det_{i,j=1,\dots,\lambda'_1} h_{\lambda_i - i + j}(\mathbf{t}), \tag{3.1}$$

where the polynomials h_j are defined with the help of the generating series

$$\exp\Big(\sum_{k\geq 1} t_k z^k\Big) = 1 + h_1(\mathbf{t})z + h_2(\mathbf{t})z^2 + \dots$$

It is convenient to put $h_0(\mathbf{t}) = 1$, $h_n(\mathbf{t}) = 0$ for $n < 0$ and $s_\emptyset(\mathbf{t}) = 1$. The functions h_j are elementary Schur polynomials in the sense that for 1-row diagrams $\lambda = (j)$ with j boxes $s_{(j)}(\mathbf{t}) = h_j(\mathbf{t})$. Equivalently, one can define

$$s_\lambda(\mathbf{t}) = \det_{i,j=1,\dots,\lambda_1} e_{\lambda'_i - i + j}(\mathbf{t}), \tag{3.2}$$

where the polynomials e_j are defined with the help of the generating series

$$\exp\Big(\sum_{k\geq 1} (-1)^{k-1} t_k z^k\Big) = 1 + e_1(\mathbf{t})z + e_2(\mathbf{t})z^2 + \dots$$

For 1-column diagrams $\lambda = (1^j)$ with j boxes $s_{(1^j)}(\mathbf{t}) = e_j(\mathbf{t})$. Equations (3.1), (3.2) are known as Jacobi-Trudi formulas. It can be proved [18] that the Schur polynomials form a basis in the space of symmetric functions of the variables x_i defined by $kt_k = \sum_i x_i^k$.

We note the Cauchy-Littlewood identity

$$\sum_\lambda s_\lambda(\mathbf{t}) s_\lambda(\mathbf{t}') = \exp\Big(\sum_{k\geq 1} k t_k t'_k\Big), \tag{3.3}$$

where the sum is over all Young diagrams including the empty one. Writing it in the form

$$\sum_\lambda s_\lambda(\mathbf{y}) s_\lambda(\tilde{\partial}) = \exp\Big(\sum_{k\geq 1} y_k \partial_{t_k}\Big),$$

where $\tilde{\partial} = \{\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \dots\}$ and applying to $s_\mu(\mathbf{t})$, we get:

$$s_\lambda(\tilde{\partial}) s_\mu(\mathbf{t})\Big|_{\mathbf{t}=0} = \delta_{\lambda\mu}\,. \tag{3.4}$$

Following [1], we introduce a generating function of the T-operators (the master T-operator) depending on the infinite number of parameters $\mathbf{t} = \{t_1, t_2, \dots\}$:

$$T(u, \mathbf{t}) = \sum_\lambda s_\lambda(\mathbf{t}) T^\lambda(u). \tag{3.5}$$

These operators commute for different values of the parameters: $[T(u, \mathbf{t}), T(u', \mathbf{t}')] = 0$. Since $T^\lambda(u) = 0$ if $\lambda'_1 > N$, the sum in (3.5) is actually restricted to diagrams with

$\lambda_1' \le N$. The T-operators $T^\lambda(u)$ can be restored from the master T-operator according to the formula

$$T^\lambda(u) = s_\lambda(\tilde\partial) T(u, \mathbf{t})\Big|_{\mathbf{t}=0}, \tag{3.6}$$

which follows from (3.4). In particular,

$$T^\emptyset(u) = T(u, 0), \tag{3.7}$$

$$T(u) = T^{(1)}(u) = \partial_{t_1} T(u, \mathbf{t})\Big|_{\mathbf{t}=0}. \tag{3.8}$$

Below we use the standard notation

$$\mathbf{t} \pm [z^{-1}] = \Big\{ t_1 \pm z^{-1},\, t_2 \pm \frac{1}{2} z^{-2},\, t_3 \pm \frac{1}{3} z^{-3},\, \ldots \Big\}, \tag{3.9}$$

$$\xi(\mathbf{t}, z) = \sum_{k=0}^{\infty} t_k z^k. \tag{3.10}$$

Eq. (3.6) implies that $T(u, 0 \pm [z^{-1}])$ is the generating series for T-operators corresponding to the 1-row and 1-column diagrams respectively:

$$T(u, [z^{-1}]) = \sum_{s \ge 0} z^{-s} T_s(u), \quad T(u, -[z^{-1}]) = \sum_{a \ge 0} (-z)^{-a} T^a(u). \tag{3.11}$$

As it was proven in [1], the CBR formulas (2.14) imply that the master T-operator obeys the bilinear identity

$$\oint_{\mathcal{C}} e^{\xi(\mathbf{t} - \mathbf{t}', z)} z^{u - u'} T(u, \mathbf{t} - [z^{-1}])\, T(u', \mathbf{t}' + [z^{-1}])\, dz = 0 \tag{3.12}$$

for all $u, u', \mathbf{t}, \mathbf{t}'$. The contour $\mathcal{C}$ encircles the cut between 0 and ∞. By standard manipulations [19, 20], one can derive from (3.12) the infinite KP and MKP hierarchies of differential (in t_k's) and differential-difference (in t_k's and u) equations. The variable u is the so-called "zero time"; it is naturally included in the extended sequence of times $t_0 = u, t_1, t_2, \ldots$. Choosing $u, u', \mathbf{t}, \mathbf{t}'$ in a special way, one can also derive from (3.12) the following bilinear equations:

$$\begin{aligned}
&(z_2 - z_3) T\left(u, \mathbf{t} + [z_1^{-1}]\right) T\left(u, \mathbf{t} + [z_2^{-1}] + [z_3^{-1}]\right) \\
&+ (z_3 - z_1) T\left(u, \mathbf{t} + [z_2^{-1}]\right) T\left(u, \mathbf{t} + [z_1^{-1}] + [z_3^{-1}]\right) \\
&+ (z_1 - z_2) T\left(u, \mathbf{t} + [z_3^{-1}]\right) T\left(u, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]\right) = 0,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&z_2 T\left(u+1, \mathbf{t} + [z_1^{-1}]\right) T\left(u, \mathbf{t} + [z_2^{-1}]\right) - z_1 T\left(u+1, \mathbf{t} + [z_2^{-1}]\right) T\left(u, \mathbf{t} + [z_1^{-1}]\right) \\
&+ (z_1 - z_2) T\left(u+1, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]\right) T\left(u, \mathbf{t}\right) = 0.
\end{aligned} \tag{3.14}$$

They are known as Hirota or Hirota-Miwa equations for the τ-function [21, 22]. In this sense the master T-operator (any of its eigenvalues) is the τ-function of the classical

MKP hierarchy (see, e.g., [23]). Equation (3.5) can be regarded as the Schur function expansion of the τ-function (see also [24]).

Note that the transformation

$$T(u,\mathbf{t}) \to C(u)\exp\Big(\sum_k c_k t_k\Big) T(u,\mathbf{t})$$

with arbitrary function $C(u)$ and arbitrary constant coefficients c_k preserves the space of τ-functions. Two τ-functions are regarded as essentially different if they are not obtained from each other by such transformation.

As a function of u, the master T-operator has the structure similar to (2.13):

$$T(u,\mathbf{t}) = \sum_{k=-L/2}^{L/2} G_k(\mathbf{t})\, e^{2k\gamma u}. \tag{3.15}$$

In particular, the highest and the lowest coefficients $G_{\pm L/2}(\mathbf{t})$ are easy to calculate. For example, the highest coefficient $G^{\lambda}_{L/2}(\mathbf{t})$ is

$$G^{\lambda}_{L/2}(\mathbf{t}) = \exp\Big(-\gamma\sum_{n=1}^{L} u_n\Big) \sum_{a_i,b_i} \mathrm{tr}_{V(\lambda)} \left(L^{+}_{a_1 b_1} L^{+}_{a_2 b_2} \dots L^{+}_{a_L b_L}\right) e^{(1)}_{a_1 b_1} e^{(2)}_{a_2 b_2} \dots e^{(L)}_{a_L b_L}$$

(one should take the first terms from each R-matrix (2.6)). Since all matrices L^{+}_{ab} here are upper triangular, the trace is equal to the product of diagonal matrices with the same diagonal elements:

$$\begin{aligned} G^{\lambda}_{L/2}(\mathbf{t}) &= \exp\Big(-\gamma\sum_{n=1}^{L} u_n\Big) \sum_{a_i} \mathrm{tr}_{V(\lambda)} \left(q^{\mathsf{h}_{a_1}+\ldots+\mathsf{h}_{a_L}} p_1^{\mathsf{h}_1} \dots p_L^{\mathsf{h}_L}\right) e^{(1)}_{a_1 b_1} \dots e^{(L)}_{a_L b_L} \\ &= \sum_{M_1,\ldots,M_L} \exp\Big(-\gamma\sum_{n=1}^{L} u_n\Big) \mathrm{tr}_{V(\lambda)}\Big((e^{\gamma M_1} p_1)^{\mathsf{h}_1} \dots (e^{\gamma M_L} p_L)^{\mathsf{h}_L}\Big) \sum_{a_i:\{M_j\}} e^{(1)}_{a_1 a_1} \dots e^{(L)}_{a_L a_L}, \end{aligned}$$

where the last sum goes over all sequences of indices $a_1, \dots a_L$ such that the number of indices equal to j is M_j. It is easy to see that

$$\sum_{a_i:\{M_j\}} e^{(1)}_{a_1 a_1} \dots e^{(L)}_{a_L a_L} = \frac{L!}{M_1! \dots M_L!} P_{M_1,\ldots,M_N},$$

where $P_{M_1,\ldots,M_N}$ is the projector to the space $\mathcal{H}(\{M_i\})$. Set $y_k = \dfrac{1}{k}\displaystyle\sum_{a=1}^{N} e^{\gamma M_a k} p_a^k$, then

$$\mathrm{tr}_{V(\lambda)} \left((e^{\gamma M_1} p_1)^{\mathsf{h}_1} \dots (e^{\gamma M_L} p_L)^{\mathsf{h}_L}\right) = s_\lambda(\mathbf{y}).$$

The calculation for $G_{-L/2}(\mathbf{t})$ is similar. Using the Cauchy-Littlewood identity (3.3), we get:

$$G_{\pm L/2}(\mathbf{t})\Big|_{\mathcal{H}(\{M_i\})} = \frac{(\pm 1)^L L!}{M_1! \dots M_L!} \exp\left(\mp\gamma\sum_{n=1}^{L} u_n + \sum_{k\geq 1}\sum_{a=1}^{N} t_k p_a^k e^{\pm\gamma k M_a}\right). \tag{3.16}$$

Let $|\omega\rangle = |\omega(\{M_i\})\rangle \in \mathcal{H}(\{M_i\})$ be an eigenstate of $T(u, \mathbf{t})$,

$$T(u, \mathbf{t})\,|\omega(\{M_i\})\rangle = |\omega(\{M_i\})\rangle\,\tau_u(\mathbf{t}; |\omega(\{M_i\})\rangle),$$

then the corresponding eigenvalue $\tau_u(\mathbf{t}; |\omega(\{M_i\})\rangle)$ can be written in the form

$$\tau_u(\mathbf{t}; |\omega(\{M_i\})\rangle) = C(\mathbf{t}) \prod_{k=1}^{L} \sinh(\gamma(u - u_k(\mathbf{t})). \tag{3.17}$$

We will call the expression in the right hand side a trigonometric polynomial (of degree L). The common multiplier $C(\mathbf{t})$ and the roots of this trigonometric polynomial depend on all the times $t_1, t_2, \ldots$ (and on $|\omega\rangle$). Comparing with (3.16), we find

$$C(\mathbf{t}) = \frac{2^L L!}{M_1! \ldots M_N!}\, \exp\Bigl(\sum_{k\geq 1} \sum_{a=1}^{N} t_k p_a^k \cosh(\gamma M_a k)\Bigr).$$

From (3.7) and (2.11) it is clear that the initial values of these roots are inhomogeneity parameters at the lattice sites: $u_i(0) = u_i$.

4 Trigonometric solutions of the MKP hierarchy

In this section we study solutions of the MKP hierarchy which are periodic in the variable $t_0 = u$ with period $2\pi i/\gamma$. We call them trigonometric solutions. For solutions of this class, the τ-function is a "trigonometric quasi-polynomial" of u, i.e., a Laurent polynomial of the variable $e^{\gamma u}$ possibly multiplied by an exponential function of u.

4.1 The construction of trigonometric solutions

By trigonometric solutions of the MKP hierarchy we mean τ-functions which are polynomials in $e^{\gamma u}$ for some γ multiplied by an exponential function of u. They can be viewed as degenerations of double-periodic (elliptic) solutions in the complex plane of the variable $u = t_0$ (they correspond to vertex models with elliptic R-matrices). The general theory of elliptic solutions for the KP hierarchy was developed in [25] and extended to the MKP hierarchy in [26]. The trigonometric degeneration simplifies the construction and makes it more explicit [27]. Here we apply it to the case of our interest. The τ-function for the trigonometric solutions will be obtained below in the form of the Casorati determinant [28].

Let $\tau_u(\mathbf{t})$ be the τ-function of the MKP hierarchy. The Baker-Akhiezer function and its adjoint are defined in the following way [19]:

$$\psi_u(\mathbf{t}, z) = z^u e^{\xi(\mathbf{t}, z)}\, \frac{\tau_u(\mathbf{t} - [z^{-1}])}{\tau_u(\mathbf{t})}, \tag{4.1}$$

$$\psi_u^*(\mathbf{t}, z) = z^{-u} e^{-\xi(\mathbf{t}, z)}\, \frac{\tau_u(\mathbf{t} + [z^{-1}])}{\tau_u(\mathbf{t})} \tag{4.2}$$

($\xi(\mathbf{t}, z)$ is given in (3.10)). In general, the ratios of the τ-functions in the right hand sides can be expanded in infinite series around $z = \infty$:

$$\psi_u(\mathbf{t}, z) = z^u e^{\xi(\mathbf{t},z)} \left(1 + \frac{w_1(\mathbf{t})}{z} + \frac{w_2(\mathbf{t})}{z^2} + \ldots \right), \tag{4.3}$$

$$\psi_u^*(\mathbf{t}, z) = z^{-u} e^{-\xi(\mathbf{t},z)} \left(\frac{w_1^*(\mathbf{t})}{z} + \frac{w_2^*(\mathbf{t})}{z^2} + \ldots \right). \tag{4.4}$$

According to the Krichever's theory of general algebro-geometric solutions [29], they can be characterized and explicitly constructed by fixing certain analytic properties of the Baker-Akhiezer function on a Riemann surface of the complex variable z (the classical spectral parameter). Recall that the quantum spectral parameter u is the zero time in the classical MKP hierarchy.

For trigonometric solutions, the Riemann surface is the Riemann sphere (compactified complex plane), which represents a genus zero algebraic curve with singularities. Correspondingly, the Baker-Akhiezer function is, in this case, a rational function on the complex z-plane multiplied by power-like and exponential factors which give the required asymptotics (the essential singularity at infinity). For non-integer u the points 0 and ∞ are branch points for the the Baker-Akhiezer function. In order to make it single-valued, one should make a cut between 0 and ∞.

We know that the second series in (3.11) truncates at $a = N$. This suggests to assume the following ansatz for the Baker-Akhiezer function, in which the series in (4.3) truncates at the N-th term:

$$\psi_u(\mathbf{t}, z) = z^u e^{\xi(\mathbf{t},z)} \left(1 + \frac{w_1(\mathbf{t})}{z} + \ldots + \frac{w_N(\mathbf{t})}{z^N} \right). \tag{4.5}$$

This explicitly defines the function $z^{-u} e^{-\xi(\mathbf{t},z)} \psi_u(\mathbf{t}, z)$ as a rational function on the extended complex plane. The multiplicity N of the pole at $z = 0$ is a discrete parameter characterizing the class of solutions to be constructed. Fix N points $p_i \in \mathbb{C}$, N non-negative integer numbers M_i such that

$$M_1 + M_2 + \ldots + M_N = L$$

and the set of parameters $b_{i,m}$ with $i = 1, \ldots, N$, $m = -\frac{1}{2}M_i, \frac{1}{2}M_i + 1, \ldots, \frac{1}{2}M_i$ (we assume that $b_{i,\pm\frac{1}{2}M_i} \neq 0$ for all i). Let us impose N conditions of the form

$$\sum_{m=-M_i/2}^{+M_i/2} b_{i,m}\, \psi_u(\mathbf{t}, p_i e^{2\gamma m}) = 0\,, \quad i = 1, \ldots, N, \tag{4.6}$$

which are supposed to hold for any values of u, t_i. The sum goes over all integer numbers between $-\frac{1}{2}M_i$ and $-\frac{1}{2}M_i$ for even M_i and over all half-integer numbers between $-\frac{1}{2}M_i$ and $-\frac{1}{2}M_i$ for odd M_i.

These conditions yield a system of N linear equations for N coefficients w_k which allows one to fix the Baker-Akhiezer function ψ. The general theory guaranties that the τ-function associated with this ψ-function according to (4.1) solves the MKP hierarchy. The points p_i and entries of the matrix $b_{i,m}$ are parameters of the solution. The coefficients w_k

appear to be rational functions of $e^{\gamma u}$ while the τ-function is a trigonometric polynomial in u (possibly multiplied by an exponential function of u). From the algebro-geometric point of view, these solutions are associated with singular Riemann surfaces with N "strings" of singular points

$$p_i e^{-\gamma M_i}, p_i e^{-\gamma(M_i-2)}, \ldots, p_i e^{\gamma(M_i-2)}, p_i e^{\gamma M_i}$$

with the center at p_i. The points of each string are glued with each other in a complicated way. Note that the parameters $b_{i,m}$ can be multiplied by any non-zero complex numbers k_i: the transformation $b_{i,m} \to k_i b_{i,m}$ does not change anything.

The family of periodic N-soliton solutions is a very particular case of this construction corresponding to $M_i = 1$ for all i. In this case conditions (4.6) become $b_{i,-1/2}\psi_u(\mathbf{t}, p_i e^{-\gamma}) = -b_{i,1/2}\psi_u(\mathbf{t}, p_i e^{\gamma})$ and the solutions are associated with the Riemann sphere with N pairs of double points $p_i e^{\gamma}$ and $p_i e^{-\gamma}$.

It is easy to see that conditions (4.6) are equivalent to the system of linear equations

$$B_i(u, \mathbf{t}) + \sum_{k=1}^{N} B_i(u-k, \mathbf{t}) w_k = 0, \tag{4.7}$$

where

$$B_i(u, \mathbf{t}) := p_i^u \sum_{m=-M_i/2}^{+M_i/2} b_{i,m}(\mathbf{t}) e^{2\gamma m u}, \quad b_{i,m}(\mathbf{t}) \equiv b_{i,m} e^{\xi(\mathbf{t}, p_i e^{2\gamma m})}. \tag{4.8}$$

The system can be solved using the Cramer's rule. This gives the following explicit expression for the Baker-Akhiezer function:

$$\psi_u(\mathbf{t}, z) = z^u e^{\xi(\mathbf{t},z)} \frac{\begin{vmatrix} 1 & z^{-1} & \ldots & z^{-N} \\ B_1(u,\mathbf{t}) & B_1(u-1,\mathbf{t}) & \ldots & B_1(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ B_N(u,\mathbf{t}) & B_N(u-1,\mathbf{t}) & \ldots & B_N(u-N,\mathbf{t}) \end{vmatrix}}{\begin{vmatrix} B_1(u-1,\mathbf{t}) & \ldots & B_1(u-N,\mathbf{t}) \\ \vdots & \ddots & \vdots \\ B_N(u-1,\mathbf{t}) & \ldots & B_N(u-N,\mathbf{t}) \end{vmatrix}}. \tag{4.9}$$

Comparing with (4.1), we conclude, using the obvious property

$$B_i(u, \mathbf{t} - [z^{-1}]) = B_i(u, \mathbf{t}) - B_i(u+1, \mathbf{t}) z^{-1}, \tag{4.10}$$

that the τ-function is given by the difference Wronskian (Casorati) determinant in the denominator:

$$\tau_u(\mathbf{t}) = (\det g)^{-u} \det_{i,j=1,\ldots,N} B_i(u-j, \mathbf{t}) \tag{4.11}$$

(here $g = \operatorname{diag}(p_1, \ldots, p_N)$ is the same matrix as in the previous sections). The factor $(\det g)^{-u}$ is put here to make $\tau_u(\mathbf{t})$ a pure trigonometric polynomial in u of the form (3.17). Comparing the highest and the lowest coefficients in (3.15) (given by equation (3.16)) with the corresponding coefficients in (4.11), we see that the parameters $b_{i,\pm\frac{1}{2}M_i}$ obey the following relations:

$$\Big(\prod_{i=1}^{N} b_{i,\pm\frac{1}{2}M_i}\Big)(\det g)^{-N} e^{\mp\gamma NL} \prod_{j<k} \Big(p_j e^{\pm\gamma M_j} - p_k e^{\pm\gamma M_k}\Big) = \frac{(\pm 1)^L L!}{M_1! \ldots M_N!}\, e^{\mp\gamma \sum_{n=1}^{N} u_n}.$$

From (4.9) it is clear that the last coefficient in (4.5), w_N, in terms of the τ-function is given by

$$w_N(u,\mathbf{t}) = (-1)^N \frac{\tau_{u+1}(\mathbf{t})}{\tau_u(\mathbf{t})}.$$

We also note the formula

$$w_1(u,\mathbf{t}) = -\partial_{t_1} \log \tau_u(\mathbf{t}) \tag{4.12}$$

for the first coefficient in (4.5), w_1, which easily follows from the obvious relation

$$\partial_{t_1} B_i(u,\mathbf{t}) = B_i(u+1,\mathbf{t}). \tag{4.13}$$

Rewriting (4.10) in the form

$$B_i(u,\mathbf{t}+[z^{-1}]) = B_i(u,\mathbf{t}) + z^{-1} B_i(u+1,\mathbf{t}+[z^{-1}]),$$

it is straightforward to check that

$$\tau_u(\mathbf{t}+[z^{-1}]) = (\det g)^{-u} \begin{vmatrix} B_1\,(u-1,\mathbf{t}+[z^{-1}]) & B_1(u-2,\mathbf{t}) & \dots & B_1(u-N,\mathbf{t}) \\ B_2\,(u-1,\mathbf{t}+[z^{-1}]) & B_2(u-2,\mathbf{t}) & \dots & B_2(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ B_N\,(u-1,\mathbf{t}+[z^{-1}]) & B_N(u-2,\mathbf{t}) & \dots & B_N(u-N,\mathbf{t}). \end{vmatrix} \tag{4.14}$$

It directly follows from the definition that

$$B_i(u,\mathbf{t}+[z^{-1}]) = p_i^u \sum_{m=-M_i/2}^{M_i/2} \frac{b_{i,m}\, z}{z - p_i e^{2\gamma m}}\, e^{2\gamma m u + \xi(\mathbf{t}, p_i e^{2\gamma m})}. \tag{4.15}$$

Expanding this in powers of z^{-1}, we get:

$$B_i(u,\mathbf{t}+[z^{-1}]) = B_i(u,\mathbf{t}) + B_i(u+1,\mathbf{t})z^{-1} + B_i(u+2,\mathbf{t})z^{-2} + \dots$$

Therefore, the expansion of $\tau_u(\mathbf{t}+[z^{-1}])$ around ∞ reads

$$\tau_u(\mathbf{t}+[z^{-1}]) = (\det g)^{-u} \sum_{s=0}^{\infty} z^{-s} \begin{vmatrix} B_1\,(u+s-1,\mathbf{t}) & B_1(u-2,\mathbf{t}) & \dots & B_1(u-N,\mathbf{t}) \\ B_2\,(u+s-1,\mathbf{t}) & B_2(u-2,\mathbf{t}) & \dots & A_2(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ B_N\,(u+s-1,\mathbf{t}) & B_N(u-2,\mathbf{t}) & \dots & B_N(u-N,\mathbf{t}) \end{vmatrix}. \tag{4.16}$$

We thus see that the adjoint Baker-Akhiezer function has the determinant representation

$$\psi_u^*(\mathbf{t},z) = z^{-u} e^{-\xi(\mathbf{t},z)} \frac{\begin{vmatrix} B_1\,(u-1,\mathbf{t}+[z^{-1}]) & B_1(u-2,\mathbf{t}) & \dots & B_1(u-N,\mathbf{t}) \\ B_2\,(u-1,\mathbf{t}+[z^{-1}]) & B_2(u-2,\mathbf{t}) & \dots & B_2(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ B_N\,(u-1,\mathbf{t}+[z^{-1}]) & B_N(u-2,\mathbf{t}) & \dots & B_N(u-N,\mathbf{t}) \end{vmatrix}}{\begin{vmatrix} B_1\,(u-1,\mathbf{t}) & B_1(u-2,\mathbf{t}) & \dots & A_1(u-N,\mathbf{t}) \\ B_2\,(u-1,\mathbf{t}) & B_2(u-2,\mathbf{t}) & \dots & A_2(u-N,\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ B_N\,(u-1,\mathbf{t}) & B_N(u-2,\mathbf{t}) & \dots & A_N(u-N,\mathbf{t}) \end{vmatrix}}. \tag{4.17}$$

Let us introduce the notation

$$\bar{B}_k(u,\mathbf{t}) := \det_{\substack{i=1,\,\dots,\not{k},\dots,N\\ j=1,\,\dots\,,\,N-1}} B_i(u+1-j,\,\mathbf{t}) \tag{4.18}$$

for the minor $M_{k,N}$ of the $N\times N$ matrix $B_i(u+1-j)$, $1\le i,j\le N$. Then, expanding the determinant in the numerator of (4.17) in the first column, we obtain:

$$\psi_u^*(\mathbf{t},z) = \frac{z^{-u}e^{-\xi(\mathbf{t},z)}}{\tau_u(\mathbf{t})}\sum_{k=1}^{N}(-1)^{k-1}\bar{B}_k(u-2,\mathbf{t})\,B_k(u-1,\mathbf{t}+[z^{-1}]),$$

or, substituting (4.15),

$$\psi_u^*(\mathbf{t},z) = \frac{z^{-u+1}e^{-\xi(\mathbf{t},z)}}{(\det g)^u\tau_u(\mathbf{t})}\sum_{i=1}^{N}(-1)^{i-1}p_i^{u-1}\sum_{m=-M_i/2}^{M_i/2}\frac{b_{i,m}e^{2\gamma m(u-1)+\xi(\mathbf{t},p_ie^{2\gamma m})}}{z-p_ie^{2\gamma m}}\,\bar{B}_i(u-2,\mathbf{t}). \tag{4.19}$$

This gives the pole expansion of the adjoint Baker-Akhiezer function. We see that in general it has simple poles at all the points forming the "strings". Below we need this formula rewritten for the function $\tau_u(\mathbf{t}+[z^{-1}])$:

$$\tau_u(\mathbf{t}+[z^{-1}]) = z(\det g)^{-u}\sum_{i=1}^{N}(-1)^{i-1}p_i^{u-1}\sum_{m=-M_i/2}^{M_i/2}\frac{b_{i,m}e^{2\gamma m(u-1)+\xi(\mathbf{t},p_ie^{2\gamma m})}}{z-p_ie^{2\gamma m}}\,\bar{B}_i(u-2,\mathbf{t}), \tag{4.20}$$

with simple poles at the same points.

4.2 Undressing Bäcklund transformations for the trigonometric solutions

As it was demonstrated in [1] for models with rational R-matrices, the main relations of the Bethe ansatz method are naturally built in the construction of rational solutions to the MKP hierarchy. The nested Bethe ansatz scheme appears to be equivalent to a chain of some special Bäcklund transformations of the initial rational MKP solution that "undress" it to the trivial solution by reducing the number of singular points in succession. All this remains valid for vertex models with trigonometric R-matrices, with the only difference that the undressing procedure should be applied to the trigonometric solutions. Technically it becomes even simpler because poles of the adjoint Baker-Akhiezer function are simple in this case. In particular, the functions $\bar{B}_k(u,\mathbf{t}=0)$ and $B_k(u,\mathbf{t}=0)$ should be identified, up to some irrelevant factors, with the (eigenvalues of) the Baxter Q-operators on the first and the last levels of nesting in the nested Bethe ansatz scheme.

Adding or removing a "string" $p_ie^{2\gamma m}$ with the center at p_i to or from the data of a trigonometric solution is a Bäcklund transformation. It sends a trigonometric τ-function to another one. We will be interested in the removing of a string that results in decreasing the degree of the trigonometric polynomial (the undressing transformations). Basically, such a transformation can be done by extracting the singular part (the residue) of the function $\tau_u(\mathbf{t}+[z^{-1}])$ at any of its simple poles which are located at the points of the string with the center at p_i. Specifically, consider the function

$$\tau_u^{[i]}(\mathbf{t}) = (-1)^{i-1}p_i^{-1}\det g\, e^{\gamma M_i(u+1)-\xi(\mathbf{t},p_ie^{-\gamma M_i})}\,\mathrm{res}_{z=p_ie^{-\gamma M_i}}\tau_{u+1}(\mathbf{t}+[z^{-1}]). \tag{4.21}$$

Equation (4.20) implies that

$$\tau_u^{[i]}(\mathbf{t}) = b_{i,-\frac{1}{2}M_i}(p_i^{-1}\det g)^{-u}\bar{B}_i(u-1,\mathbf{t}),$$

i.e., up to the irrelevant constant factor, it has exactly the same determinant form as $\tau_u(\mathbf{t})$ with the string with the center at p_i removed. Therefore, it is a τ-function, i.e., it satisfies the same Hirota equations as $\tau_u(\mathbf{t})$ does and $\tau \to \tau^{[i]}$ is indeed a Bäcklund transformation. The degree of the trigonometric polynomial $\tau_u^{[i]}(\mathbf{t})$ is $L - M_i$. Note that the residue in (4.21) is taken at the very left edge of the string. This has an advantage that the coefficient $b_{i,-\frac{1}{2}M_i}$ is non-zero by definition and thus the result of the transformation does not vanish identically (the same holds for the very right edge).

The procedure can be continued until one obtains a polynomial of degree 0. The inductive definition is as follows. Fix a set $I_n = \{i_1, \ldots, i_n\} \subset \{1, 2, \ldots, N\}$. Suppose we have a τ-function $\tau_u^{[i_1 i_2 \ldots i_{n-1}]}(\mathbf{t})$ obtained at the $(n-1)$-th step, then the τ-function at the n-th step is defined as

$$\begin{aligned}\tau_u^{[i_1 i_2 \ldots i_n]}(\mathbf{t}) \;&=\; (-1)^{i_n-1}\Big(\prod_{j\in\{1,\ldots,N\}\setminus I_n} p_j\Big)(b_{i_n,-\frac{1}{2}M_{i_n}})^{-1}e^{\gamma M_{i_n}(u+1)-\xi(\mathbf{t},p_{i_n}e^{-\gamma M_{i_n}})}\\ &\quad\times \mathrm{res}_{z=p_{i_n}e^{-\gamma M_{i_n}}}\,\tau_{u+1}^{[i_1\ldots i_{n-1}]}(\mathbf{t}+[z^{-1}]).\end{aligned} \tag{4.22}$$

This function has the determinant representation

$$\tau_u^{[i_1 i_2 \ldots i_n]}(\mathbf{t}) = \Big(\prod_{i\in I_n} b_{i,-\frac{1}{2}M_i}\Big)\Big(\prod_{j\in\{1,\ldots,N\}\setminus I_n} p_j^{-u}\Big)\det_{\substack{i=\{1,\ldots,N\}\setminus I_n\\ j=1,\,\ldots\,,\,N-n}} B_i(u-j,\,\mathbf{t}). \tag{4.23}$$

As it is shown in detail in [1], the "undressing" chain of Bäcklund transformations

$$\tau_u(\mathbf{t}) \to \tau_u^{[i_1]}(\mathbf{t}) \to \tau_u^{[i_1]}(\mathbf{t}) \to \ldots \to \tau_u^{[i_1\ldots i_N]}(\mathbf{t}) \to 0$$

is equivalent to the nested Bethe ansatz scheme, with the τ-functions $\tau_u^{[i_1\ldots i_n]}(\mathbf{t})$ being eigenvalues of the master T-operators on higher levels of the nesting procedure. In particular, $\tau_u^{[i_1\ldots i_n]}(\mathbf{t})$ at $\mathbf{t} = 0$ are eigenvalues of the Baxter's Q-operators. They are trigonometric polynomials in u of decreasing degree. This implies the system of Bethe equations for their zeros.

5 Zeros of the master T-operator as the Ruijsenaars-Schneider particles

As we have seen, eigenvalues of the master T-operator are trigonometric polynomials in the spectral parameter u of the form (3.17). The roots of each eigenvalue have their own dynamics in the times t_i. This dynamics is known [26] to be given by the trigonometric Ruijsenaars-Schneider model [6]. The inhomogeneity parameters u_i are coordinates of the Ruijsenaars-Schneider particles at $t_i = 0$: $u_i = u_i(0)$.

Here we derive, following [26], the equations of motion for zeros of the trigonometric τ-function (the master T-operator) with respect to the first time flow $t_1 = t$. Our starting point is the differential-difference equation for the Baker-Akhiezer function:

$$\partial_{t_1}\psi_u(\mathbf{t},z) = \psi_{u+1}(\mathbf{t},z) + \partial_{t_1}\log\frac{\tau_{u+1}(\mathbf{t})}{\tau_u(\mathbf{t})}\;\psi_u(\mathbf{t},z). \tag{5.1}$$

which follows from the definition and from the Hirota equations.

It is clear from (4.1) that $\psi_u(\mathbf{t})$ has simple poles at $u = u_j(\mathbf{t})$. Let us introduce the function

$$\Phi(u,\zeta) = \frac{\sinh(\gamma(u+\zeta))}{\sinh(\gamma u)\sinh(\gamma\zeta)} = \coth(\gamma u) + \coth(\gamma\zeta)$$

and adopt the following pole ansatz for the Baker-Akhiezer function:

$$\psi_u(\mathbf{t}, z) = z^u \sum_{j=1}^{L} s_j(\mathbf{t}, z, \zeta))\Phi(u - u_j(\mathbf{t}, \zeta). \tag{5.2}$$

Here ζ plays the role of an auxiliary spectral parameter. Substituting this ansatz into (5.1), one is able to derive the equations of motion together with their Lax representation. Skipping further details of the calculations, we give the results. The double poles at $u = u_j$ cancel automatically. Cancelation of simple poles at $u = u_j - 1$ yields:

$$\gamma \dot{u}_j \sum_k \Phi(u_{jk} - 1, \zeta) s_k = z s_j, \quad j = 1, \ldots, L,$$

where $\dot{u}_j := \partial_{t_1} u_j$, $u_{jk} := u_j - u_k$. Cancelation of simple poles at $u = u_j$ yields:

$$\dot{s}_j = \gamma \dot{u}_j \sum_{k \neq j} \Phi(u_{jk}, \zeta) s_k + \gamma \Big[\coth(\gamma\zeta)\, \dot{u}_j + \sum_{k \neq j} \dot{u}_k (\coth \gamma(u_{jk}) - \coth(\gamma(u_{jk}{+}1))\Big] s_j.$$

Finally, comparison of the constant terms at $u \to \pm\infty$ (we assume that γ is real positive) yields the condition

$$\sum_j \dot{s}_j = z \sum_j s_j$$

which does not add any new constraint because in fact follows from the previously obtained relations. The conditions obtained above can be written in the matrix form as

$$\mathcal{L}(\zeta)\, \mathbf{s} = z\mathbf{s}, \qquad \dot{\mathbf{s}} = \mathcal{M}(\zeta)\, \mathbf{s}, \tag{5.3}$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_L)^{\mathrm{t}}$ and the matrices $\mathcal{L}(\zeta)$, $\mathcal{M}(\zeta)$ are defined as

$$\mathcal{L}_{jk}(\zeta) = \gamma \dot{u}_j \Phi(u_{jk} - 1, \zeta), \tag{5.4}$$

$$\mathcal{M}_{jk}(\zeta) = \gamma \Big[(\coth(\gamma\zeta) {-} \coth\gamma)\dot{u}_j + \sum_{l \neq j} V(u_{jl})\Big]\delta_{jk} + \gamma(1 - \delta_{jk})\dot{u}_j \Phi(u_{jk}, \zeta), \tag{5.5}$$

where

$$V(u) := \coth(\gamma u) - \coth(\gamma(u+1)).$$

The compatibility of equations (5.3) implies the Lax equation

$$\dot{\mathcal{L}}(\zeta) = [\mathcal{M}(\zeta), \mathcal{L}(\zeta)]. \tag{5.6}$$

A direct calculation shows that it is equivalent to the equations of motion for the Ruijsenaars-Schneider system:

$$\begin{aligned} \ddot{u}_j &= \gamma \dot{u}_j \sum_{k \neq j} \dot{u}_k \Big(V(u_{jk}) {-} V(u_{kj})\Big) \\ &= -2\gamma \sinh^2\!\gamma \sum_{k \neq j} \frac{\dot{u}_j \dot{u}_k \cosh(\gamma u_{jk})}{\sinh(\gamma(u_{jk}{-}1))\sinh(\gamma u_{jk})\sinh(\gamma(u_{jk}{+}1))}\,. \end{aligned} \tag{5.7}$$

In the course of the calculation, the following identities are useful:

$$\Phi(u-1,\zeta)\Phi(v,\zeta)-\Phi(u,\zeta)\Phi(v-1,\zeta)=\Phi(u+v-1,\zeta)\Big(V(-u)-V(-v)\Big),$$

$$\partial_u\Phi(u-1,\zeta)=\gamma\Big(\coth(\gamma\zeta)-\coth\gamma-V(-u)\Big)\Phi(u-1,\zeta)-\gamma\Phi(-1,\zeta)\Phi(u,\zeta).$$

We also note that the system (5.6) is a Hamiltonian system with the Hamiltonian

$$\mathcal{H}_1=\sum_{j=1}^{L}e^{v_j}\prod_{k\neq j}\left(\frac{\sinh(\gamma(u_{jk}+1))\sinh(\gamma(u_{jk}-1))}{\sinh^2(\gamma u_{jk}))}\right)^{1/2} \tag{5.8}$$

and the canonically conjugate variables v_j, u_j with the Poisson brackets $\{v_j, u_k\}=\delta_{jk}$. There are also higher Hamiltonians $\mathcal{H}_j$ in involution which generate the higher flows with respect to t_j.

The spectral curve is given by the equation

$$\det_{L\times L}\Big(\mathcal{L}(\zeta)-z\Big)=0. \tag{5.9}$$

One can show that this curve is the Riemann sphere with points of the strings $p_i e^{2\gamma m}$ being glued in a complicated way. The coefficients of the characteristic polynomial in the l.h.s. are integrals of motion for the Ruijsenaars-Schneider system.

Finally, let us stress the specific way of posing the problem in the context of the Ruijsenaars-Schneider system that corresponds to solution of the vertex model or quantum spin chain. The standard mechanical problem is: given initial coordinates and velocities of the particles $u_j(0)$, $\dot{u}_j(0)$, find the time evolution $u_j(t)$. By contrast, in order to find eigenvalues of the transfer matrix, one should pose the problem in the following non-standard way: given initial coordinates $u_j=u_j(0)$ and values of all higher integrals of motion $\mathcal{H}_j$, find initial velocities $\dot{u}_j(0)$. Indeed, the initial velocities allow one to restore the transfer matrix via residues at its poles:

$$\mathrm{res}_{u=u_k}\left.\frac{T(u)}{T^{\emptyset}(u,0)}\right|_{|\omega(\{M_i\})\rangle}=-\dot{u}_k(0). \tag{5.10}$$

The solution is not unique: different possible solutions to this problem correspond to different eigenstates of the transfer matrix in the sector $\mathcal{H}(\{M_i\})$.

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