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Homogenization in the Problem of Long Water Waves over a Bottom Site with Fast Oscillations

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Abstract—The system of equations of gravity surface waves is considered in the case where the basin's bottom is given by a rapidly oscillating function against a background of slow variations of the bottom. Under the assumption that the lengths of the waves under study are greater than the characteristic length of the basin bottom's oscillations but can be much less than the characteristic dimensions of the domain where these waves propagate, the adiabatic approximation is used to pass to a reduced homogenized equation of wave equation type or to the linearized Boussinesq equation with dispersion that is "anomalous" in the theory of surface waves (equations of wave equation type with added fourth derivatives). The rapidly varying solutions of the reduced equation can be found (and they were also found in the authors' works) by asymptotic methods, for example, by the WKB method, and in the case of focal points, by the Maslov canonical operator and its generalizations.

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1. INTRODUCTION

In linear problems for partial differential equations, the homogenization methods work in situations where their coefficients are rapidly oscillating functions. In numerous publications concerned with homogenization methods, both serious theoretical mathematical problems and their applications were considered; here we mention only [1]–[4]. As a rule, they are used to construct asymptotic solutions of the initial equation whose leading term is already a sufficiently smooth (but not a rapidly oscillating) function. On the other hand, in many physical problems, it is of interest to consider the situations where the leading term of the asymptotic solution is also a rapidly varying function. In this case, the initial problem contains several different scales, and it is reasonable to use the adiabatic approximation to solve the problem. In the present paper, these methods are used to study surface waves in the case where the basin's bottom is represented by a rapidly oscillating function against a background of slow variations in the basin's bottom. Moreover, it is also assumed that the lengths of the waves under study are greater than the characteristic length of the basin's bottom oscillations but can be much less than the characteristic dimensions of the domain where these waves propagate.

We consider an incompressible ideal vortex-free liquid in a gravity field, neglecting the temperature and the molecular-diffusion and dissipative effects. For simplicity, we eliminate the effects caused by wave reflection from the basin shores. We consider only a domain Ω in space with horizontal $x = (x_1, x_2)$ and vertical z coordinates and assume that the unperturbed surface of the liquid is described by the equation $z = 0$ and the free surface of the liquid is determined by the equation $z = \eta(x, t)$. We assume that the basin's bottom is given by the equation $z = -H(x)$, $H(x) > 0$. In particular, such problems

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arise in the study of tsunami waves (see, e.g., [5]); then the function η, which is the excess over the free surface, describes a long wave on the ocean surface over rapidly varying regions of the bottom.

If the function H regularly depends on x (in dimensionless variables, which will be introduced below), then the problem solution is determined by asymptotic formulas obtained in [6]–[8] and appealing to the Hamiltonian system in the four-dimensional phase space with Hamiltonian $H|p|$. Here we consider the case where H is a rapidly oscillating function against a background of slow variations in the basin's bottom. In the situation under study, the theory developed in the works cited above cannot be used any more, and it is already necessary to use homogenization type methods. In such situations, we use the homogenization version developed in [9]–[11] (see also [12]–[14]).

In Sec. 2, we study the system of equations for the velocity potential Φ in the linear approximation of small amplitude waves. In the corresponding dimensionless system, there is a parameter $h = d/l$, where d is the characteristic depth of the basin and l is the characteristics length (in the variables x) of slow variations in the basin's bottom. In what follows, h is assumed to be a small parameter, which allows us to use asymptotic methods to solve the problem. Then the problem is reduced to solving a certain two-dimensional pseudodifferential equation for the function $\psi = \Phi|_{z=0}$, where Φ is the velocity potential. In Sec. 3, the version of the homogenization method developed in [9], [11] is applied to the equation obtained for ψ, which allows us to obtain the reduced equation to which we can apply the WKB method, and if there are focal points, the Maslov canonical operator and its generalizations.

As was already noted, this allows us to study the waves whose length is much less than the characteristic length of slow variations in the basin's bottom but greater than the characteristic length of the basin's bottom oscillations. In Sec. 4, we obtain approximate formulas for the coefficients of the reduced equation in the case where the function H has a small oscillating part.

2. REDUCTION OF THE PROBLEM TO SOLVING A TWO-DIMENSIONAL PSEUDODIFFERENTIAL EQUATION

In the linear approximation of small amplitude waves, we have the following system for the velocity potential [15]–[17]:

$$\Phi_{zz} + \Delta\Phi = 0 \qquad \text{for} \quad -H(x) < z < 0, \tag{2.1}$$

$$(\Phi_z + \nabla_x H \cdot \nabla_x \Phi)|_{z=-H(x)} = 0, \tag{2.2}$$

$$(\Phi_t + g\eta)|_{z=0} = 0, \qquad (\eta_t - \Phi_z)|_{z=0} = 0. \tag{2.3}$$

Here $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2)$, $\Delta = \nabla_x^2$, and g is the free fall acceleration.

We pass from the variables (x, z, t) to the dimensionless variables $x' = x/l$, $z' = z/d$, $t' = \sqrt{gd}\,t/l$ and denote

$$H' = \frac{H(lx')}{d}, \qquad \Phi'(x', z', t') = \frac{\Phi(lx', dz', lt'/\sqrt{gd})}{a\sqrt{gd}}, \qquad \eta'(x', t') = \frac{\eta(lx', lt'/\sqrt{gd})}{a},$$

where d is the characteristic depth of the basin, l is the characteristic length in the variables x, and a is the wave amplitude. Then system (2.1)–(2.3) becomes (we omit the primes on the new variables):

$$\Phi_{zz} + h^2\Delta\Phi = 0 \qquad \text{for} \quad -H(x) < z < 0, \tag{2.4}$$

$$\left(\Phi_z + h^2\nabla_x H\left(x, \frac{\theta(x)}{\varepsilon}\right) \cdot \nabla_x \Phi\right)\Big|_{z=-H(x)} = 0, \tag{2.5}$$

$$(h\Phi_t + \eta)|_{z=0} = 0, \qquad (h\eta_t - \Phi_z)|_{z=0} = 0, \tag{2.6}$$

where $h = d/l$. In what follows, we assume that h is a small parameter and seek the approximate (asymptotic as $h \to +0$) solution of this system. About the behavior of the function H describing the basin's bottom profile (in dimensionless variables), we assume that it has the shape $H(x, \theta(x)/\varepsilon)$, where $H(x, y)$ is a smooth function 2π-periodic in the variables y_1 and y_2 ($y = (y_1, y_2)$), $\theta(x) = (\theta_1(x), \theta_2(x))$ is a smooth vector function, and the phases θ_j are not collinear, i.e., the matrix θ_x composed of the rows $((\theta_1)_{x_k}, (\theta_2)_{x_k})$, $k = 1, 2$, is nondegenerate. The fact that the phases Θ_x nonlinearly depend on x means that there is a weak variation in the frequencies of spatial oscillations of the bottom profile. The one-phase case where $\theta(x)$ is a scalar function and y is a single variable, can be considered similarly with several obvious changes. The parameters h and ε are assumed to be small; moreover, we assume that they are related as $h = \varepsilon^2$.

Problems of such type arise, in particular, in the study of tsunami waves (see, e.g., [5]); then the function η, which is the excess over the free surface, describes a long wave on the ocean surface over rapidly varying regions of the bottom. The condition $H(x, \eta) > 0$ means that the influence of the shore boundaries is not considered.

To construct the solutions of system (2.4)–(2.6), we use the considerations discussed in [18], [19] and the techniques of ordered operators [20]–[22]. In what follows, we show that system (2.4)–(2.6) can be reduced to a single equation for the function $\psi(x, t) = \Phi(x, z, t)|_{z=0}$, and this is already a pseudodifferential equation. After deriving this equation, its specific physically interesting solutions can be obtained directly by using the homogenization procedure and the Maslov asymptotic method.

Numerous monographs and papers deal with pseudodifferential equations (with a parameter), the techniques of ordered operators, the Maslov asymptotic theory, and their applications (see, e.g., [20]–[22] and the references therein). But, for the reader's convenience, we here repeat the necessary definitions and formulas of this theory.

Recall that, in the two-dimensional case, the h-pseudodifferential operator $\widehat{L}$ with symbol $L(x, p, h)$ is determined by the formula

$$\widehat{L}\psi = L\biggl(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}, h\biggr)\psi = \frac{1}{(-2\pi ih)}\int_{\mathbb{R}^2} e^{ipx/h} L(x, p, h)\widetilde{\psi}(p)\,dp, \tag{2.7}$$

where $\widetilde{\psi}$ is the Fourier transform of the function $\psi(x)$:

$$\widetilde{\psi}(p) = \frac{1}{(2\pi ih)}\int_{\mathbb{R}^2} e^{-ipx/h}\psi(x)\,dx, \qquad p = (p_1, p_2).$$

The index over an argument shows the order of application of the required operator. The existence of such operators guarantees, for example, that the following estimates [22] are satisfied:

$$\left|\frac{\partial^{|m|+|l|}L(x, p, h)}{\partial x^m \partial p^l}\right| \le C_{m,l}(1 + |x|)^M (1 + |p|)^M$$

for some integer M and arbitrary multiindices m and l.

Generalizing the approach in [18], [19], we seek the solution of Eqs. (2.4), (2.5) in the form

$$\Phi = \widehat{R}\psi, \qquad \psi = \Phi|_{z=0}, \qquad \widehat{R} = R\biggl(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}, \frac{\theta(\overset{2}{x})}{\varepsilon}, z, \varepsilon\biggr), \tag{2.8}$$

where the function $R(x, p, y, z, \varepsilon)$ is periodic in y_1 and y_2 with period 2π. The function $R(x, p, y, z, \varepsilon)$ is called the *symbol* of the operator R. The meaning of the introduced new variables y is that they allow one to regularize the coefficients of Eq. (2.5); namely, the obtained coefficients already smoothly depend on ε as $\varepsilon \to +0$.

Substituting Φ in the form (2.8) into Eq. (2.4), we obtain

$$\biggl(\frac{\partial^2}{\partial z^2} - (ih\nabla_x)^2\biggr)R\biggl(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}, \frac{\theta(\overset{2}{x})}{\varepsilon}, z, \varepsilon\biggr)\psi(x) = 0. \tag{2.9}$$

Let us find the symbol of the operator in the left-hand side of (2.9). The condition that this symbol is zero is sufficient for (2.4) to be satisfied. To perform this, it is required to "pull" the operator $-ih\partial/\partial x$ through $\widehat{R}$. Let us explain this operation. In the general case of an h-pseudodifferential operator $\widehat{L}$ with symbol $L(x, p, h)$, we differentiate the integrand in (2.7) to obtain

$$\biggl(-ih\frac{\partial}{\partial x}\biggr)\widehat{L}\psi = \frac{1}{(-2\pi ih)}\int_{\mathbb{R}^2} e^{ipx/h}\biggl(p - ih\frac{\partial}{\partial x}\biggr)L(x, p, h)\widetilde{\psi}(p)\,dp.$$

Thus, the symbol of the operator $(-ih\partial/\partial x)\widehat{L}$ has the form $(p - ih\partial/\partial x)L(x, p, h)$. We note that the obtained formula is a special case of the well-known composition formulas for h-pseudodifferential operators, but we need only this special case. We twice "pull" $(-ih\partial/\partial x)$ through $\widehat{R}$ in (2.9) and obtain the symbol of the operator $(-ih\partial/\partial x)^2\widehat{R}$ in the form

$$\left(p - ih\nabla_x - i\frac{h}{\varepsilon}\theta_x\nabla_y\right)^2 R(x,p,y,z,\varepsilon).$$

As a result, we see that the symbol of the operator in the left-hand side of (2.9) becomes

$$R_{zz} - \left(p - ih\nabla_x - i\frac{h}{\varepsilon}\theta_x\nabla_y\right)^2 R(x,p,y,z,\varepsilon).$$

Equating this symbol with zero, as well as the symbol of the operator in Eq. (2.5) with the obvious condition $R|_{z=0} = 1$, we obtain the boundary-value problem in the case $h = \varepsilon^2$ (cf. [18], [19]):

$$R_{zz} - (p - \varepsilon^2\nabla_x - i\varepsilon\theta_x\nabla_y)^2 R = 0 \qquad \text{for} \quad -H(x,y) < z < 0, \quad R|_{z=0} = 1, \tag{2.10}$$

$$R_z + i\langle(\varepsilon^2\nabla_x + \varepsilon\theta_x\nabla_y)H, (p - i\varepsilon^2\nabla_x - i\varepsilon\theta_x\nabla_y)R\rangle|_{z=-H(x,y)} = 0. \tag{2.11}$$

We let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the real scalar product of vectors. Representing R as the asymptotic series $R = R^0 + \varepsilon R^1 + \varepsilon^2 R^2 + \cdots$ and equating the coefficients of equal powers of ε with zero, from (2.10), (2.11) we obtain a chain of ordinary differential equations in z for determining R^j. The variables (x,p,y) enter these equations as parameters, in particular, for R^0 and R^1, we have

$$R^0_{zz} - p^2R^0 = 0 \qquad \text{for} \quad -H(x,y) < z < 0, \quad R^0|_{z=0} = 1, \quad R^0_z|_{z=-H(x,y)} = 0, \tag{2.12}$$

$$R^1_{zz} - p^2R^1 = -2i\langle p, \theta_x\nabla_y H\rangle R^0 \qquad \text{for} \quad -H(x,y) < z < 0, \tag{2.13}$$

$$R^1|_{z=0} = 0, \qquad (R^1_z + i\langle p, \theta_x\nabla_y H\rangle R^0)|_{z=-H(x,y)} = 0. \tag{2.14}$$

Solving (2.12)–(2.14), we obtain

$$R^0 = \frac{\cosh[(z+H)|p|]}{\cosh(H|p|)}, \tag{2.15}$$

$$R^1 = \frac{i\langle p, \theta_x\nabla_y H\rangle}{\cosh^3(H|p|)}[H\sinh(z|p|)\sinh(H|p|) - z\cosh(z|p|)\cosh(H|p|)]. \tag{2.16}$$

The solution of the corresponding equations for R^2 is rather cumbersome, but for further calculations, it suffices to obtain R^2 up to $O(p^2)$ as $p \to 0$. With the relation $R^0 = 1 + O(p^2)$ taken into account, we can rewrite the system for determining R^2 as

$$R^2_{zz} = O(p^2) \qquad \text{for} \quad -H(x,y) < z < 0,$$
$$R^2|_{z=0} = 0, \qquad R^2_z|_{z=-H(x,y)} = -i\langle p, \nabla_x H\rangle + O(p^2),$$

so that, up to $O(p^2)$, we obtain a sufficiently simple formula for R^2:

$$R^2 = -i\langle p, \nabla_x H\rangle z + O(p^2). \tag{2.17}$$

Similar argument easily shows that $R^j = O(|p|)$ for all $j \geq 2$.

Now we consider Eqs. (2.6) which imply

$$(h^2\Phi_{tt} + \Phi_z)|_{z=0} = 0.$$

Representing Φ as $\Phi = \widehat{R}\psi$ (see formula (2.8)), we obtain the pseudodifferential equation

$$h^2\psi_{tt} + \widehat{L}\psi = 0, \tag{2.18}$$

where the h-pseudodifferential operator $\widehat{L} = \widehat{R}_z|_{z=0}$ has symbol $L = L^0 + \varepsilon L^1 + \varepsilon^2 L^2 + \cdots$, and

$$L^j(x,p,y) = R^j_z(x,p,y,z)|_{z=0}.$$

With relations (2.15), (2.16), (2.17), $R^j = O(|p|)$ for $j \geq 2$, and $|p|\tanh(H|p|) = Hp^2 + O(p^4)$ taken into account, we see that the symbol L has the form

$$L(x,p,y,\varepsilon) = Hp^2 - i\varepsilon\langle\theta_x\nabla_y H, p\rangle - i\varepsilon^2\langle\nabla_x H, p\rangle + V_0(x,p,y)p^4$$

$$+ \varepsilon\langle \theta_x \nabla_y H, p\rangle p^2 V_1(x,p,y) + \varepsilon^2 \langle V_2(x,p,y,\varepsilon)p,p\rangle + \varepsilon^3\langle V_3(x,p,y,\varepsilon),p\rangle,$$

where V_2 is a matrix-valued symbol, V_3 is a vector-valued symbol, and V_0 and V_1 are scalar symbols. Now, if we pass from the operators $\widehat{p} = -ih\partial/\partial x = -i\varepsilon^2\partial/\partial x$ to the operators $\widehat{p}_\varepsilon = -i\varepsilon\partial/\partial x$, so that $p = \varepsilon p_\varepsilon$, then Eq. (2.18) can be written as

$$\psi_{tt} = \left\langle \nabla_x, H\left(x, \frac{\theta(x)}{\varepsilon}\right)\nabla_x\right\rangle\psi - V\left(\overset{2}{x}, -i\varepsilon\overset{1}{\nabla}_x, \frac{\theta(\overset{2}{x})}{\varepsilon}, \varepsilon\right)\psi, \tag{2.19}$$

where

$$V(x,p_\varepsilon,y,\varepsilon) = V_1(x,\varepsilon p_\varepsilon,y,\varepsilon)p_\varepsilon^4 + \langle V_2(x,\varepsilon p_\varepsilon,y,\varepsilon)p_\varepsilon,p_\varepsilon\rangle + \langle V_3(x,\varepsilon p_\varepsilon,y,\varepsilon),p_\varepsilon\rangle. \tag{2.20}$$

Thus, we obtain the following assertion.

Lemma 1. *Solving system* (2.4)–(2.6) *can be reduced to solving Eq.* (2.19) *for the function* $\psi = \Phi(x,z,t)|_{z=0}$, *and the symbol* $V(x,p_\varepsilon,y,\varepsilon)$ *has the form* (2.20).

We cannot determine the explicit symbols V_j, although we could find finitely many terms of their asymptotic expansions by solving an appropriate number of equations for R^j, which would lead to very cumbersome formulas. But, in what follows, we see that the specific form of these symbols does not affect the leading term of the asymptotic expansions of the solutions to Eq. (2.19), which we consider below.

A differential operator with a rapidly oscillating coefficient $H(x,\theta(x)/\varepsilon)$ is contained in Eq. (2.19). As was already noted in the introduction, the homogenization methods, as a rule, are used to construct asymptotic solutions whose leading term is already a sufficiently smooth (but not rapidly oscillating) function. On the other hand, in many physically interesting problems, there are situations where the leading term of the asymptotic solution is also a rapidly varying function. In such situations, we use the version of the homogenization method developed in [9], [11]. The goal in the further calculations is to obtain equations

$$\varepsilon^2 w_{tt} = -\mathcal{L}(\overset{2}{x}, -i\varepsilon\overset{1}{\nabla}_x, \varepsilon)w$$

or their symbols $\mathcal{L}(x,p,\varepsilon)$, whose coefficients already regularly depend on the parameter ε, whose solutions w can be used to express several asymptotic solutions of Eq. (2.19), and hence of the initial system (2.4)–(2.6). These equations will be called *homogenized* equations, and the procedure of their derivation, the *homogenization*.

3. HOMOGENIZED EQUATIONS

3.1. General Scheme for Deriving the Homogenized Equations

We briefly describe the approach used to derive the homogenized equations, which was developed in [12], [14], [9]. The solution of the problem is first represented as

$$\psi = \Psi\left(x, \frac{\theta(x)}{\varepsilon}, t, \varepsilon\right), \tag{3.1}$$

where the function $\Psi(x,y,t,\varepsilon)$ 2π-periodic in y_1 and y_2 satisfies the equation

$$\varepsilon^2\Psi_{tt} = -\widehat{\mathcal{H}}\Psi, \tag{3.2}$$

$$\widehat{\mathcal{H}} = \langle(-i\varepsilon\nabla_x - i\nabla_y^\theta), H(x,y)(-i\varepsilon\nabla_x - i\nabla_y^\theta)\rangle + \varepsilon^2 V(\overset{2}{x}, -i\varepsilon\overset{1}{\nabla_x} - i\nabla_y^\theta, \overset{2}{y}, \varepsilon), \tag{3.3}$$

where ∇_y^θ from now on denotes $\theta_x\nabla_y$.

The operator V in the right-hand side of Eq. (3.3) is understood as follows. We expand the symbol $V_j(x,\varepsilon p_\varepsilon,y,\varepsilon)$ in formal power series in p_ε and ε and then replace p_ε by the operators $-i\varepsilon\nabla_x - i\theta_x\nabla_y$. After appropriate simplifications, we write the operator V as the sum of terms

$$V(\overset{2}{x}, -i\varepsilon\overset{1}{\nabla_x} - i\nabla_y^\theta, \overset{2}{y}, \varepsilon) = \sum b_{j,k,m}(x,y)\varepsilon^j(-i\varepsilon\nabla_x)^k(-i\nabla_y)^m,$$

where k and m are multiindices, $k = (k_1, k_2)$, $m = (m_1, m_2)$, and $n + 4 \geq |k| + |m|$ ($|k| = k_1 + k_2$, $|m| = m_1 + m_2$). Thus, we can write the operator V in the right-hand side of (3.3) as the ε-pseudodifferential operator

$$V(\overset{2}{x}, \overset{1}{-i\varepsilon\nabla_x - i\nabla_y^\theta}, \overset{2}{y}, \varepsilon) = \widetilde{V}(\overset{2}{x}, -i\overset{1}{\varepsilon}\nabla_x, y, -i\nabla_y, \varepsilon)$$

with an operator-valued symbol admitting the asymptotic expansion

$$\widetilde{V}(\overset{2}{x}, p_\varepsilon, y, -i\nabla_y, \varepsilon) = \widetilde{V}_0(\overset{2}{x}, p_\varepsilon, y, -i\nabla_y) + \varepsilon\widetilde{V}_1(\overset{2}{x}, p_\varepsilon, y, -i\nabla_y) + \cdots,$$

where

$$\widetilde{V}_j(\overset{2}{x}, p_\varepsilon, y, -i\nabla_y) = \sum_{k,m} b_{j,k,m}(x, y) p_\varepsilon^k (-i\nabla_y)^m.$$

Accordingly, we can write the operator $\widehat{\mathcal{H}}$ in (3.3) as the ε-pseudodifferential operator

$$\widehat{\mathcal{H}} = \mathcal{H}(\overset{2}{x}, -i\overset{1}{\varepsilon}\nabla_x, y, -i\nabla_y, \varepsilon)$$

with an operator-valued symbol admitting the asymptotic expansion

$$\mathcal{H}(x, p_\varepsilon, y, -i\nabla_y, \varepsilon) = \mathcal{H}_0(x, p_\varepsilon, y, -i\nabla_y) + \varepsilon\mathcal{H}_1(x, p_\varepsilon, y, -i\nabla_y) + \cdots, \tag{3.4}$$

and $\mathcal{H}_j(x, p_\varepsilon, y, -i\nabla_y)$ are differential operators with respect to y.

This equation belongs to the class of equations known in the mathematical literature (see [20], [23]) as equations with operator-valued symbols. The meaning of the introduced new variables y is that they allow one to regularize the coefficients of Eq. (2.19) so that the obtained coefficients already regularly depend on ε as $\varepsilon \to 0$.

The use of the adiabatic approximation in operator form [14], [23], [9] allows one to obtain several solutions of Eq. (3.2). We seek the solutions Ψ of Eq. (3.2) in the form of action of some (so far unknown) pseudodifferential operator on the new (also so far unknown) function

$$\Psi(x, y, t, \varepsilon) = \widehat{\chi} w \equiv \chi(\overset{2}{x}, -i\overset{1}{\varepsilon}\nabla_x, y, \varepsilon) w(x, t, \varepsilon). \tag{3.5}$$

Here $\widehat{\chi}$ is the "intertwining" pseudodifferential operator with symbol admiting the asymptotic expansion

$$\chi(x, p_\varepsilon, y, \varepsilon) = \chi_0(x, p_\varepsilon, y) + \varepsilon\chi_1(x, p_\varepsilon, y) + \cdots. \tag{3.6}$$

We assume that the function w satisfies the "effective" (reduced or "homogenized") equation[1]:

$$\varepsilon^2 w_{tt} = -\mathcal{L}(\overset{2}{x}, -i\overset{1}{\varepsilon}\nabla_x, \varepsilon) w, \tag{3.7}$$

whose symbol $\mathcal{L}$ admits the regular expansion in ε:

$$\mathcal{L}(x, p_\varepsilon, \varepsilon) = \mathcal{L}_0(x, p_\varepsilon) + \varepsilon\mathcal{L}_1(x, p_\varepsilon) + \cdots. \tag{3.8}$$

The function $H_{\text{eff}}(p_\varepsilon, x) = \mathcal{L}_0(x, p_\varepsilon)$ is called the (classical) *effective Hamiltonian*. If we obtain $\mathcal{L}(x, p_\varepsilon)$, then the construction of (some) solutions of Eq. (3.2), and hence of the initial system of Eqs. (2.1)–(2.3), reduces to solving Eq. (3.7). It is impossible exactly to determine both $\mathcal{L}(x, p_\varepsilon)$, and the solution w of Eq. (3.7), and hence we deal only with the corresponding asymptotics. Our considerations are adapted to asymptotics based on the semiclassical approximation or on its generalizations. If we deal with real applications, then it suffices (see below and [9]–[11], [23]–[26]) to determine $\mathcal{L}_j(x, p_\varepsilon)$ and even their expansions in p_ε; in this case, the algorithm for determining fast varying asymptotic solutions is based only on operations with $\mathcal{L}_j(x, p_\varepsilon)$; the calculations use the corresponding differential equation only because it can be reconstructed from the symbol $\mathcal{L}(x, p_\varepsilon) = \mathcal{L}_0(x, p_\varepsilon) + \cdots + O(\varepsilon^3)$.

The operators $\widehat{\chi}$ and $\widehat{\mathcal{L}}$ must be related as

$$\widehat{\mathcal{H}}\widehat{\chi} = \widehat{\chi}\widehat{\mathcal{L}}, \tag{3.9}$$

[1] In the physical literature, the change $p \to -i\varepsilon\nabla_x$ in the function $\mathcal{L}(x, p, \varepsilon)$ is called the *Peierls substitution* [27].

which implies the equation

$$\chi(\overset{2}{x}, p_\varepsilon - \overset{1}{i}\varepsilon\nabla_x, y, \varepsilon)\mathcal{L}(x, p_\varepsilon, \varepsilon) = \mathcal{H}\biggl(\overset{2}{x}, p_\varepsilon - \overset{1}{i}\varepsilon\nabla_x, y, -i\frac{\partial}{\partial y}, \varepsilon\biggr)\chi(x, p_\varepsilon, y, \varepsilon) \tag{3.10}$$

for their symbols (see Lemma 1 in [9]), where

$$\begin{aligned}\mathcal{H}(x, p_\varepsilon, y, -i\nabla_y, \varepsilon) &= \mathcal{H}_0(x, p_\varepsilon, y, -i\nabla_y, \varepsilon)\\ &\qquad + \varepsilon\mathcal{H}_1(x, p_\varepsilon, y, -i\nabla_y, \varepsilon) + \varepsilon^2\widetilde{V}(x, p_\varepsilon, y, -i\nabla_y, \varepsilon),\\ \mathcal{H}_0(x, p_\varepsilon, y, -i\nabla_y, \varepsilon) &= \langle (p_\varepsilon - i\nabla_y^\theta), H(y, x)(p_\varepsilon - i\nabla_y^\theta)\rangle,\\ \mathcal{H}_1(x, p_\varepsilon, y, -i\nabla_y, \varepsilon) &= -\langle \nabla_x, H(y, x)\nabla_y^\theta\rangle - i\langle \nabla_x, H(y, x)p_\varepsilon\rangle.\end{aligned}$$

3.2. Calculation of the Symbols by Perturbation Series in the Parameter ε and Small Momentum Variables

Equation (3.10) can be solved by perturbation theory methods. For the problems considered in the present paper, the solution is determined as an expansion in a small parameter ε and a (small) variable p_ε. As was already noted, to construct the leading term of the asymptotics, it suffices to consider finitely many terms of the corresponding expansions.

First, we use the perturbation series in the parameter ε. For the leading terms χ_0 and $\mathcal{L}_0(x, p_\varepsilon)$ of expansions (3.6) and (3.8), we obtain the following family of problems, which depend on x and p_ε and are 2π-periodic in y_1 and y_2:

$$\mathcal{H}_0\chi_0(x, p_\varepsilon, y) = \mathcal{L}_0(x, p_\varepsilon)\chi_0(x, p_\varepsilon, y), \qquad \mathcal{H}_0 = \langle (p_\varepsilon - i\nabla_y^\theta), H(y, x)(p_\varepsilon - i\nabla_y^\theta)\rangle. \tag{3.11}$$

Problem (3.11) has infinitely many solutions, i.e., infinitely many "modes" (eigenfunctions and eigenvalues). We are interested in the solution with the minimal eigenvalue. For such a solution, we have the following expansions in the momentum variables p_ε:

$$\mathcal{L}_0(x, p_\varepsilon) = \mathcal{L}_0^{(2)}(x, p_\varepsilon) + \mathcal{L}_0^{(4)}(x, p_\varepsilon) + O(|p_\varepsilon|^6), \qquad \mathcal{L}_1(x, p_\varepsilon) = \mathcal{L}_1^{(1)}(x, p_\varepsilon) + O(p_\varepsilon^2), \tag{3.12}$$

where $\mathcal{L}_j^{(k)}$ are homogeneous polynomials in p_ε of degree k. We write the corresponding formulas obtained in [9], [10].

For any function $f(x, p_\varepsilon, y)$ 2π-periodic in the variables y_1 and y_2, i.e., for a function defined on the torus $\mathbb{T} = \{y_1 \in [0, 2\pi],\ y_2 \in [0, 2\pi]\}$, we denote the homogenized value by

$$\langle f\rangle_{\mathbb{T}} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(x, p_\varepsilon, y)\,dy_1\,dy_2. \tag{3.13}$$

In what follows, it is convenient to introduce the space $L_2(\mathbb{T})$ in the variables y with "normed" scalar product by setting

$$(g, f)_{L_2(\mathbb{T})} = \frac{1}{(2\pi)^2}\int_{\mathbb{T}} \overline{g}(y)f(y)\,dy \tag{3.14}$$

for any functions $g(y)$ and $f(y)$, where the bar denotes complex conjugation. In the one-phase case, the integrals in (3.13) and (3.14) are taken over the circle $y \in [0, 2\pi]$ and the multiplier $(2\pi)^{-2}$ is replaced by $(2\pi)^{-1}$.

Further, we introduce the operators

$$\Delta_y^\theta = \langle \nabla_y^\theta, H(x, y)\nabla_y^\theta\rangle, \qquad D = \langle p_\varepsilon, \nabla_y^\theta\rangle$$

and consider the equation (the problem on the cell)

$$\Delta_y^\theta F = f, \tag{3.15}$$

where $f(x, p_\varepsilon, y)$ is a smooth 2π-periodic (in the variables y_j) function with zero mean. This equation has a unique (smooth) solution with zero mean. This solution is denoted by $F(x, p_\varepsilon, y) =$

$f(x, p_\varepsilon, y)/\Delta_y^\theta$. Let $H_0(x) = \langle H(x,y)\rangle_{\mathbb{T}}$ and $\widetilde{a}(x,y) = H(x,y) - H_0(x)$. In addition to the operator D, we introduce the operator $Q = DH - HD$. We let $g_0(x,y)$, $g_1(x,p_\varepsilon,y)$, and $g_2(x,p_\varepsilon,y)$ denote the solutions with zero mean on the cell:

$$g_0 = \frac{1}{\Delta_y^\theta}\widetilde{a}, \qquad g_1 = \frac{1}{\Delta_y^\theta}(D\widetilde{a}), \qquad g_2 = \frac{1}{\Delta_y^\theta}(Qg_1 - \langle Qg_1\rangle_{\mathbb{T}}), \qquad \langle g_{0,1,2}\rangle_{\mathbb{T}} = 0. \tag{3.16}$$

We note that $g_1(x,p_\varepsilon,y)$ is a linear homogeneous function of p_ε and $g_2(x,p_\varepsilon,y)$ is a second-order homogeneous polynomial in p_ε.

Lemma 2. *For x lying in a compact set K and for sufficiently small p_ε, the minimal eigenvalue $\mathcal{L}_0(x,p_\varepsilon)$ of the operator $\mathcal{H}_0$ is nondegenerate and analytic in p_ε. Then the functions $\chi_0(x,p_\varepsilon,y)$ and $\chi_1(x,p_\varepsilon,y)$ can be chosen to be analytic in p_ε, so that expressions* (3.12) *hold with*

$$\mathcal{L}_0^{(2)}(x,p_\varepsilon) = p_\varepsilon^2 H_0 - \langle HDg_1\rangle_{\mathbb{T}}, \qquad \mathcal{L}_1^{(1)}(x,p_\varepsilon)) = i\langle\langle\nabla_x, H\nabla_y^\theta g_1\rangle\rangle_{\mathbb{T}} - i\langle\nabla_x, H_0 p_\varepsilon\rangle, \tag{3.17}$$

$$\mathcal{L}_0^{(4)}(p_\varepsilon,x) = p_\varepsilon^4\langle g_0\widetilde{a}\rangle_{\mathbb{T}} + 2p_\varepsilon^2\langle g_1 Qg_0\rangle_{\mathbb{T}} + \langle g_1^2\rangle_{\mathbb{T}}\langle Qg_1\rangle_{\mathbb{T}} + p_\varepsilon^2\langle g_1^2\widetilde{a}\rangle_{\mathbb{T}} + \langle g_2 Qg_1\rangle_{\mathbb{T}}, \tag{3.18}$$

$$\chi_0 = 1 - ig_1(y,x,p_\varepsilon) + p_\varepsilon^2 g_0(y,x) - g_2(x,p_\varepsilon,y) - \frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}} + O(|p_\varepsilon|^3), \tag{3.19}$$

$$\left\| 1 - ig_1(y,x,p_\varepsilon) + p_\varepsilon^2 g_0(y,x) - g_2(x,p_\varepsilon,y) - \frac{1}{2}\langle g_1^2\rangle_{\mathbb{T}} \right\|_{L_2(\mathbb{T})} = 1 + O(|p_\varepsilon|^3). \tag{3.20}$$

Proof. Formulas (3.17) were proved in [9], and formulas (3.18)–(3.20) were proved in [10]. For the waves whose length is greater than the characteristic length of oscillations of the basin's bottom, we seek the solution of the reduced equation (3.7) by using the WKB method. $\square$

3.3. WKB-Solutions of Homogenized Equations

Let w have the form

$$w = A(x,t)e^{iS(x,t)/\mu}, \qquad \mu = \varepsilon^k, \quad 1 > k > 0. \tag{3.21}$$

When using the WKB method, it is convenient to pass from the operators $\widehat{p}_\varepsilon = -i\varepsilon\nabla_x$ to the operators $\widehat{p}_\mu = -i\mu\nabla_x$ so that $p_\varepsilon = (\varepsilon/\mu)p_\mu$. In the case $0 < k \le 2/3$, we only consider the terms $\mathcal{L}_0^{(2)}(x,p_\varepsilon)$, $\mathcal{L}_0^{(4)}(x,p_\varepsilon)$, and $\mathcal{L}_1^{(1)}(x,p_\varepsilon)$ in the expansion of the symbol $\mathcal{L}$ by writing Eq. (3.7) in the form

$$\mu^2 w_{tt} = -\mathcal{L}_0^{(2)}(\overset{2}{x}, -i\overset{1}{\mu}\nabla_x)w - \frac{\varepsilon^2}{\mu^2}\mathcal{L}_0^{(4)}(\overset{2}{x}, -i\overset{1}{\mu}\nabla_x)w - \mu\mathcal{L}_1^{(1)}(\overset{2}{x}, -i\overset{1}{\mu}\nabla_x)w. \tag{3.22}$$

The coefficient ε^2/μ^2 of $\mathcal{L}_0^{(4)}$ in (3.22) is equal to $\mu^{(2-2k)/k}$, and $(2-2k)/k \ge 1$ for $0 < k \le 2/3$. Therefore, the second term in the right-hand side of (3.22) can be taken into account in the transport equation for A by writing it in the form

$$\begin{aligned} 2S_tA_t + S_{tt}A = \langle(\mathcal{L}_0^{(2)})_p(x,\nabla_x S), \nabla_x A\rangle + \frac{1}{2}ASp((\mathcal{L}_0^{(2)})_{pp}(x,\nabla_x S)S_{xx}) \\ + i\mathcal{L}_1^{(1)}(x,\nabla_x S) + i\mu^{2/k-3}\mathcal{L}_0^{(4)}(x,\nabla_x S), \end{aligned}$$

i.e., the term corresponding to $\mathcal{L}_0^{(4)}$ are added to the standard transport equation, where only $\mathcal{L}_0^{(2)}$ is taken into account, and then A depends on x, t, and μ. We obtain the standard Hamiltonian–Jacobi equation for the phase S:

$$S_t^2 = \mathcal{L}_0^{(2)}(x,\nabla_x S).$$

The orders (with respect to the parameter μ) of the terms of the expansion of $\mathcal{L}$ in p_ε and ε, which do not enter the right-hand side of (3.22), are greater than one, and they are neglected when the leading term of the WKB-approximation is calculated.

If the second term on the right-hand side of Eq. (3.22) is neglected in the transport equation but is taken into account in the equation for the phase, which is written in the form

$$S_t^2 = \mathcal{L}_0^{(2)}(x, \nabla_x S) + \mu^{2/k-2}\mathcal{L}_0^{(4)}(x, \nabla_x S),$$

then we obtain the WKB-approximation (3.21) for all $0 < k < 4/5$ (in this case, A and S depend on x, t, and μ). In the case $1 > k \geq 4/5$, the approximate equation (3.22) can no longer be used, because, for such k, the terms with $j \geq 6$ begin to play a significant role in the expansion (3.12) of the symbol $\mathbb{L}_0$ in homogeneous polynomials in p (the closer k to one, the greater the number of the expansion terms to be considered). Thus, for waves with $\mu = \varepsilon^k$, $0 < k \leq 4/5$, an approximate solution of the reduced Eq. (3.7) can be obtained by using the WKB method with Eq. (3.22).

We note that a similar approach was used in [11] (also see [6], [8]) to study the waves generated by a localized source in the initial data of the form

$$w|_{t=0} = W\left(\frac{x}{\mu}\right), \qquad w_t|_{t=0} = 0,$$

where $W(z)$ is a function sufficiently fast decreasing as $x \to \infty$.

3.4. Smooth Solutions (Long Waves) and the Standard "Homogenized" Equation

In the case of "long" waves, where the solution $w(x, t, \varepsilon)$ of Eq. (3.7) is a smooth function regularly depending on ε (i.e., can be represented as an asymptotic power series in ε with coefficients smooth in x and t), our approach leads to ordinary homogenized equations (see [9]). For the leading term of the corresponding expansion of w in a power series in ε, only the terms $\mathcal{L}_0^{(2)}(x, p_\varepsilon)$, $\mathcal{L}_1^{(1)}(x, p_\varepsilon)$, and $\mathcal{L}_2^{(0)}(x)$ are taken into account in the homogenized equation. In [9], Eq. (2.19) was considered in the case where V is the operator of multiplication by the function $V(x, \theta(x)/\varepsilon)$, and the following formula was obtained for $\mathcal{L}_2^{(0)}(x)$:

$$\mathcal{L}_2^{(0)}(x) = \langle V(x, y)\rangle_{\mathbb{T}}. \tag{3.23}$$

As was shown in [9], the homogenized equation has the form

$$w_{tt} = -\mathcal{L}_0^{(2)}(\overset{2}{x}, -i\overset{1}{\nabla}_x)w - \mathcal{L}_1^{(1)}(\overset{2}{x}, -i\overset{1}{\nabla}_x)w - \mathcal{L}_2^{(0)}(x)w. \tag{3.24}$$

The proof of formula (3.23), which was given in [9], shows that the following formula for V in (2.19) can be obtained similarly:

$$\mathcal{L}_2^{(0)}(x) = \langle \widetilde{V}(x, p_\varepsilon, y, -i\nabla_y, \varepsilon)1\rangle_{\mathbb{T}}. \tag{3.25}$$

It follows from the above definition of the operator V in (3.3) that its operator-valued symbol $\widetilde{V}(x, p_\varepsilon, y, -i\nabla_y, \varepsilon)$ with $\varepsilon = 0$ and $p_\varepsilon = 0$ satisfies the relation

$$\widetilde{V}(x, 0, y, -i\nabla_y, 0) = V_1(x, 0, y, 0)(-i\nabla_y^\theta)^4 \\ + \langle V_2(x, 0, y, 0)(-i\nabla_y^\theta), -i\nabla_y^\theta\rangle + \langle V_3(x, 0, y, 0), -i\nabla_y^\theta\rangle.$$

Therefore, in our case, formula (3.25) implies that $\mathcal{L}_2^{(0)}(x) = 0$ and the homogenized equation has the form

$$w_{tt} = -\mathcal{L}_0^{(2)}(\overset{2}{x}, -i\overset{1}{\nabla}_x)w - \mathcal{L}_1^{(1)}(\overset{2}{x}, -i\overset{1}{\nabla}_x)w. \tag{3.26}$$

It follows from formulas (3.17) for $\mathcal{L}_0^{(2)}$ and $\mathcal{L}_1^{(1)}$ that the operators in the right-hand side of (3.26) have the form

$$\mathcal{L}_0^{(2)}(\overset{2}{x}, -i\overset{1}{\nabla}_x) = -H_0\sum_k \frac{\partial^2 w}{\partial x_k^2} + \sum_{k,j}\langle H\nabla_y^\theta b_j\rangle_{\mathbb{T}}^k \frac{\partial^2 w}{\partial x_k \partial x_j},$$

$$\mathcal{L}_1^{(1)}(\overset{2}{x}, -i\overset{1}{\nabla}_x) = -\sum_k \left(\frac{\partial}{\partial x_k} H_0\right)\frac{\partial w}{\partial x_k} + \sum_{k,j}\left(\frac{\partial}{\partial x_k}\langle H\nabla_y^\theta b_j\rangle_{\mathbb{T}}^k\right)\frac{\partial w}{\partial x_j}.$$

Therefore, the homogenized equation (3.26) can be written as

$$w_{tt} = \sum_{k,j}\frac{\partial}{\partial x_k}\left((\delta_{i,j}H_0 - \langle H\nabla_y^\theta b_j\rangle_{\mathbb{T}}^k)\frac{\partial w}{\partial x_j}\right), \qquad \text{where} \quad b_j(x,y) = \frac{1}{\Delta^\theta}(\theta_{1x_j}H_{y_1} + \theta_{2x_j}H_{y_2}); \tag{3.27}$$

here $\langle H\nabla_y^\theta b_j\rangle_{\mathbb{T}}^k$ is the kth component of the vector $\langle H\nabla_y^\theta b_j\rangle_{\mathbb{T}}$, $H_0 = \langle H(x,y)\rangle_{\mathbb{T}}$, and $\delta_{i,j}$ is the Kronecker delta.

Thus, we arrive at the following assertion.

Theorem 1. *The homogenization of Eq.* (2.19) *in the case of smooth solutions* (*long waves*) *leads to Eq.* (3.27).

4. THEORY OF PERTURBATIONS IN THE SMALL OSCILLATING PART OF THE FUNCTION H

The formulas given in Lemma 2 contain the inverse of the operator Δ_y^θ; for example, to determine g_1, it is necessary to solve the equation

$$\Delta_y^\theta g_1 = D\widetilde{a}, \qquad \langle g_1\rangle_{\mathbb{T}} = 0 \tag{4.1}$$

on the torus $\mathbb{T}$. Such an explicit inversion of the operator Δ_y^θ is possible, but only in exceptional cases. In the one-phase case, the obtained formulas can in principle be realized in quadratures, but they are rather cumbersome (see [10]). Therefore, from the standpoint of asymptotic methods, we can deal only with perturbation theory. Effective asymptotic formulas can be obtained only if the oscillating part $\widetilde{a}$ is not large as compared to the homogenized nonoscillating part $H_0(x)$; therefore, we introduce one more parameter δ and set $\widetilde{a} = \delta a$. Recall that $\widetilde{a} = H(x,y) - H_0(x)$, $H_0(x) = \langle H(x,y)\rangle_{\mathbb{T}}$, and $H_0(x) > 0$. In several problems, it is expedient to consider only the first nontrivial corrections in the expansions in the parameter δ of the functions $\mathcal{L}_0^{(2)}$ and $\mathcal{L}_0^{(4)}$. Such formulas were obtained in [10]. Here we propose another method for deriving these formulas. In [10], the formulas were first obtained for the expansions of g_j in power series in δ up to $O(\delta^2)$, and then these expansions were substituted into formulas (3.17), (3.18). Here we first obtain the expansion of $\mathcal{L}_0$ in power series in δ,

$$\mathcal{L}_0 = \mathcal{L}_{0,0} + \delta\mathcal{L}_{0,1} + \delta^2\mathcal{L}_{0,2} + O(\delta^3), \tag{4.2}$$

and then pass in this formula to the corresponding expansions in p_ε. Formula (3.18) is not used in this approach.

In the situation under study, the expansion in the parameters δ and the momentum coordinates p_ε can be found by applying the perturbation methods to Eq. (3.11). We use the following well-known elementary formulas of the perturbation theory for self-adjoint operators (see, e.g., [28]).

Let $\mathcal{A}$ and $\mathcal{B}$ be self-adjoint operators in the space $\mathbf{H}$ equipped with the inner product $(\,\cdot\,,\,\cdot\,)$, and let λ and ϕ be a nondegenerate eigenvalue and a normed eigenfunction of the operator $\mathcal{A} + \delta\mathcal{B}$:

$$(\mathcal{A} + \delta\mathcal{B})\phi = \lambda\phi, \qquad (\phi,\phi) = 1. \tag{4.3}$$

If λ and ϕ are analytic in δ in a neighborhood of $\delta = 0$ and

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + (\delta^3), \qquad \lambda = \lambda_0 + \delta\lambda_1 + \delta^2\lambda_2 + O(\delta^3),$$

then

$$\lambda_1 = (\phi_0, \mathcal{B}\phi_0), \qquad \lambda_2 = (\phi_1, (\mathcal{B} - \lambda_1)\phi_0), \tag{4.4}$$

where

$$\phi_1 = -(\mathcal{A} - \lambda_0)^{-1}(\mathcal{B} - \lambda_1)\phi_0 \tag{4.5}$$

and $(\mathcal{A}-\lambda_0)^{-1}$ is the inverse of the operator $\mathcal{A}$ in the subspace of the space $\mathbf{H}$ that is orthogonal to ϕ_0. In our case,

$$\mathcal{A}=\langle(p_\varepsilon-i\nabla_y^\theta),H_0(x)(p_\varepsilon-i\nabla_y^\theta)\rangle,\qquad \mathcal{B}=\langle(p_\varepsilon-i\nabla_y^\theta),a(x,y)(p_\varepsilon-i\nabla_y^\theta)\rangle,\qquad \lambda=\mathcal{L}_0. \quad (4.6)$$

For the space $\mathbf{H}$ we take the space $L_2(\mathbb{T})$ with "normed" scalar product (3.14). All eigenfunctions of the operator $\mathcal{A}$ can easily be obtained and are equal to $\exp(i\langle\nu,y\rangle)$, where ν is the column vector with integer-valued components (ν_1,ν_2). We are interested only in $\phi_0=\chi_{0,0}=1$. Then the corresponding eigenvalue is $\lambda_0=\mathcal{L}_{0,0}=H_0(x)p_\varepsilon^2$. Therefore,

$$\lambda_1=\mathcal{L}_{0,1}=(1,\langle(p_\varepsilon-i\nabla_y^\theta),a(x,y)(p_\varepsilon-i\nabla_y^\theta)\rangle 1)=p_\varepsilon^2\langle a\rangle_{\mathbb{T}}=0. \quad (4.7)$$

It follows from (4.4) and (4.5) that

$$\lambda_2=\mathcal{L}_{0,2}=-((\mathcal{A}-\lambda_0)^{-1}(\mathcal{B}-\lambda_1)\phi_0,(\mathcal{B}-\lambda_1)\phi_0).$$

Since $(\mathcal{B}-\lambda_1)\phi_0=\langle p_\varepsilon-i\nabla_y^\theta,p_\varepsilon\rangle a$, we have

$$\lambda_2=\mathcal{L}_{0,2}=-((\mathcal{A}-\lambda_0)^{-1}\langle p_\varepsilon-i\nabla_y^\theta,p_\varepsilon\rangle^2a,a). \quad (4.8)$$

Here we used the fact that the operators $\mathcal{A}$ and $\langle p_\varepsilon-i\nabla_y^\theta,p_\varepsilon\rangle$ commute, and hence the operators $(\mathcal{A}-\lambda_0)^{-1}$ and $\langle p_\varepsilon-i\nabla_y^\theta,p_\varepsilon\rangle$ also commute.

Now we expand the right-hand side of (4.8) in homogeneous polynomials in p_ε. We represent the operator $\mathcal{A}-\lambda_0$, where $\mathcal{A}$ is given by formula (4.6), in the form

$$\mathcal{A}-\lambda_0=-H_0\langle\nabla_y^\theta,\nabla_y^\theta\rangle+2H_0\langle -i\nabla_y^\theta,p_\varepsilon\rangle=-H_0\langle\nabla_y^\theta,\nabla_y^\theta\rangle(1-S),\qquad S=2\frac{\langle -i\nabla_y^\theta,p_\varepsilon\rangle}{\langle\nabla_y^\theta,\nabla_y^\theta\rangle}.$$

Using the standard Neumann series to invert the operator $(1-S)$ and the fact that the operators $\langle\nabla_y^\theta,\nabla_y^\theta\rangle$ and S commute, we obtain the expansion of the operator $(\mathcal{A}-\lambda_0)^{-1}$ in homogeneous polynomials in the variables p_ε:

$$(\mathcal{A}-\lambda_0)^{-1}=-H_0^{-1}\langle\nabla_y^\theta,\nabla_y^\theta\rangle(1+S+S^2+\cdots).$$

We substitute this expansion and the expression

$$\langle p_\varepsilon-i\nabla_y^\theta,p_\varepsilon\rangle^2=(p_\varepsilon^2+\langle -i\nabla_y^\theta,p_\varepsilon\rangle)^2=p_\varepsilon^4+2p_\varepsilon^2\langle -i\nabla_y^\theta,p_\varepsilon\rangle+\langle -i\nabla_y^\theta,p_\varepsilon\rangle^2 \quad (4.9)$$

into formula (4.8), remove the brackets in the obtained product of the series $1+S+S^2+\cdots$ and the polynomial in the right-hand side of (4.9), an then obtain the desired expansion of $\lambda_2=\mathcal{L}_{0,2}$ in homogeneous polynomials in p_ε. Since any derivative of odd order in y is an antisymmetric operator, it follows that the expansion of λ_2 contains only the terms with p raised to even powers. Thus, we obtain

$$\lambda_2=\mathcal{L}_{0,2}=H_0^{-1}(\langle\nabla_y^\theta,\nabla_y^\theta\rangle^{-1}(\langle -i\nabla_y^\theta,p_\varepsilon\rangle^2+(p_\varepsilon^2+S\langle -i\nabla_y^\theta,p_\varepsilon\rangle)^2)a,a)+O(|p_\varepsilon|^6). \quad (4.10)$$

We substitute the earlier obtained values $\mathcal{L}_{0,0}=H_0(x)p_\varepsilon^2$ and $\mathcal{L}_{0,1}=0$ (see formula (4.7)) and the expression for $\mathcal{L}_{0,2}$ in (4.10) into the expansion (4.2), separate the homogeneous components of the second- and fourth degree in (4.10), and obtain the approximate formulas for $\mathcal{L}_0^{(2)}$ and $\mathcal{L}_0^{(4)}$:

$$\mathcal{L}_0^{(2)}=H_0p_\varepsilon^2-\delta^2R(x,p)+O(\delta^3),\qquad R=-H_0^{-1}(\langle\nabla_y^\theta,\nabla_y^\theta\rangle^{-1}\langle -i\nabla_y^\theta,p_\varepsilon\rangle^2a,a), \quad (4.11)$$

and

$$\mathcal{L}_0^{(4)}=-\delta^2M(x,p)+O(\delta^3),\qquad M=-H_0^{-1}\biggl(\langle\nabla_y^\theta,\nabla_y^\theta\rangle^{-1}\biggl(p_\varepsilon^2+2\frac{\langle -i\nabla_y^\theta,p_\varepsilon\rangle^2}{\langle\nabla_y^\theta,\nabla_y^\theta\rangle}\biggr)^2a,a\biggr), \quad (4.12)$$

where $O(\delta^3)$ denotes a homogeneous polynomial in p of the corresponding degree with coefficients of order $O(\delta^3)$. If we consider only the terms of order of δ^2, then we can rather easily express $\mathcal{L}_0^{(2)}$ and $\mathcal{L}_0^{(4)}$ in terms of the Fourier coefficients of the function a. We expand the function a in the Fourier series and substitute it into (4.11) and (4.12). Since the exponentials $\exp(i\langle\nu,y\rangle)$ form an orthonormal system in the space $L_2(\mathbb{T})$, we have the following assertion.

Lemma 3. *Assume that $H(x,y) = H_0(x) + \delta a(x,y)$ and $H_0(x) > 0$ for x belonging to a compact set K. Assume also that $a(x,y)$ is a smooth real 2π-periodic (in the variables y_j) function with zero mean, i.e., $\langle a\rangle_{\mathbb{T}} = 0$, and with the following expansion in Fourier series:*

$$a(x,y) = \sum_{\nu\neq 0} a_\nu(x)\exp(i\langle \nu, y\rangle). \tag{4.13}$$

Here the column vector (multiindex) $\nu = (\nu_1, \nu_2)$ takes values on the integral lattice. Then

$$\mathcal{L}_0^{(2)} = H_0 p_\varepsilon^2 - \delta^2 R(x,p) + O(\delta^3), \qquad R = \frac{1}{H_0}\sum_{\nu\neq 0}\frac{\langle\theta_x\nu, p_\varepsilon\rangle^2}{\langle\theta_x\nu,\theta_x\nu\rangle}|a_\nu|^2, \tag{4.14}$$

and

$$\mathcal{L}_0^{(4)} = -\delta^2 M(x,p) + O(\delta^3), \qquad M = \frac{1}{H_0}\sum_{\nu\neq 0}\biggl(p_\varepsilon^2 - 2\frac{\langle\theta_x\nu, p_\varepsilon\rangle^2}{\langle\theta_x\nu,\theta_x\nu\rangle}\biggr)^2\frac{|a_\nu|^2}{\langle\theta_x\nu,\theta_x\nu\rangle}. \tag{4.15}$$

In the "one-phase" case, where θ is a scalar function, y is a scalar variable, and ν is an index, the formulas become somewhat simpler:

$$R = \frac{1}{H_0}\frac{\langle\theta_x, p_\varepsilon\rangle^2}{\theta_x^2}\sum_{\nu\neq 0}|a_\nu|^2, \quad M = \frac{1}{H_0\theta_x^2}\biggl(p_\varepsilon^2 - 2\frac{\langle\theta_x, p_\varepsilon\rangle^2}{\theta_x^2}\biggr)^2\sum_{\nu\neq 0}\frac{|a_\nu|^2}{\nu^2}.$$

Now let us approximately calculate the symbol $\mathcal{L}_1^{(1)}$ in (3.17) and the coefficients of the homogenized equation (3.27).

Theorem 2. *Let the conditions of Lemma 2 be satisfied. Then the symbol $\mathcal{L}_1^{(1)}$ admits the expansion*

$$\mathcal{L}_1^{(1)} = i\delta^2\biggl\langle\nabla_x, \frac{1}{H_0}\sum_{\nu\neq 0}\frac{\theta_x\nu\langle\theta_x\nu, p_\varepsilon\rangle}{\langle\theta_x\nu,\theta_x\nu\rangle}|a_\nu|^2\biggr\rangle - i\langle\nabla_x, H_0 p_\varepsilon\rangle + O(\delta^3), \tag{4.16}$$

where $O(\delta^3)$ is a linear homogeneous (in p) function with coefficients of order $O(\delta^3)$. The coefficients of the homogenized equation (3.27) admit the expansion

$$\langle H\nabla_y^\theta b_j\rangle_{\mathbb{T}}^k = \delta^2 B_{k,j}(x) + O(\delta^3), \tag{4.17}$$

where

$$B_{k,j}(x) = \frac{1}{H_0}\sum_{\nu\neq 0}\frac{(\theta_x\nu)^k(\theta_x\nu)^j}{\langle\theta_x\nu,\theta_x\nu\rangle}|a_\nu|^2 \tag{4.18}$$

and the superscript k or j denotes the corresponding component of the vector. Thus, up to $O(\delta^3)$, the homogenized equation (3.27) can be written as

$$w_{tt} = \sum_k\frac{\partial}{\partial x_k}\biggl(H_0(x)\frac{\partial w}{\partial x_k}\biggr) - \delta^2\sum_{k,j}\frac{\partial}{\partial x_k}\biggl(B_{k,j}(x)\frac{\partial w}{\partial x_j}\biggr). \tag{4.19}$$

Proof. Formula (4.16) can be obtained directly from the second formula in (3.17) with g_1 replaced by the expansion of g_1 in a power series in δ (see [10]). Formula (4.1) shows that, in the first approximation, we have

$$g_1 = \delta H_0^{-1}\langle\nabla_y^\theta, \nabla_y^\theta\rangle^{-1}Da + O(\delta^2), \tag{4.20}$$

where $O(\delta^2)$ is a linear homogeneous (in p) function with coefficients of order $O(\delta^2)$. After some obvious transformations, we obtain

$$\mathcal{L}_1^{(1)} = i\delta^2\langle\nabla_x, H_0^{-1}\langle a\nabla_y^\theta\langle\nabla_y^\theta,\nabla_y^\theta\rangle^{-1}Da\rangle_{\mathbb{T}}\rangle - i\langle\nabla_x, H_0 p_\varepsilon\rangle + O(\delta^3). \tag{4.21}$$

Now, just as in the proof of Lemma 1, we expand the function a in its Fourier series and then obtain formula (4.16) from (4.21). Equation (4.19) is obtained from (4.14) and (4.16) by transformations similar to those used to obtain Eq. (3.27) from formulas (3.17). We can also derive formulas (4.17), (4.18) by replacing b_j in the left-hand side of (4.17) by the expansion

$$b_j = \delta H_0^{-1}\langle\nabla_y^\theta, \nabla_y^\theta\rangle^{-1}(\nabla_y^\theta a)^j + O(\delta^2),$$

which follows from (4.20) and $g_1 = b_1p_1 + b_2p_2$. As a result, we obtain

$$B_{k,j} = -\delta^2\langle a(\nabla_y^\theta)^k\langle\nabla_y^\theta, \nabla_y^\theta\rangle^{-1}(\nabla_y^\theta)^j a\rangle_{\mathbb{T}}. \tag{4.22}$$

To obtain formula (4.18), it is now sufficient to pass in (4.22) to the expansion of a in the Fourier series.

The formulas in Lemma 3 are especially effective in the case, where a has the form of a finite sum similar to (4.13). Formulas (4.14), (4.15), (4.16), and (4.18) also have finitely many terms, and hence the calculation of the right-hand sides of these formulas in such cases reduces to finitely many algebraic operations with Fourier coefficients for a and with elements of the matrix θ_x.

If we add the terms corresponding to $\mathcal{L}_0^{(4)}$ to the homogenized equation, then we obtain the linearized Boussinesq equation with variable coefficients, where the so-called dispersion effects are taken into account. An equation of such a type is obtained when studying the solutions of system (2.4)–(2.6) with slowly varying bottom and for certain wave lengths. Then the sign of the term containing p raised to the fourth power (or the fourth derivatives), which is similar to $\mathcal{L}_0^{(4)}$, is positive in contrast to the sign of $L_0^{(4)}$. Therefore, from the standpoint of the theory of water waves, the dispersion effects due to the fast oscillation profile of the bottom turn out to be anomalous. This problem and the consequences of the appearance of anomalous dispersion in the case of localized solutions of the linearized Boussinesq equation were discussed in [11], [29]. $\square$

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