

HIGHER SCHOOL OF ECONOMICS
NATIONAL RESEARCH UNIVERSITY

Alexander Smirnov

**FINANCIAL BUBBLES AS LARGE
ASSET PRICE DEVIATIONS**

Working Paper WP7/2011/05
Series WP7

Mathematical methods
for decision making in economics,
business and politics

Moscow
2011

УДК 336.144.36
ББК 65.261
S68

Editors:

Aleskerov Fuad, Mirkin Boris, Podinovskiy Vladislav

S68 **Smirnov, A.** Financial Bubbles as Large Asset Price Deviations : Working paper WP7/2011/05 [Text] / A. Smirnov ; National Research University "Higher School of Economics". – Moscow: Publishing House of the Higher School of Economics, 2011. – 36 p. – 150 copies.

Constructed a model of financial markets transition from laminar to turbulent regime. Such a transition manifests itself in emergence and evolution of financial bubbles that ultimately burst in subsequent crises. The bubble emergence is analyzed in a stochastic context of "large asset price deviations" from their fundamental value. Irrational herding of financial investors, in particular, bears responsibility for asset prices divergence which in the model takes place at the critical point of liquidity issuance. Bursting of a bubble is viewed as a result of financial investors interactions producing qualitative changes to the aggregate system behavior. The latter formally is represented as singularity of a Bernoullian differential equation that could be studied via percolation process in the financial market. Investigation of a system's behavior near the point of singularity seems to be of vital importance since it provides essential clues to our understanding of "how markets fail".

УДК 336.144.36
ББК 65.261

JEL Classification: E63, G13, H63, P24.

Key words: financial bubble, credit meltdown, singularity, complex system.

Smirnov, Alexander. D., Professor, National Research University "Higher School of Economics", Pokrovsky Boulevard 11, Moscow 101987, Russia, tel. 772 9590*2175, e-mail: adsmir@hse.ru ; adsmir@gmail.com

Смирнов, А. Д. Финансовые пузыри как дивергенция цен активов : препринт WP7/2011/05 [Текст] / А. Д. Смирнов ; Нац. исслед. ун-т «Высшая школа экономики». – М. : Изд. дом Высшей школы экономики, 2011. – 36 с. – 150 экз. (на англ. яз.).

Рассматривается модель зарождения и развития финансовых пузырей и кризисов, основанная на концепции современной финансовой системы. Большие отклонения цен активов от их фундаментальной стоимости характеризуют финансовый пузырь, динамика которого представлена стохастическими дифференциальными уравнениями. Модель характеризует фазы поведения инвесторов, которые соответствуют циклу Х. Минского, и позволяет рассчитать различные характеристики финансового пузыря.

Классификация JEL: E63, G13, H63, P24.

Ключевые слова: финансовый пузырь, кредитный кризис, сингулярность, сложная система.

Смирнов Александр Дмитриевич – заслуженный деятель науки РФ, доктор экономических наук, профессор, действительный член Российской академии естественных наук, Национальный исследовательский университет «Высшая школа экономики». 101987 Москва, Покровский бульвар, 11. Тел: 772 9590*2175.

E mail: adsmir@hse.ru , adsmir@gmail.com.

**Препринты Национального исследовательского университета
«Высшая школа экономики» размещаются по адресу: <http://www.hse.ru/org/hse/wp>**

© Smirnov A., 2011

© Оформление. Издательский дом
Высшей школы экономики», 2011

For whoever knows the ways of Nature will more easily notice its deviations and, on the other hand, whoever knows its deviations will more accurately describe her ways.

Sir Francis Bacon, Novum Organum

Global financial crisis of 2007–2009 had brought about huge losses imposed upon financial institutions; it overloaded customers with houses, disrupted the world trade and production, and dramatically increased unemployment¹. In intellectual sphere financial crisis was accompanied by drastic disillusionment with the “market fundamentalism” (Soros, 2006) including its theoretical projection – the “representative agent” model. Barrage of criticisms in their address was understandable in view of spectacular failure of a theory to predict the upcoming crisis timely. This failure was, possibly, due to two major reasons. First, being calmed down by prolong period of the so called “Great Moderation”, economists, to some extent, lost their interest in the analysis of extreme (or “fat tail”) events like bubbles or crises. Paradoxically enough, the mere subject of crises was effectively excluded from the domain of economic and financial theory (Lucas, 2003). Once these phenomena had been declared virtually nonexistent, no wonder that the “representative agent” model was doomed to demonstrate its inadequacy to the complexity and uncertainty of a modern financial system.

More important, as it became increasingly noticeable, the rational agent model, by itself, appeared to be incapable to identify and filter out extreme events similar to bubbles or crises. The core of that problem lies in unrestrained reliance of the model on reductionism. But the reductionist methodology, as it is known, claims, unconditionally, the existence of similarity

¹ The author is grateful to the participants of seminars at the London School of Economics and the National Research University – Higher School of Economics (Moscow), and especially to Professors Avinash Dixit, Charles Goodhart, Fuad Aleskerov and Emil Ershov, though nobody, except the author, would bear the responsibility for possible inconsistencies and errors.

between any system and its (typical) element. This assertion while being quite relevant under some conditions cannot be justified generally, though. Yet, in a guise of a “representative” investor, it has been dominant in financial science. This dominance has been continued for several decades, in spite of the fact that its inadequacy was well known and extensively documented (Mandelbrot, 2005). In our point of view, these circumstances explained precisely the *causa sine* of impotence of that model. On the one hand, methodology of reductionism appeared to be largely at odds with the reality, for, as a rule, economic and financial markets are scale free only under some special conditions. On the other hand, such an assertion, rather evidently, contradicts to the modern complex systems theory (Stanley et al., 2003). The latter is focused on the analysis of the so called “critical points” where a general system transforms its quality and behavior. Since financial system evolves largely as a laminar flow while infrequently, under some specific conditions, producing a burst of turbulence, the endogenous mechanisms of such “switching” are of great importance. Possibly, they are working as transformation of heterogeneous (“normal”) markets into homogeneous ones, transformation, which is typically manifested by emergence of financial bubbles.

Devastating damage, both theoretical and practical, has evoked and initiated an unprecedented search for the “New Economic Paradigm” (Stiglitz, 2010) including comprehensive revision of mere foundations of financial science. In our view, the new paradigm could be constructed on the solid basis of the complex system methodology. The latter provides a researcher with a wide spectrum of methods and models (*Encyclopedia*, 2009). An important avenue within such an intellectual thrust seems to be an investigation of the mechanism of financial bubbles. Financial bubbles and crises are complex, hierarchically organized phenomena with many different features that are prominent either on a macro- or a microlevel. Macro aspect of financial activity, contrary to microfinance, has for different reasons remained a relatively underdeveloped part of modern finance. Yet the ideas of J.M. Keynes, I. Fischer, H. Minsky, H. Simon, B. Mandelbrot as well as

some of the contemporary researchers, with their emphasis on the “animal spirit”, “herding” or “irrational exuberance” has formed, in effect, foundations of macrofinancial theory including analysis of bubbles and crises.

Financial bubbles were historically always forerunners of crises. The modern financial system analysis (Rajan, 2005) takes the view on bubbles in a context of “large asset price deviations” from their fundamental value. Similar ideas were developed and investigated in (Turner, 2010) who wrote that “all liquid financial markets are susceptible to unstable divergence from equilibrium values”. Such a general system approach takes essentially into account market interactions which are responsible for qualitative changes in the aggregate system behavior. Irrationality or herding of financial investors, in particular, bears responsibility for asset price divergence that under particular conditions brings about a system’s collapse. The possibility of persistent deviations of asset prices from their fundamental value was shown in (Campbell and Shiller, 1992) for AR (1) stochastic processes. This paper makes an attempt to describe prices divergence at the critical point (where a system becomes singular) by appealing to investors’ actions and motivations. Investigation of a system’s behavior around the point of singularity is of vital importance since it would provide essential clues to our understanding of “how markets fail” (Cassidy, 2009).

The model overview

The proposed model aims to describe the financial market transition from laminar to turbulent regimes. Such a transition manifests itself in emergence and evolution of a financial bubble that ultimately bursts in a subsequent crisis. The model elaborates and quantifies some important aspects of a “large asset prices deviation” – theoretical concept concerned with the modern financial system. According to it the phenomenon of asset price divergence at the critical point of liquidity issuance reveals a nonzero probability of a system collapse, the latter being an outcome of financial

investors herding and “fat tail” financial events. Some important features of these complex processes are reproduced in the proposed model.

Initial conditions in the model were given by the Keynesian two-component system consisting of money and debt. Though being very simple, this model provides a thorough description of the major impact of the quantitative easing , QE, policy under which the huge influx of money circulates in the financial markets mainly leaving the real ones relatively immune to the unprecedented monetary stimulus. The debt market is “broad” and “deep” with many heterogeneous buyers and sellers. Under “normal” circumstances its dynamics go on along as debt monetization process. The latter was described via the debt value equation which reflects interactions between money and debt. Debt value in aggregate is considered as a function of random liquidity issuance, the latter being subject to the lognormal distribution (geometric Brownian motion). Random process of liquidity issuance by the central bank brings about changes to the value of financial claims; hence all the model variables depend upon money issuance except the par value which is assumed to be constant. Debt purchases were represented via call option written on the debt expected value with its face value being a strike price. The total debt guarantees, in their turn, were viewed as a put option also being written on its expected value. Call and put options were imbedded into debt making its market value a structured financial product. It has a dual representation: either as a difference between its expected value and the call option, or equivalently, as the par value minus the put-to-default option. The above said statement is nothing more than the model interpretation of a well known put – call equivalence theorem. A modified representation of that equivalence (using the basic accounting equation) allows performing asset and equity value estimations. Value of equity in the system appears to be equal to the sum of put and call options.

The model performance is facilitated by persistent drift in liquidity issuance that brings changes to all financial variables except the par value. At the critical point of liquidity issuance financial system might arrive dually, under two different regimes. By definition, normal regime of financial mar-

ket implies the absence of investors' herding. Hence, in the "normal" regime the system has possessed enough of capital that makes the probability of a crisis strictly less than unity. Otherwise, random process of liquidity issuance being coupled with herding might produce "irrational exuberance" of investors that drives the system towards its inevitable collapse. Financial process becomes autocatalytic that ends up at the critical point as a leverage singularity.

It is demonstrated in the following that system develops in different phases that roughly correspond to the Minsky financial cycle. Three phases of investors' behavior depicted by H. Minsky are easily identified in a sequence of hedge finance, speculation, and the Ponzi game ultimately ending in the total collapse. Application of a stochastic calculus along the lines of (Dixit and Pindyck, 1994) makes the relevant variables power functions that permits routine calculations. From the formal point of view processes of herding in the financial market were represented as a solution to the "trivial" dynamic programming problem. In the last part of the paper financial bubble is viewed as the percolation of the financial market. Models of such type are seemed adequate in investigating of microfinancial interactions among investors (Smirnov, 2007, 2008, and 2010) but they are forming a different approach which is beyond the scope of the paper. The model proposed was focused primarily on "large price deviations", their origins and consequences.

Financial market dynamics

As stated above, initial conditions are defined for a "normal" system performance. Under these conditions financial market dominated by rational investors and has a simple two-component representation. At any time, t the total value of assets, $A(t)$, is equal to the sum of money, $M(t)$, and to the expected value of debt, $B(t)$:

$$(1) \quad A(t) = M(t) + B(t),$$

where each variable is a continuous and at least twice differentiable function of time. In any infinitesimally short period of time all the borrowers in aggregate are to service their debt at the market rate of return, μ , subject to $dA = \mu B dt$, while creditors are to agree receiving periodical (coupon) income, $dM = m(t)dt$, and to acquire new debt, dB . Hence the creditor-borrower balance would correspond to the following equation:

$$(2) \quad \mu B(t)dt = m(t)dt + dB.$$

Given initial debt, $B(0)$, equation (2) can be solved with regard to the future debt value:

$$(3) \quad B(t) = B(0) \exp[\mu t] - \int_0^t m(u) \exp[-\mu(u - t)] du.$$

The future debt value given by (3) might increase indefinitely in the future. Thus its amount to be redeemed, $B(t) = 0$, gives the value of a continuously compounded annuity:

$$(4) \quad B(0) = \int_0^t m(u) \exp[-\mu u] du$$

by making up the future flow of continuously accrued coupon payments to its market value.

By definition, as it follows from equation (2), the risk-adjusted rate of return, is equal to the sum of current yield, $\delta = \frac{m}{B}$, and the rate of capital appreciation (loss), $a = \frac{dB}{B}$:

$$(5) \quad \mu = \delta + a,$$

while, on the other hand, as it is known from the CAPM theory, the same risk-adjusted rate of return might be decomposed into the sum of riskless rate, r , and risk premium, $\lambda\sigma$:

$$(6) \quad \mu = r + \lambda\sigma$$

where λ is a unit risk price.

In the following the expected debt value is considered to be twice differentiable function of money issuance (density at any moment of time, s_t) alone, $B(t, s_t) = B(s_t)$. Hence debt maturity profile is irrelevant to the model. Fig. 1 shows the weighted index of market liquidity (Gieve, 2006) whose changes in time resembled a standard stochastic process. Hence

money issuance is supposed to be a random process depending upon time t , $s = s_t$. Continuous decomposition the of money issuance rate of change into its deterministic and pure stochastic components makes the latter a simple geometric Brownian motion:

$$(7) \frac{ds}{s_t} = a dt + \sigma dz_t$$

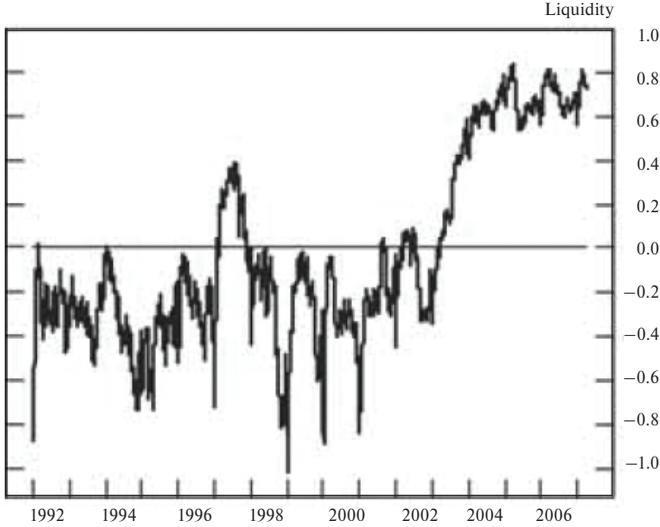


Fig. 1. The weighted index of market liquidity measures for 1992–2006 (Gieve, 2006)

where a is drift and σ is volatility parameter for money issuance, s_t . Stochastic differential equation (7) can be solved along the standard procedures that give rise to the following random process for money issuance:

$$(8) s_t = s_0 \exp [(a - 0.5\sigma^2)t + \sigma z_t]$$

where the term $z_t = \int_0^t dz_u$ is the Ito integral of random noise, and

$$(9) \langle s_t \rangle = s_0 \exp [at]$$

serves as a representation of the expected money issuance.

Thus central bank monetary policy, such as quantitative easing, QE, is performed in accordance with (8) subject to a stochastic noise while market

participants expect money issuance in amounts given by (9). In accordance with the logic described above, financial investors, due to money issuance, persistently acquire new debt in a random fashion. Its total value evolves according to the following stochastic equation:

$$(10) dB = [\mu B(s_t) - s_t]dt + \sigma B(s_t)dz_t.$$

Generally, volatilities of debt and liquidity processes in (7) and (10) are different but this technical detail is avoided for the moment. The debt value (10) and the liquidity dynamics (7) equations together with conditions (5) and (6) form the basic structure of the stochastic money – debt model to be developed further.

Options of new debt and debt protection

Assume that expected debt value could be decomposed into par and embedded option to purchase new debt by debt holders:

$$(11) B(s_t) = F + [B(s_t) - F].$$

Since investors are not under the obligation to buy they, in effect, do possess an option of debt purchase which is expressed as a plain-vanilla call option:

$$(12) f(s_t) = [B(s_t) - F, 0]^+$$

being written on the expected debt value $B(s_t)$ with par, F , as a strike price. The upfront option premium might be zero or very small. Buyers exercise their option to purchase new debt if expected value exceeds the par, and do nothing otherwise. For example, anticipating increase in liquidity due to the central bank “easy money” policy they would reasonably try to benefit from interest rates decreases that make the expected debt larger than its par value. This is typical behavior of investors who expect decreasing interest rates in the future or perform “flight to the quality” under market distress and “quantitative easing”. Since investors are not obligatory to make purchases, they could refuse to buy new debt in the case of interest rates increases. The above said assumption makes total debt similar to a callable bond. Evidently, equation (11) in terms of liquidity issuance represents behavior of a ra-

tional investor that hinges around the expected interest rates as was depicted in (Keynes, 1936).

The expected debt value being stripped of the option to buy new debt makes the market debt value a simple structured product of a following form:

$$(13) D(s_t) = B(s_t) - [B(s_t) - F, 0]^+$$

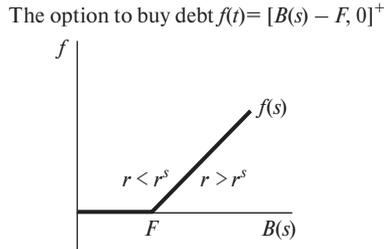


Fig. 2. Matured option to purchase new debt

On the other hand, bond holders are concerned with possibility of losses due to increasing interest rates. Anticipating stochastic money issuance bond holders protect their wealth by acquiring debt guaranties that are widely traded in financial markets. From the well-known reasoning (Merton, 1976) it follows that the value of a debt guaranty has a put-to-default option representation:

$$(14) P(s_t) = [F - B(s_t), 0]^+.$$

Market debt value, again, becomes a simple structured product of the following form:

$$(15) D(s_t) = F - [F - B(s_t), 0]^+.$$

Since put and call options have the same strike price (and maturity profile) equations (13) and (15) being taken together describe investors' behavior via the "chooser" option which allows them to benefit from the large changes in the debt value. Next, combined equations (13) and (15) give us

the market value of aggregate debt, $D(s_t)$, that satisfies conditions of a put-call equivalence theorem:

$$(16) B(s_t) - f(s_t) = D(s_t) = F - P(s_t).$$

It is easy to show for maturing option contracts that adding and subtracting both put and call values from r.h.s. and l.h.s. of equalities (16) would lead to a following representation of total financial assets, $A(s_t)$:

$$(17) B(s_t) + P(s_t) = A(s_t) = F + f(s_t).$$

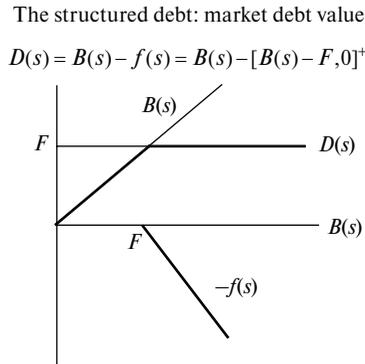


Fig. 3. Market value of a debt at options maturity

Next, by subtracting (16) from (17) due to fulfillment of the basic accounting equation

$$(18) A(s_t) = D(s_t) + E(s_t),$$

we arrive at the following definition of the capital (equity) value:

$$(19) E(s_t) = f(s_t) + P(s_t),$$

where $E(s_t)$ is the value of the owner's capital for financial system as a whole. Note, that due to the definition of the debt purchase option (12), the equity in the system (19) is different from its analogue in the well-known Merton model. Equation (19) implies that options even not being exercised do have positive value due to some additional abilities of investors provided by the well developed financial system. These possibilities are responsible

for the Pigou effects upon the total wealth that might have disastrous consequences, if being exaggerated.

In the following it is important to distinguish among three quantities: nominal, F , market, $D(s_t)$, and expected, $B(s_t)$, debt value. In the model the first of them is assumed to be independent of liquidity issuance, and hence constant. Market value, as seen in Fig. 3, cannot exceed its par value, while the expected value (see Fig. 2) depending upon liquidity issuance anticipated by investors might take any value thus becoming larger, smaller or equal to the par.

The expected debt valuation

In an uncertain financial market the aggregate debt is evolved stochastically in accordance with equation (10). Taking into account liquidity dynamics as it was represented by (7) the debt infinitesimal change being transformed along the Ito lemma gives us the following stochastic equation

$$(20) \quad dB = [as_t B'(s_t) + 0.5\sigma^2 s_t^2 B''(s_t)]dt + \sigma s_t B'(s_t) dz_t$$

where debt derivatives are taken with respect to the liquidity issuance s_t . From (5) and (6) it follows that

$$(21) \quad r - \delta = a - \lambda \sigma ,$$

and coefficients for deterministic and random components in equations (10) and (20) could be equated into the following pairs:

$$(22) \quad \mu B(s_t) - s_t = a s_t B'(s_t) + 0.5\sigma^2 s_t^2 B''(s_t)$$

$$(23) \quad \sigma B(s_t) = \sigma s_t B'(s_t).$$

Simultaneous transformations of (22) and (23) bring about the following inhomogeneous second order differential equation with respect to function (s_t) :

$$(24) \quad 0.5\sigma^2 s_t^2 B(s_t)'' + (r - \delta)s_t B(s_t)' - rB(s_t) + s_t = 0$$

which is an analogue to the well known Black-Sholes equation.

The expected debt value function $B(s_t)$ as a solution to (24) takes the following form:

$$(25) \quad B(s_t) = B_1 s_t^{\beta_1} + B_2 s_t^{\beta_2} + \frac{1}{\delta} s_t,$$

where $\beta_1 < 0$ and $\beta_2 > 1$ are real and distinct roots of the characteristic equation (26) corresponding to the homogeneous part of (24):

$$(26) \quad 0.5\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r = 0.$$

Since $\beta_1 < 0$ the first component of (25) for the very small money issuance goes to infinity. Hence, in order to preserve the economic sense of solution, the constant B_1 in (25) should be chosen as zero. This is so called “the absorption” condition requiring zero debt value in the absence of money issuance: its violation would lead to the unlimited debt growth that should be excluded from the model. The second constant in (25) is taken as zero, $B_2 = 0$, due to representation of the expected debt as its fundamental value which also excludes situation of the unlimited growth of the latter. Hence the expected debt value, $B(s_t)$, becomes the particular solution to equation (24) of the following form:

$$(27) \quad B(s_t) = \frac{1}{\delta} s_t.$$

The expected debt value, as it follows from (27), is just the perpetual (capitalized) value of the future stream of coupon payments being discounted by the current yield δ . Interestingly enough to note that it is an anticipation of money growth, or the future excess liquidity, that makes rational investors to be complacent with the current yield, δ , though it is smaller than the risk adjusted rate, μ . It becomes evident after taking expectation of the money issuance in accordance with equation (9): while investors use precisely the risk adjusted rate, μ , to discount flow of future payments they do anticipate increases in the future money issuance. Hence evaluating (at point $t = 0$) conditional expectation gives the same as in (27) formula for the expected debt value:

$$(28) \quad B_0 \equiv \langle B_0 \rangle = \int_0^\infty \langle s_t \rangle \exp(-\mu t) dt = s_0 \int_0^\infty \exp[-(\mu - a)] dt = \frac{1}{\delta} s_0.$$

According to (28) investors consider debt as a perpetuity which, being combined with anticipated future excess liquidity, would contribute to the

persistent asset overvaluation. This process explains the subsequent “large price deviations” phenomenon that takes place at the critical point of money issuance.

Investors’ new debt portfolio

In the model financial investors combine their decisions to buy new and guarantee existing debt with their decisions to hedge portfolios. The dominant group of investors is assumed to be able to fulfill this task, in other words, there is enough market participants who take the opposite position. Let the incremental portfolio, $\Phi(s_t)$, consisting of money issuance and new debt, be represented as follows:

$$(29) \quad \Phi(s_t) = \theta_1 s_t + \theta_2 f(s_t),$$

where θ_1, θ_2 are the weights of new money and new debt, respectively. This incremental portfolio could be made riskless, if investors are to choose special values of constants, namely, $\theta_1 = -f'(s_t)$ and $\theta_2 = 1$. With these constants, and due to (7), infinitesimal change $d\Phi$ to the incremental portfolio (29) becomes riskless:

$$(30) \quad d\widehat{\Phi}(s_t) = 0.5\sigma^2 s_t^2 f''(s_t) dt.$$

Evaluation of riskless return on the hedged portfolio requires the latter to be decreased by the amount of money payment, $\theta_1 \delta s_t dt$, being lost due to hedging. These requirements give rise to the following equation:

$$(30) \quad r[\theta_1 s_t + \theta_2 f(s_t) - \theta_1 \delta s_t] dt = 0.5\sigma^2 s_t^2 f''(s_t) dt.$$

By using the hedging values of constants and dividing (31) through by dt we arrive at the following equation for the riskless portfolio held by investors:

$$(32) \quad 0.5\sigma^2 s_t^2 f''(s_t) + (r - \delta) s_t f'(s_t) - r f(s_t) = 0.$$

The important and rather unexpected result of these transformations is that the “new debt” equation (32) has the same parameters (and the same characteristic equation) as the homogeneous part of the debt value equation (24). Thus, by performing the same procedures of its solving as before, we get the value of the option to buy new debt as a function:

$$(33) \quad f(s_t) = K_1 s_t^{\beta_1} + K_2 s_t^{\beta_2}.$$

Due to the absorption condition the first constant in the r.h.s. of (33) has to be zero. Hence the option to purchase new debt (33) becomes the power function of a random liquidity s_t :

$$(34) \quad f(s_t) = K s_t^\beta$$

where $K \equiv K_1 > 0, \beta \equiv \beta_1 > 1$. Since call option is exercised in the money, investors, quite naturally, behave so as to maximize the value of option (34). That could be done if money issuance starts to increase due to the “easy money” policy of the central bank, or the “quantitative easing, QE, that facilitates changes in asset values.

Large asset prices deviation

Formally, random growth of liquidity to some undefined upper boundary $s = s^*$ can be represented as a “trivial” solution to the dynamic programming problem (Dixit, Pindyck, 1998). In order to find point $s = s^*$ the second order differential equation (32) has to be complemented with three boundary conditions. They consist of the initial value condition, $f(0) = 0$, together with the value-matching condition:

$$(35) \quad f(s^*) = F - B(s^*),$$

and the smooth-pasting condition (in derivatives with respect to random variable s_t):

$$(36) \quad f(s^*)' = B(s^*)'.$$

Upon substitution of the expected debt value (27) and the new debt value (34) into equations (35) and (36), point $s_t = s^*$ could be found as the following quantity:

$$(37) \quad s^* = \frac{\beta}{\beta-1} \delta F.$$

The free boundary point $s = s^*$ has an important economic meaning since it delivers maximum value to the debt purchase option. The latter at this point appears to be “in the money” and should be exercised but whether it is exercised or not depends upon the broad market conditions to be stud-

ied later. Investors' purchases of debt in additional amounts are influenced by the increase in liquidity up to its upper boundary or critical level at $s = s^*$. Hence this point could be treated as a critical point of money issuance in two senses: it is a point of maximum value of an option and, as it will be seen soon, extreme or "fat tail" events happen here. Money issuance at the free boundary point simultaneously maximizes the expected value of debt, $B(s)$, up to amount of

$$(38) \quad B(s^*) = F + f(s^*).$$

Due to put-call equivalence theorem (16) this is possible if the put value would go to the zero, $P(s^*) = 0$. Remember that the market value of debt is restricted from above by its nominal value, $D(s^*) = F$.

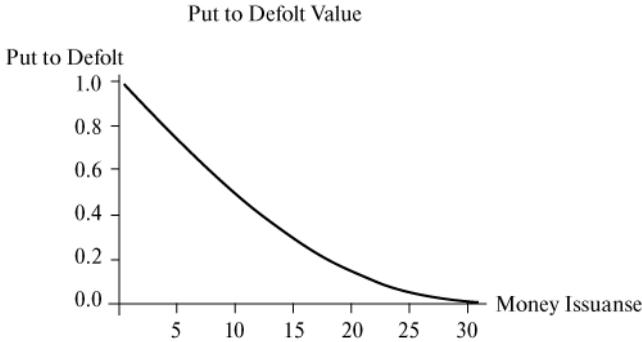


Fig. 4. Value of debt protection

Behaviour of a model in economic terms might be described as follows. The central bank, in a conduct of "easy money" policy (or the quantitative easing), issues money according to (8) thus changing its quantity in the system. Additional liquidity is anticipated by investors according to (9). Since, under the circumstances, interest rates are expected to decrease in the future, investors are trying to benefit on higher debt value. In accordance with (11) they buy new debt thus forming greater demand and starting persistent asset price growth along (27). Under these circumstances investors rebalance their portfolios by decreasing debt guarantees while maximizing the

value of new debt purchases (35) – (36). Simultaneously investors hedge risks by eliminating money – major source of uncertainty – from their incremental portfolios (31). It is important to recall that they do it in the atmosphere of unanimous optimism being supported by persistent asset prices increases (growing asset prices are “financial narcotics”, in the words of W. Baffett). Along the blowing bubble investors maximize and exercise their options to buy new debt.

Critical point without herding

To purchase new debt investors in aggregate have to spend their money hence to decrease their value of debt guaranties. Additional demand for new debt supports the growing price of new debt thus inducing investors, in accordance with (35) and (36), to substitute market debt value for its expected value. These coherent actions imply a persistent process of the debt overvaluation. It follows from (38) that expected value exceeds its market value for any positive value of a call option. Hence at the critical point the asset value increases at the rate of

$$(39) \quad \frac{B(s^*)}{D(s^*)} = \frac{1}{\delta} \frac{\beta}{\beta-1} \delta F: F = \frac{\beta}{\beta-1} > 1,$$

where the magnitude $\frac{\beta}{\beta-1}$ defines the scale of the asset prices divergence at the critical point. By definition, it implies a short run Pigou effect upon the total wealth under normal conditions in the financial market. It should be noted that under some conditions this effect might be totally spurious thus triggering losses incurred to investors in the time of crisis.

Under “normal” conditions, as it follows from (17), the total assets value at the point s^* is equal to:

$$(40) \quad A(s^*) = F + f(s^*),$$

since, by definition, the market debt value at the critical point equals to its nominal value: $D(s^*) = F$. What is the amount of financial equity in the system at the critical point? The answer depends upon the hypothesis of herding.

Assuming no herding, or prevalence of rational market participants, at the critical point of liquidity issuance s^* rational investors would not exercise their call option thus keeping a strictly nonzero value of their own capital:

$$(41) \quad E(s^*) = A(s^*) - D(s^*) = \frac{1}{\delta} \frac{\beta}{\beta-1} \delta F - F = \frac{1}{\beta} F = f(s^*) > 0.$$

Equation (41) implies that investors while maximizing the call option are rational, in other words, are cautious enough as to keep in the money form an additional value to their nominal debt assets. Consequently, under the “no herding” condition due to investors’ accumulation of equity, the “distance-to-default” magnitude, or the system “survival”, at the critical point amounted to the quantity:

$$(42) \quad \Pr[\textit{survival} \equiv \textit{Distance} - \textit{to} - \textit{Default}] = \frac{A(s^*) - D(s^*)}{A(s^*)} = 1 - \frac{\beta-1}{\beta} = \frac{1}{\beta}.$$

Alternatively, without herding or “irrational exuberance”, the probability of financial default, however large, should be strictly less than unity:

$$(43) \quad 0 < \Pr[\textit{default}] = \frac{D(s^*)}{A(s^*)} < 1.$$

Since, by definition, $\Pr[\textit{default}] = 1 - \Pr[\textit{survival}]$, the default probability in the model should be equal to the following quantity:

$$(44) \quad \Pr[\textit{default}] = \frac{\beta-1}{\beta}.$$

Such a scenario was investigated in (Smirnov, 2005) as a preferable, though not realized in practice, outcome of the government debt collapse in Russia in August 1998.

Herding at critical point

The coherent behavior of investors that are hedging simultaneously their portfolios, though being an optimal one, brings about some unexpected consequences. This is the central part of the proposed model performance. As (Chan et al, 2005) pointed out, the 1998 default on Russian government

debt induced a dramatic increase in market correlations. Instead of being negligibly small in normal times they turned virtually overnight to plus one – a phenomenon they termed the “phase lock-in”. In other words, if herding takes place, collective behavior of investors would bring about dramatic changes to the market. The nature of herding *per se* might be captured via models of a percolation in financial markets (Stauffer, 2001) but it is a different avenue of financial studies that is not pursued in this paper.

The process of hedging in the model would imply that every one of financial investors while maximizing the option to buy new debt, should, in effect, substitute the market debt value, $D(s_t)$, for its expected value, $B(s_t)$. In spite of the finite scale of a bubble given by (39), the market leverage performed by investors, $D(s)/E(s)$, might grow indefinitely, since $D(s^*) = A(s^*) = B(s^*)$. It was noted and studied extensively in (Adrian and Shin, 2008) that increasing asset prices are followed by growing leverage in the market. At the critical point, $s_t = s^*$, according to (38) and (40) the expected debt value becomes equal to the total assets value which forces amount of capital in the system to diminish virtually to the zero:

$$(45) \quad E(s^*) = A(s^*) - B(s^*) = 0.$$

Hence the ongoing process of herding among investors implies that *a posteriori* probability of default equals to one:

$$(46) \quad \Pr [default] = \frac{B(s^*)}{A(s^*)} = 1,$$

that makes crisis to be a virtually inevitable event. From the economic point of view, it might be concluded that persistent increasing of the money issuance (excess liquidity) coupled with herding would lead to the systemic collapse. The latter is the sudden and dramatic decline in asset prices that might take place at the once benign point of money issuance $s = s^*$ were the herding features of investors behaviour become pronounced enough to dominate the market.

As it was stressed in (Rajan, 2005), “[the] prolonged deviations from fundamental value are possible because relatively few resources will be deployed to fight the herd”. Unfortunately, few would want to go up against

the trend that is originated by enormous mass of traders. Evidently, at the critical point investors trying desperately to get higher asset value are doomed to increase leverage as it had happened with investment banks on the eve of financial meltdown in 2007.

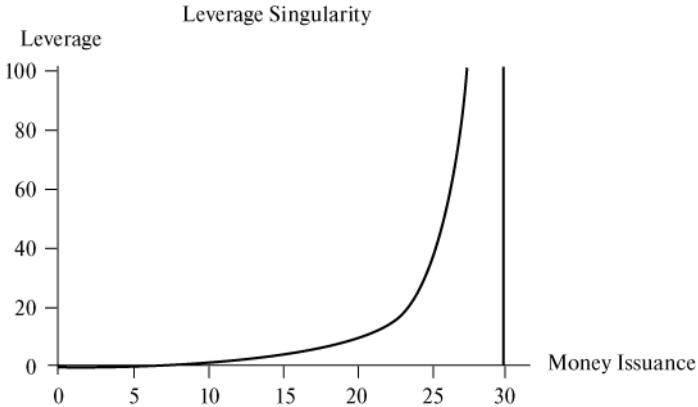


Fig. 5. Leverage singularity

As a direct consequence of such a development, financial leverage ratio

$$(47) \frac{A(s^*)}{E(s^*)} \rightarrow \infty,$$

theoretically, as shown in Fig. 5, should increase indefinitely at the critical point which makes the occurrence of a financial crisis to be a virtually inevitable event.

The Minsky point

As follows from the above said, it is important to distinguish between the trajectory of “normal” increases in the debt value which is represented by function $D(s_t)$, and the debt overvaluation process going on along trajectory $B(s_t)$. This problem has an important real life equivalent of the early detection of the asset price overvaluation. When the system has passed through the point of bifurcation, the divergence of the expected debt value

from its market value becomes noticeable and significant. In effect, it is the point where the financial bubble emerges, hence, it might be called as” the Minsky” point. As learned from the history of finance, the failure to depict properly the Minsky point would have had ominous consequences for the market: after it has passed over to reverse the avalanche of financial assets value seems virtually impossible. Economists debated this issue for a long time: it is enough to recall the critique of Greenspan’s policy by P. Krugman (Krugman, 2008) for its inability to prevent the bubble growth in the housing and credit markets.

It is reasonable to identify the Minsky point with intersection tween $D(s_t)$ or $B(s_t)$ with trajectory of the debt protection, $P(s_t)$. Assuming that equation

$$(48) \quad D(s_m) = P(s_m)$$

takes place then the Minsky point $s_t = s_m$ might be found as the solution to

$$(49) \quad F = 2 P(s_m).$$

On the other hand, if there is an equation

$$(50) \quad B(s_m) = P(s_m)$$

then its solution might be found as

$$(51) \quad F + f(s_m) = 2 P(s_m).$$

Evidently, the discrepancy between (49) and (51) is rather small since the option to buy new debt is out of the money for the small liquidity issuance.

Another view on system singularity

The system singularity which is aftermath of zero equity condition at the critical point, $E(s^*) = 0$, might be deduced alternatively, via investors’ expectations, along the following reasoning. Remind, that according to equation (1) at any point of time the sum of expected values for money and debt is given by

$$(52) \quad \langle A(t) \rangle = \langle M(t) \rangle + \langle B(t) \rangle.$$

The expected money aggregate, $\langle M(t) \rangle$, at time t is equal, by definition, to

$$(53) \quad \langle M(t) \rangle = \int_0^t \langle s_u \rangle du = \int_0^t s_0 \exp[au] du = \frac{s_0}{a} (\exp[at] - 1)$$

due to expected money issuance given by equation (9). At the same time, by taking the expected debt value from (27) we get

$$(54) \quad \langle B(t) \rangle = \frac{1}{\delta} \langle s_t \rangle = \frac{1}{\delta} s_0 \exp[at].$$

Hence, by adding (53) and (54), the expected asset value at time t becomes to be expressed as follows:

$$(55) \quad \langle A(t) \rangle = \frac{s_0}{a} \left(\frac{\mu}{\delta} \exp[at] - 1 \right).$$

Taking the time of a crisis as $t^* = 0$, which takes place at the critical point of money issuance, $s^* = s_0$, we get the total asset value (55) as

$$(56) \quad \langle A(t^*) \rangle = \langle A(s^*) \rangle = \langle B(s^*) \rangle.$$

Hence, analysis of investors' expectations leads to the same result as before: the value of total assets at the critical point (due to herding) consists of expected debt only. As such, the result (56) would have suggested also the ergodic character of financial processes but this assertion is in need of the further exploration.

Yet one more important comment should be made with regard to the "absolute" quantities of debt, money and value of total assets. Comparing equation (1) being evaluated at the time of a crisis $t^* = 0$

$$(57) \quad \langle A(t^*) \rangle = \langle M(t^*) \rangle + \langle B(t^*) \rangle$$

with equation (17) evaluated at the critical point $s = s^*$

$$(58) \quad A(s^*) = B(s^*) + P(s^*)$$

we have to conclude that

$$(59) \quad \langle M(t^*) \rangle = P(s^*) = 0$$

since $P(s^*) = 0$ due to (38). Since total money aggregate is a nonzero quantity, equality (59) being taken literally, forms a logical contradiction. In fact, this controversy is a spurious one, and it could be explained as a mere manifestation of the system singularity. Any financial crisis, or event, that takes place at the critical point, is a crisis of liquidity. Hence equality $\langle M(t^*) \rangle = P(s^*) = 0$ is just a demonstration of the fact that at the critical

point amount of liquidity is negligible comparing to the total debt value. Money expectation is zero at the critical point while the random quantity of money is not. Crisis, by definition, destroys the system which becomes singular at the critical point which is manifested by “zero-money” condition (59).

Numerical primer

The model described above is illustrated numerically using the following, quite realistic, parameters. Assume a system with nominal debt of 400 billion of dollars, $F = \$400bn$, riskless rate of return per annum, $r = 0.05$, and annual risk-adjusted interest rate, $\mu = 0.07$. The latter equals to the sum of current yield, $\delta = 0.045$, and annual capital gain, $a = 0.025$. Amount of risks (per annum) in the system being measured by its volatility is equal to the quantity, $\sigma = 0.15$.

The characteristic equation of such a system:

$$0.5 \times 0.15^2 \beta (\beta - 1) + (0.05 - 0.045) \beta - 0.05 = 0$$

has two distinct real roots: $\beta_2 = -0.099$, and $\beta_1 = 2.404 > 1$ of which only the positive root has an economic meaning. It is shown in Fig. 6. It is interesting to note that in general case financial processes in the model are fractals and are not too far from the so called “cubic law” (Lux, 2006).

The graph depicted in Fig. 7 demonstrates the system behavior and its major characteristics. The critical point of money issuance being defined as in (32) is equal in our example to

$$s^* = \frac{2.4}{1.4} \times 0.045 \times 400 = \$30.86bn.$$

This quantity defines the expected value of a debt at the critical point:

$$B(s^*) = \frac{1}{0.045} \times 30.86 = \$685.75bn,$$

and the value of new debt:

$$f(s^*) = 0.0761 \times 30.86^{2.4} = \$285.74bn$$

where constant $K = 0.0761$. Hence at the critical point $s^* = 30.86$ (billion of dollars) the expected value of debt equals to \$685.75 bn, the latter being

the sum of the nominal debt (\$400bn) plus new debt (\$285.7bn) with a small error.

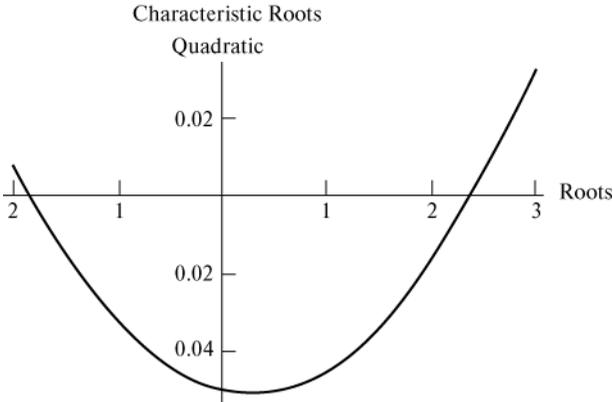


Fig. 6. Characteristic equation and roots

Were total assets $A(s^*)$ to be comprised of nominal debt, $F = \$400bn$, and equity, $E(s^*) = \$285.7bn$, that is totaling to the same amount as before, \$685.7bn, the system though being fragile would have survived with probability of $\Pr[DtD] = \frac{285.7}{685.75} = 0.42$. Alternatively, the same magnitude of the “distance-to-default” probability could be given by the quantity $\Pr[DtD] = \frac{1}{\beta} = 0.42$.

At the critical point with zero total equity, $E(s^*) = 0$, being a direct consequence of herding, investors would expect their debt at the amount of \$685.7bn. In this case, the system’s default would have happened for sure with probability of 1.0.

In the model the system’s losses, incurred as a result of a crisis, are amounted to the nominal debt, $F = \$400bn$. It is due to the model definition of a crisis which is associated with no debt guaranties, $P(s^*) = 0$. After the crisis the shocked investors resumed their activity very slowly and cautiously. These features are reflected in Fig. 7 via rapid growth of debt

guaranties in the crisis aftermath. Hence the value of owners' equity at the point $s = \hat{S}$ becomes equal to the total asset value:

$$(60) A(\hat{S}) = E(\hat{S}) = f(\hat{S}) + P(\hat{S})$$

which is an evident exaggeration. While being in no contradiction to the reality it nevertheless reflects some crudeness of the model as well.

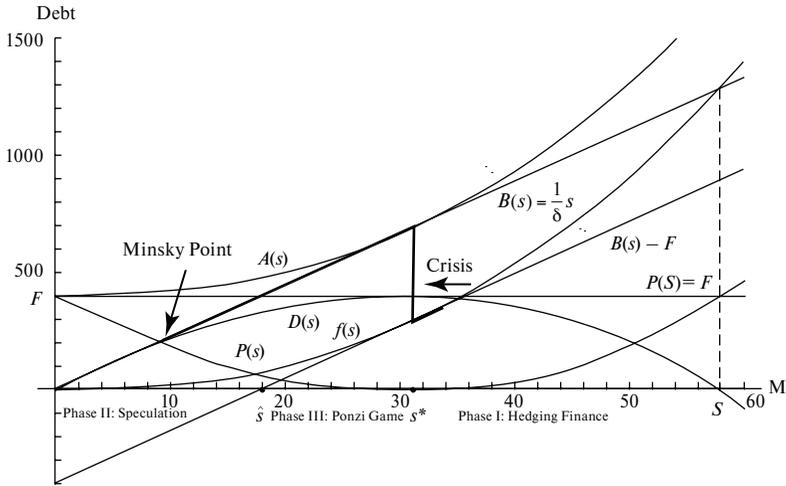


Fig. 7. The Minsky phases in finance

The Minsky point, $s_t = s_m$ is defined by the equality of debt guaranties to the market (49) or the expected (51) debt value, and equals to either 9.8 bn or 9.3 bn , respectively. Note, that in (Cassidy, 2009) the so called Minsky point was identified with the beginning of financial market meltdown. As shown in Fig. 7 the system approaches the Minsky point through speculation (phase II). At this point rational behavior of investors is being transformed into reckless Ponzi game (phase III in the Minsky parlance) and the process becomes autocatalytic eventually ending up in total crisis. Phases II and III together form financial bubble that burst at the point of the critical money issuance $s = s^*$.

Interesting feature of the model proposed is that it gives the same amount of assets at the critical point $s = s^*$ the latter being evaluated by different methods. Thus from equation (40), being evaluated at the critical point, we estimate financial assets $A(s^*)$ which (for the numerical primer parameters) is amounted to \$685.74bn:

$$A(s^*) = \$400 + \$285.74 = \$685.74bn.$$

Using the same numerical parameters, we get the total asset value at the moment of crisis via equation (51) being evaluated at the moment of crisis, $t^* = 0$, as

$$\langle A^* \rangle = \frac{30.86}{0.025} \left(\frac{0.07}{0.045} - 1 \right) = \$685.78$$

which is virtually the same. This similarity of numerical results might be argued as an indication of the model consistency.

The model reveals important feature of a crisis aftermath: the central bank has to increase liquidity further in order to overcome “zero money” condition (59). This corresponds closely to monetary policy of central banks during the current financial meltdown. Total losses in the model are equal to the debt par value due to the zero guaranties, $(s^*) = 0$. After the collapse, behavior of investors becomes very cautious: they prefer to form debt guaranties in the first place thus making the system safer than before the crisis. Gradually investors’ confidence improves while their actions become more arrogant, and the cycle repeats itself. It is easy to find the striking resemblance of the model behavior to the well known cycle in finance as described by H. Minsky (2008).

Financial bubble singularity

It should be noted that the proposed model described just an emergence of a financial bubble. The origins of the latter have been hidden in the normal conditions that allow for the investors’ exercising the appropriate call and put options implying persistent substitution of the market debt value, $D(s_t)$, for its expected value, $B(s_t)$. At the critical point the market might

arrive dually depending upon the existence of herding. Without herding investors would orient themselves around the market debt value (41), hence the probability of a total default is strictly less than unity (44). The herding process has an important implication: to be consistent, investors have to use the expected instead of the market debt value, as a guideline, because the former serves as a better benchmark in the atmosphere of everybody's frenzy being fed by accelerating price increases. Asset prices bifurcate at the Minsky point that initiates the process of asset prices deviation. At the critical point, where the price divergence reaches its maximum (40), investors, all of a sudden, realize that their own capital becomes zero because expected debt becomes equal to the total asset value (45). That awareness smashes everybody's confidence in a system: market participants start to sell *en masse*, asset prices drop dramatically, and the system collapses. *A posteriori* probability of a total default (crisis) becomes equal to one (46) while financial leverage starts to grow indefinitely, and the system collapses.

By implying persistent substitution of the market debt value for its expected value, the model characterizes an emergence of a financial bubble. The author believes that its origins are rooted in the normal market conditions as was noted by (Cooper, 2008). Since investors are allowed for exercising the appropriate call and put options, the model suggests the existence of the finite, however large, value of the debt outstanding at the critical point of money issuance. In the model the important feature of such a process is the existence of the finite, however large, amount of the debt, $B(s^*)$, at the critical point of money issuance, $s_t = s^*$. This feature, though empirically quite correct since in reality debt outstanding is always a finite amount, theoretically poses the main deficiency of the model.

Looking at this angle, linear function $B(s_t)$ serves as a poor representation of asset prices dynamics near the critical point since the latter is a highly nonlinear one being chosen by the herd of investors. Hence, formally, it seems to be more preferable to represent asset prices dynamics near the critical point as a singular process. Such an approach implies that the es-

sence of herding is not only in the asset prices overvaluation but in the transformation of investors' behavior into a highly nonlinear process of autocatalytic type. Thus, the epicenter of all events has to be focused on the new debt acquiring where investors compete and mimic each other in the process of herding. The pronounced mimicry of investors forces both the debt value and its derivatives (both in financial and mathematical sense of this term) to increase which explains its ultimate singularity.

Historically, financial bubbles were always precursors of crises which emerged almost always in the aftermath of a bubble burst (Kindleberger, 2000). Bursting bubble, in its turn, could be represented formally via debt singularity that appears due to herding. Singularity takes place for the systems of the infinite dimension while empirically all the systems are of finite dimensionality. Such a contradiction, well known in the natural sciences, manifests itself in the instable characteristic scale while samples are increasing.

Theoretically, the essence of herding is shown up not only as asset prices overvaluation but primarily through the transformation of investors' behavior into a highly nonlinear process of autocatalytic type. Their pronounced mimicry not only forces the debt value to increase but increases its speed (first derivative) as well. The asset prices start to behave both in nonlinear and almost deterministic manner. The stock quotations become a commonplace: the story goes that in 1929 Joe Kennedy (the father of the future US President) liquidated his portfolio when he heard that a shoeshine boy was giving stock tips. The trajectory of the blown asset prices becomes the only one along which the singularity might take place. In other words, in the process of herding, more precisely, in the small neighborhood of the critical point, investors completely ignore the possibility of the "normal" debt reimbursement which is implied by finite amount of the par debt. Thus, the epicenter of all events shifts to the process of acquiring of the new debt where investors compete and mimic each other. Such a process becomes a wholly irrational one for investors completely ignore the mere notion of the fair price of an asset. Taking these considerations into ac-

count, the new debt function (34) has to be modified in order to describe the singular process. The simplest modification of this sort is the following:

$$(61) (s_t) = K s_t^\beta + h * (s^* - s_t)^{-\gamma} ,$$

where the herding parameter is

$$h = \begin{cases} 1, & \text{if herding;} \\ 0, & \text{if no herding;} \end{cases}$$

and $\gamma = 2.39$ is one of the percolation invariant constants (Stauffer,2009).

As it might be seen from Fig. 8, herding modifies the new debt function (34) significantly only in the small neighborhood of the critical point s^* .

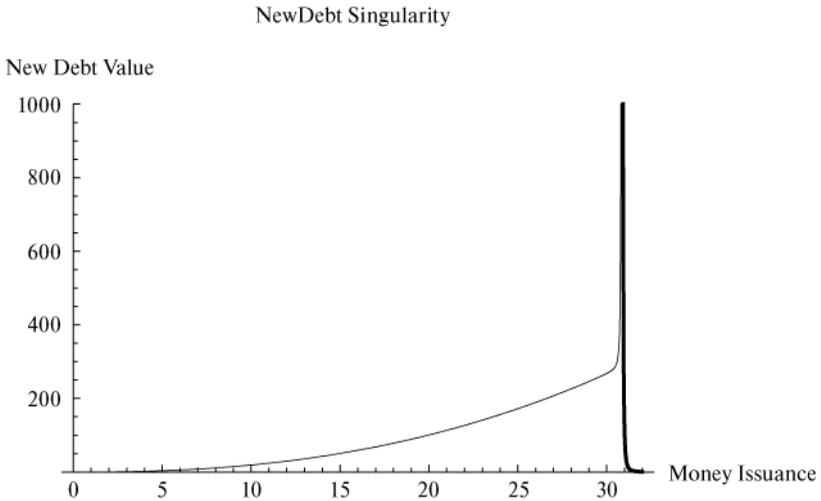


Fig. 8. The new debt function singularity

In our numerical example the new debt function was taken as

$$f(s_t) = 0.076 s_t^{2.4} + h * (30.81 - s_t)^{-2.39} .$$

It adds to the new debt value just 0.000276 at $s_t = 0$, 1.473 at $s_t = 30$ and 11.493 at $s_t = 30.5$, but afterwards increases very fast. Thus in the vicinity of the critical point the new debt function (61) dominated entirely

by its second component and thus loses completely all the qualities of the call option. Ignoring in (61) the first component and differentiating we get

$$(62) \frac{df}{ds} \sim (s^* - s_t)^{-\gamma-1}$$

where (\sim) is the sign of asymptotic equality. According to (62) the process of acquiring the new debt is accelerated quickly in the small neighborhood of the critical point at which singularity takes place.

Bubbles occur when investors develops an enthusiasm for particular class of assets like stocks in the late 1990ties or houses in the beginning of 2000ties. Quickly blowing financial bubbles could be studied via models of financial percolation. Percolation is a huge body of knowledge with a large spectrum of applications from physics to chemistry, to earthquakes to avalanches to forest fires (Stauffer, 2009). In finance percolation models are useful in describing interactions of investors via geometric configurations of sites being formed randomly on a large 2D grid. Monte Carlo simulation of percolation models shows that in the vicinity of a critical point these interactions might lead to formation of a huge spanning cluster of sites that transforms the quality of the financial system. The latter is due to a sudden increase of the “connectedness” among the hitherto independent financial investors (Smirnov, 2007, 2010). It follows that near the critical point financial bubble starts to expand in a highly nonlinear manner, probably first noticed by J.M. Keynes in his description of “speculation” and “enterprise” in financial market (Keynes, 1936). The model demonstrates that since investors acquire new debt unboundedly, the total debt value at the critical point becomes infinite, and the bubble bursts very quickly.

It follows that near the critical point $s = s^*$ financial bubble starts to expand in a highly nonlinear manner which bursts very quickly as represented in Fig. 9. Being stimulated by herding investors acquire all new debt unboundedly, hence the total debt value at the critical point becomes infinite. In reality that signifies the burst of a bubble or the system singularity at the critical point. Looking at the different angle, however large amount of money becomes, in fact, negligibly small comparing to the infinitely large

debt value. The shortage of liquidity which is a financial crisis *per se*, is a result of the eventual bursting of a financial bubble that takes place at the critical point of money issuance. To overcome the consequences of a crisis, the money issuance in the model should be increased even further than before the crisis. That precisely had been done by major central banks in the aftermath of credit crunch 2007–2009.

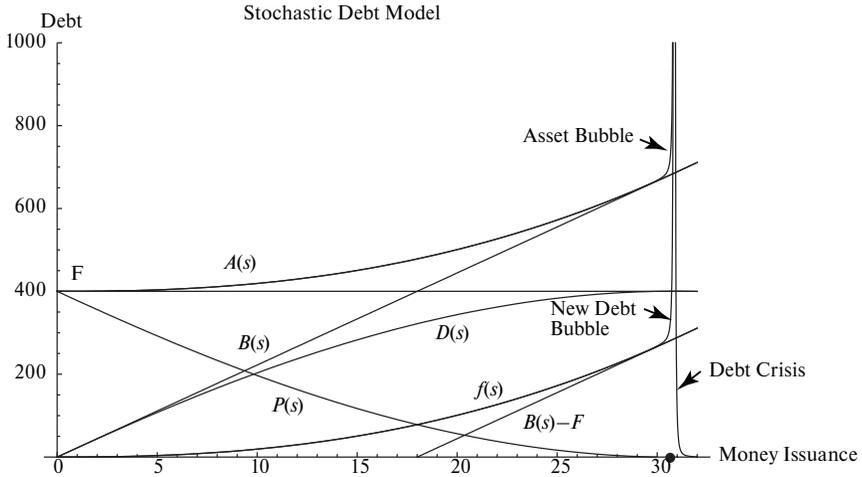


Fig. 9. Financial bubble and crisis

The trajectory of the blown bubble becomes the only one, and asset prices increase along it in nonlinear and almost deterministic manner. In the process of herding investors, quite in accordance with “the greater fool theory”, completely ignore possibilities of “normal” reimbursement of the par debt finite amount. Volumes and prices of the new debt acquiring quickly accelerate, especially in the small neighborhood of the critical point. These considerations could be implemented as a new debt function being a solution to a Bernoulli differential equation. As it is well known, a simple Bernoulli process contains singularity. The latter represents the bubble burst which is inevitable result of herding. The bubble singularity can be ex-

plained alternatively via growing leverage in the market characterized by increasing asset prices. This phenomenon was thoroughly explained in (Adrian and Shin, 2008).

Looking from the different angle, singularity could be explained as a natural consequence of interactions between debt and money. At the critical point however large, but finite amount of money becomes negligibly small comparing to the infinitely large debt value. The subsequent acute shortage of liquidity which, in effect, is a financial crisis *per se*, appears as a result of eventual bursting of a financial bubble that takes place at the critical point. In the model to overcome the consequences of a crisis, money issuance should be increased even further than before the crisis. In that aspect, as it seems, the model could explain paradoxical, at first glance, behavior of major central banks during the credit crunch 2007-09. In spite of the fact that “excess liquidity” had been considered as one of the major causes of the crisis, instead of evaporating its amount they dramatically increased asset side of their balance sheets in a prolonged process of “quantitative easing“.

References

Adrian T., Shin H.S. (2008) Liquidity and financial contagion, Banque de France // Financial Stability Review. No. 11 (February).

Campbell J., Shiller R. (1988) Stock prices, earnings, and expected dividends // Journal of Finance. No. 43. P. 661–676.

Cassidy J. (2009) How Markets Fail. The Logic of Economic Calamities. L.: Allen Lane.

Chan N., Getmansky M., Haas S. et al. (2005) Systemic Risk and Hedge Funds: NBER Working Paper 11200.

Dixit A., Pindyck R. (1994) Investment under Uncertainty. Princeton: Princeton University Press.

Encyclopedia of Complexity and Systems Science. Springer, Berlin/Heidelberg, 2009.

Johansen A. (1997) *Discrete Scale Invariance and other Cooperative Phenomena in spatially Extended Systems with Threshold Dynamics* (PhD Thesis) / Niels Bohr Institute, University of Copenhagen.

Keynes J.M. (1936) *The General Theory of Employment, Interest and Money*. N.Y.: Harcourt-Brace.

Kindleberger C. P. (2000) *Manias, Panics and Crashes: A History of Financial Crises*, N.Y.: J. Wiley.

Krugman P. (2008) *The Return of Depression Economics*. L.: Penguin Books.

Lucas R. (2003) *Macroeconomic Priorities* // *American Economic Review*. Vol. 93. Iss. 1. March 2003.

Lux T. (2006) *Financial Power Laws: Empirical Evidence, Models, and Mechanism* // *Power Laws in Social Sciences* / C. Cioffi (ed.). Cambridge: Cambridge University Press.

Mandelbrot B., Hudson R. (2005) *The (mis)Behaviour of Markets. A Fractal View of Risk, Ruin and Reward*, L.: Profile Books.

Minsky H. (2008) *Stabilizing an Unstable Economy*. N.Y.: McGraw Hill.

Rajan R. (2005) *Has Financial Development Made the World Riskier? // The Greenspan Era: Lessons for Future*. Wyoming: Jackson Hole.

Smirnov A.D. (2005) *The Process of Debt Monetization: working paper*, WP2/2005/03. M.: SU HSE.

Smirnov A.D. (2007) *Global Debt Monetization: Redemption or Crisis? // Economic Journal of HSE*. Vol. 11. No. 4.

Smirnov A.D. (2010) *Macrofinance: Models of Bubbles and Crises: working paper*, WP2/2010/03. M.: SU HSE.

Soros G. (1998) *The Crisis of Global Capitalism, Open Society Endangered*. L.: Little, Brown and Company. P. 41–42.

Stanley H.E., Gopikrishnan P., Plerou V. et al. (2003) *Patterns and Correlations in Economic Phenomena Uncovered Using Concepts of Statistical Physics* // *Lecture Notes in Physics (Series)*. Vol. 621/2003. Springer, Berlin/Heidelberg.

Stauffer D. (2001) Percolation Models of Financial Market Dynamics // Modeling Complexity in Economic and Social Systems / F. Schweitzer (ed.). World Scientific, River Edge.

Stauffer D. (2009) Classical Percolation // Lecture Notes in Physics (Series). Vol. 762/2009. Springer, Berlin/Heidelberg.

Stiglitz J. Needed: a new economic paradigm // Financial Times. August 20, 2010.

Turner A. The Future of Finance, Speech at the Conference in London, 14 July, 2010.

Turner A. et al. (2010) The Future of Finance. The LSE Report, London.

Препринт WP7/2011/05

Серия WP7

Математические методы анализа решений
в экономике, бизнесе и политике

Смирнов Александр Дмитриевич

Финансовые пузыри как дивергенция цен активов

Зав. редакцией оперативного выпуска *А.В. Заиченко*
Технический редактор *Ю.Н. Петрина*

Отпечатано в типографии
Национального исследовательского университета
«Высшая школа экономики» с представленного оригинал-макета
Формат 60×84 ¹/₁₆. Тираж 150 экз. Уч.-изд. л. 2,2
Усл. печ. л. 2,1. Заказ № . Изд. № 1349

Национальный исследовательский университет
«Высшая школа экономики»
125319, Москва, Кочновский проезд, 3
Типография Национального исследовательского университета
«Высшая школа экономики»
Тел.: (499) 611-24-15