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The topological classification of structurally stable 3-diffeomorphisms with two-dimensional basic sets

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Abstract

In this paper we consider a class of structurally stable diffeomorphisms with two-dimensional basic sets given on a closed 3-manifold. We prove that each such diffeomorphism is a locally direct product of a hyperbolic automorphism of the 2-torus and a rough diffeomorphism of the circle. We find algebraic criteria for topological conjugacy of the systems.

Keywords: structural stability, surface basic set, topological classification
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(Some figures may appear in colour only in the online journal)

1. Introduction and formulation of the results

We consider a diffeomorphism f on a closed 3-manifold M^3 which satisfies Smale's axiom A (*A-diffeomorphism*). According to Smale's spectral theorem [27], the nonwandering set $NW(f)$ of f can be represented as a finite union of pairwise disjoint closed invariant sets, called *basic sets*, each of which contains a dense trajectory.

It is known that the existence of a basic set with dimension 3 or 2 imposes strong constraints on the topology of M^3 and the dynamic of f. Indeed, if $NW(f)$ contains the basic set $\mathcal{B}$ with $\dim \mathcal{B} = 3$ then f is an Anosov diffeomorphism and the manifold M^3 is the torus $\mathbb{T}^3$. Topological classification of Anosov diffeomorphisms on $\mathbb{T}^3$ was obtained by Franks and Newhouse in [5, 21].

If dim $\mathcal{B} = 2$, then, due to [22], $\mathcal{B}$ is either an attractor or a repeller. It follows from [4] that any two-dimensional attractor (repeller) of A-diffeomorphism $f : M^3 \to M^3$ is either an expanding attractor (contracting repeller) or a surface attractor (surface repeller).

It follows from [11] and [18] that any manifold M^3 which admits structurally stable diffeomorphism $f : M^3 \to M^3$ with a two-dimensional expanding attractor (contracting repeller), is diffeomorphic to the torus $\mathbb{T}^3$ and, moreover, f is topologically conjugated with the diffeomorphism obtained from the Anosov diffeomorphism by the generalized surgery operation. According to [9] each connected component of the two-dimensional surface basic set for A-diffeomorphism $f : M^3 \to M^3$ is homeomorphic to the torus and a restriction of some degree of f to this component is topologically conjugate with the Anosov diffeomorphism.

In this paper we consider class G of all A-diffeomorphisms $f : M^3 \to M^3$ such that for each $f \in G$ the nonwandering set $NW(f)$ consists of two-dimensional surface basic sets. We prove that each f from G is an ambient Ω-conjugate with some model diffeomorphism of the mapping torus. It follows from the information above that any structurally stable diffeomorphism $f : M^3 \to M^3$ with two-dimensional basic sets automatically belongs to the class G. We prove that, in this case, f is topologically conjugated with the model and we give algebraic criteria for the topological conjugacy of two models.

Let us represent torus $\mathbb{T}^2$ as the factor group $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and denote the *neutral element* of the group $\mathbb{T}^2$ by $O \in \mathbb{T}^2$. Recall that an algebraic automorphism $\widehat{C} : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus is a map defined by a matrix $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which belongs to the set $GL(2, \mathbb{Z})$ of integer matrices with determinant ± 1, i.e. $\widehat{C}(x, y) = (ax + by, cx + dy) \pmod 1$. $\widehat{C}$ is called *hyperbolic* if the absolute value of each eigenvalue does not equal 1, herewith the matrix C is also called *hyperbolic*. Denote by $\mathcal{C}$ the set of the hyperbolic matrices from $GL(2, \mathbb{Z})$. Set $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $-Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathcal{J} = \mathcal{C} \cup Id \cup (-Id)$.

One says that M_τ is a *mapping torus* if M_τ, derived from $\mathbb{T}^2 \times [0, 1]$ by the identification of points $(z, 1)$ and $(\tau(z), 0)$, where $\tau : \mathbb{T}^2 \to \mathbb{T}^2$, is a homeomorphism.

The next theorem (see its proof in section 2) singles out the set of all manifolds which admit diffeomorphisms from G[3].

Theorem 1. *Let a manifold M^3 admit a diffeomorphism f from the class G. Then M^3 is diffeomorphic to a mapping torus $M_{\widehat{J}}$, where $J \in \mathcal{J}$.*

For description dynamics of the diffeomorphism from G let us denote by Φ_+ (Φ_-) the set of all locally direct products $\phi_+ = \widehat{C}_+ \otimes \varphi_+$ ($\phi_- = \widehat{C}_- \otimes \varphi_-$) of a hyperbolic automorphism $\widehat{C}_+ (\widehat{C}_-)$ of the 2-torus and a rough diffeomorphism of the circle $\varphi_+ (\varphi_-)$ preserving (changing) orientation, see section 3 for details. Each such difffeomorphism $\phi_+ \in \Phi_+$ is uniquely defined by parameters $\{J_+, C_+, n, k, l\}$ and each difffeomorphism $\phi_- \in \Phi_-$ is uniquely defined by parameters $\{J_-, C_-, q, \nu\}$. Set $\Phi = \Phi_+ \cup \Phi_-$.

The following result provides an algebraic criteria for topological conjugacy of the diffeomorphisms from Φ (see proof in section 4).

[3] In [14] it was considered an irreducible manifold M^3 (i.e. any cylindrically embedded 2-sphere in M^3 bounds a 3-ball) which admits a diffeomorphism $f : M^3 \to M^3$ with an *invariant Anosov torus* (i.e. there exists a smooth f-invariant submanifold in M^3 which is homeomorphic to the 2-torus and such that the induced action of f in the fundamental group of the 2-torus is hyperbolic). Under these assumptions the authors of [14] obtained results similar to those from theorem 1. Notice that in theorem 1 we don't use the assumption on the irreducibility of M^3.

Theorem 2.

1. *Two diffeomorphisms* $\phi_+; \phi'_+ \in \Phi_+$ *with parameters* $\{J_+, C_+, n, k, l\}; \{J'_+, C'_+, n', k', l'\}$ *are topologically conjugated if and only if* $n = n', k = k'$*, there exists a matrix* $H \in GL(2, \mathbb{Z})$ *such that* $C_+H = HC'_+$ *and at least one of the following assertions holds:*
    - $J_+H = HJ'_+$ *and* $l = l'$*,*
    - $J_+^{-1}H = HJ'_+$ *and either* $l = l' = 0$ *or* $l = k' - l'$*.*
2. *Two diffeomorphisms* $\phi_-; \phi'_- \in \Phi_-$ *with parameters* $\{J_-, C_-, q, \nu\}; \{J'_-, C'_-, q', \nu'\}$ *are topologically conjugated if and only if* $J_- = J'_-$*,* $q = q'$*,* $\nu = \nu'$ *and there exists a matrix* $H \in GL(2, \mathbb{Z})$ *such that* $C_-H = HC'_-$*.*
3. *There are no topologically conjugated diffeomorphisms* $\phi_+ \in \Phi_+$*,* $\phi_- \in \Phi_-$*.*

Let us recall that the two diffeomorphisms $f : M^3 \to M^3$, $f' : M'^3 \to M'^3$ are said to be *ambient* Ω*-conjugated* if there exists a homeomorphism $h : M^3 \to M'^3$ such that $h(NW(f)) = NW(f')$ and $hf|_{NW(f)} = f'h|_{NW(f)}$.

Theorem 3. *Any diffeomorphism from the class* G *is ambient* Ω*-conjugated to some diffeomorphism from the class* Φ*.*

See section 5 for the proof of Theorem 3.

The next theorem is the main result of the paper and is proved in section 6. The main difficulty of the proof is the nontrivial investigation of the asymptotic behavior of invariant two-dimensional manifolds of the nonwandering set of a diffeomorphism $f \in G$. Assuming structural stability, it has a surprising property: the union of closures of all arcs in the intersection of the two-dimensional manifolds forms an invariant one-dimensional foliation whose leaves are transversal to all basic sets. This fact is a crucial argument for the construction of a conjugating homeomorphism in the proof of theorem 4.

Theorem 4. *Each structurally stable diffeomorphism from class* G *is topologically conjugate with a diffeomorphism from class* Φ*.*

Notice that in class G there are diffeomorphisms which are not topologically conjugated to any diffeomorphism from class Φ (see section 8, where the corresponding example is constructed).

2. The structure of the ambient manifold admitting diffeomorphisms of class *G*

Let M^3 be a closed 3-manifold. Recall ([27]) that a diffeomorphism $f : M^3 \to M^3$ satisfies *axiom A* if the following conditions hold: (1) the non-wandering set $NW(f)$ is hyperbolic[4]; (2) the periodic points are dense in $NW(f)$. By [17, 24] and [25], axiom A and the strict transversality condition are necessary and sufficient conditions for the structural stability of f. *The strict transversality condition* means that all intersections of the stable and unstable manifolds of any nonwandering point are transversal.

[4] A closed f-invariant set $\Lambda \subset M^3$ is said to be *hyperbolic* if there exists continuous Df-invariant decomposition of the tangent subbundle $T_\Lambda M^3$ into the direct sum $E^s_\Lambda \oplus E^u_\Lambda$ of the stable and unstable subbundles such that $\|Df^k(v)\| \leqslant C\lambda^k\|v\|$, $\|Df^{-k}(w)\| \leqslant C\lambda^k\|w\|$, $\forall v \in E^s_\Lambda, \forall w \in E^u_\Lambda, \forall k \in \mathbb{N}$ for some fixed numbers $C > 0$ and $\lambda < 1$. The hyperbolicity condition implies the existence of stable and unstable manifolds denoted by W^s_x and W^u_x for each point $x \in \Lambda$, which are defined as follows: $W^s_x = \{y \in M^3 : d(f^k(x), f^k(y)) \to 0, k \to +\infty\}$, $W^u_x = \{y \in M^3 : d(f^k(x), f^k(y)) \to 0, k \to -\infty\}$, where d is the metric on Λ induced by the Riemannian metric on $T_\Lambda M^3$.

By [1, 2] each basic set $\mathcal{B}$ can be represented as a finite union $B_1\cup\cdots\cup B_{k_\mathcal{B}}$, $k_\mathcal{B}\geqslant 1$, of closed subsets such that $f^{k_\mathcal{B}}(B_i)=B_i$, $f(B_i)=B_{i+1}$ $(B_{k_\mathcal{B}+1}=\mathcal{B}_1)$. We call the sets $B_1,\dots,B_{k_\mathcal{B}}$ *the periodic components*[5] of the set $\mathcal{B}$; the number $k_\mathcal{B}$ being their *period*.

Let $\mathcal{B}$ be a basic set of a diffeomorphism f. Set $a=\dim\,E^s_\mathcal{B}$, $b=\dim\,E^u_\mathcal{B}$ and call the pair (a,b) the *type* of the basic set $\mathcal{B}$. A basic set $\mathcal{B}$ of f is called an *attractor* if $\mathcal{B}$ has a closed neighborhood U such that $f(U)\subset\operatorname{int}\,U$, $\bigcap_{j\geqslant 0}f^j(U)=\mathcal{B}$. A basic set $\mathcal{B}$ is called a *repeller* of f if it is an attractor for the diffeomorphism f^{-1}.

According to [22], the following facts take place for any A-diffeomorphism $f: M^3\to M^3$ (detailed proofs can be found in [10]).

Statement 1.

- *A basic set $\mathcal{B}$ of f is an attractor (repeller) if and only if* $\mathcal{B}=\bigcup_{x\in\mathcal{B}}W^u(x)\big(\mathcal{B}=\bigcup_{x\in\mathcal{B}}W^s(x)\big)$;
- *if a basic set $\mathcal{B}$ of f has topological dimension 2 then it is either an attractor or a repeller;*

Recall that, due to [28], an attractor $\mathcal{B}$ of f is said to be *expanding* if the topological dimension $\dim\mathcal{B}$ equals the dimension of W^u_x, $x\in\mathcal{B}$. *A contracting repeller* of a diffeomorphism f is an expanding attractor of f^{-1}. According to [7], a basic set $\mathcal{B}$ of a diffeomorphism $f: M^3\to M^3$ is called a *surface basic set* if it is contained in an f-invariant closed surface $M^2_\mathcal{B}$ (not necessarily connected) topologically embedded in M^3. The surface $M^2_\mathcal{B}$ is called *the support* of $\mathcal{B}$.

The following statement on surface basic sets follows from [9].

Statement 2. *For any two-dimensional surface attractor (repeller) $\mathcal{B}$ of A-diffeomorphism $f: M^3\to M^3$ the following holds:*

- *$\mathcal{B}$ has type* $(2, 1)$ $((1, 2))$ *and, therefore, is not an expanding attractor (an contracting repeller).*
- *$\mathcal{B}$ coincides with its support and is a finite union of manifolds tamely embedded*[6] *in M^3 and homeomorphic to the 2-torus*[7].
- *the restriction of $f^{k_\mathcal{B}}$ to any connected component of the support is conjugated to some hyperbolic automorphism of the torus.*

The next fact follows from [5] (see also [10]).

Statement 3. *Let $h:\mathbb{T}^2\to\mathbb{T}^2$ be a homeomorphism topologically conjugated with an Anosov diffeomorphism, with a fixed point $x_0\in\mathbb{T}^2$ and acting in the fundamental group $\pi_1(\mathbb{T}^2,x_0)$ by a hyperbolic matrix H. Then there is a unique isotopic to identity homeomorphism $g:\mathbb{T}^2\to\mathbb{T}^2$ such that $gh=\widehat{H}g$ and $g(x_0)=O$.*

Let us recall that we denote by G the class of all A-diffeomorphisms $f: M^3\to M^3$ such that for each $f\in G$ the nonwandering set $NW(f)$ consists of two-dimensional surface basic sets. Denote by $\mathcal{A}$ (by $\mathcal{R}$) the union of all attractors (repellers) from $NW(f)$. The next lemma is a

[5] In [2], the sets $B_1,\dots,B_{k_\mathcal{B}}$ are called C-dense components but in our opinion, it is more natural to call them the periodic components (see [10]).

[6] A topological embedding $\lambda: X\to Y$ (a homeomorphism to the image) of an m-manifold X into an n-manifold Y $(m\leqslant n)$ is said to be *locally flat at the point* $\lambda(x)$, $x\in X$, if the point $\lambda(x)$ is in the domain of such a chart (U,ψ) of the manifold Y that $\psi(U\cap\lambda(X))=\mathbb{R}^m$, here $\mathbb{R}^m\subset\mathbb{R}^n$ is the set of points for which the last $n-m$ coordinates are equal to 0. An embedding λ is said to be *tame* and the manifold X is said to be *tamely embedded* if λ is locally flat at every point $x\in X$. Otherwise the embedding λ is said to be *wild* and the manifold X is said to be *wildly embedded*.

[7] It should be emphasized that the support of a two-dimensional surface basic set is not necessarily smooth (the corresponding example is given in [15]). Due to [19], 2-torus B is tamely embedded in M^3 if and only if there is a topological embedding $\eta:\mathbb{T}^2\times[-1,1]\to M^3$ such that $\eta(\mathbb{T}^2\times\{0\})=B$.

base for studying the topology of M^3. This result was proved in [8], and for completeness we give its proof in section 2.

Lemma 1. *For any diffeomorphism $f \in G$ the sets $\mathcal{A}$, $\mathcal{R}$ are non-empty and the boundary of each connected component V of the set $M^3 \setminus (\mathcal{A} \cup \mathcal{R})$ consists of precisely one periodic component $A \subset \mathcal{A}$ and one periodic component $R \subset \mathcal{R}$. Herewith $cl\ V$ is homeomorphic to the manifold $\mathbb{T}^2 \times [0, 1]$.*

The proof of lemma 1 will follow from the propositions 1 and 2.

Proposition 1. *For any diffeomorphism $f \in G$ the sets $\mathcal{A}$, $\mathcal{R}$ are nonempty and the boundary of each connected component of the set $M^3 \setminus (\mathcal{A} \cup \mathcal{R})$ consists of precisely one periodic component of an attractor and one periodic component of a repeller.*

Proof. We first show that the sets $\mathcal{A}$, $\mathcal{R}$ are not empty. Assume the contrary: $\mathcal{A} = \varnothing$ for definiteness. According to [27] (corollary 6.3 to the theorem 6.2) the manifold M^3 is represented in the form $M^3 = \bigcup_i W^s_{\mathcal{B}_i} = \bigcup_i W^u_{\mathcal{B}_i}$, where $\mathcal{B}_i$ is the basic set of diffeomorphism f from the decomposition $NW(f) = \bigcup_i \mathcal{B}_i$. Then $M^3 = \bigcup_{\mathcal{B}_i \subset \mathcal{R}} W^s_{\mathcal{B}_i}$. According to [22], for any point $z \in \mathcal{R}$ the stable manifold $W^s(z)$ belongs to the set $\mathcal{R}$. Therefore $M^3 \subset \mathcal{R}$, but it is impossible because the set $\mathcal{R}$ is two-dimensional. Thus the sets $\mathcal{A}$, $\mathcal{R}$ are not empty.

Let us consider the set $M^3 \setminus (\mathcal{A} \cup \mathcal{R})$ and denote by V its a connected component. Let us notice that $V \subset \bigcup_{z \in \mathcal{A}} W^s(z)$ and $V \subset \bigcup_{z \in \mathcal{R}} W^u(z)$. Then there exist only one connected component A of an attractor from the set $\mathcal{A}$ and only one connected component R of a repeller from $\mathcal{R}$, such that $V \subset \bigcup_{z \in A} W^s(z)$ and $V \subset \bigcup_{z \in R} W^u(z)$. Therefore $cl\ V = A \cup V \cup R$ and $\partial V = A \cup R$. $\square$

The following two lemmas are used in the proof of proposition 2.

Lemma 2. *Let Q and P be connected domains in an n-manifold M and the boundary of P consist of two disjoint sets S_1, S_2, such that $S_1 \subset Q$, $S_2 \cap (Q \cup \partial Q) = \varnothing$. Then if S_1 bounds a domain $Q_1 \subset Q$, then $\partial Q \subset P$.*

Proof. Set $P_1 = P \cup Q_1 \cup S_1$. Since $P_1 \cap Q \neq \varnothing$ and the boundary S_2 of domain P_1 has no common points with the closure $cl\ Q$, then $Q \cup \partial Q \subset P_1$. From the equality $\partial Q \cap (Q_1 \cup S_1) = \varnothing$ it follows that $\partial Q \subset P$. $\square$

Lemma 3. *Let P_1, P_2 and Q be topological spaces such that there exist homeomorphisms $h_1 : Q \times [0, 1] \to P_1$ and $h_2 : Q \times [0, 1] \to P_2$. Then*

(a) if $P_1 \cap P_2 = h_1(Q \times \{1\}) = h_2(Q \times \{0\})$, then there exists a homeomorphism $H : Q \times [0, 1] \to P_1 \cup P_2$;

(b) if $P_1 \cap P_2 = h_1(Q \times \{0, 1\}) = h_2(Q \times \{0, 1\})$, then there exists a continuous map $H : Q \times [0, 1] \to P_1 \cup P_2$ such that the restrictions $H|_{Q \times (0,1)}$, $H|_{Q \times \{0\}}$ and $H|_{Q \times \{1\}}$ are homeomorphisms.

Proof. In the case (a) we define a homeomorphism $h_{1,2} : Q \to Q$ by the formula $h_2^{-1}(h_1(q, 1)) = (h_{1,2}(q), 0)$ for any point $q \in Q$ and a homeomorphism $H_{1,2} : Q \times [0, 1] \to Q \times [0, 1]$ by the formula $H_{1,2}(q, t) = (h_{1,2}(q), t)$. Let $H_2 = h_2 H_{1,2} : Q \times [0, 1] \to P_2$. Then the required homeomorphism $H : Q \times [0, 1] \to P_1 \cup P_2$ is defined by the formula $H(q, t) = \begin{cases} h_1(q, 2t), t \in \left[0, \frac{1}{2}\right] \\ H_2(q, 2t - 1), t \in \left[\frac{1}{2}, 1\right]. \end{cases}$

In the case (b) without loss of generality we can suppose that $h_1(Q\times\{1\}) = h_2(Q\times\{0\})$. Then the map H, which was constructed in the case a), has one-to-one correspondence on the set $Q\times[0,1)$ and $H(Q\times\{0\}) = H(Q\times\{1\})$. By construction, the map H is continuous and its restrictions $H|_{Q\times(0,1)}$, $H|_{Q\times\{0\}}$, $H|_{Q\times\{1\}}$ are homeomorphisms. □

Let B be a connected component of $\mathcal{A}\cup\mathcal{R}$. According to statement 2 the surface B is a torus cylindrically embedded in M^3. Then there exist a closed neighborhood $U(B)$ and a homeomorphism h_B such that: $h_B : U(B)\to\mathbb{T}^2\times[-1,1]$, and besides, $h_B(B) = \mathbb{T}^2\times\{0\}$. Set $V_B = h_B^{-1}(\mathbb{T}^2\times(0,1))$ and $T_B = h_B^{-1}(\mathbb{T}^2\times\{1\})$. Since the number of connected components of $\mathcal{A}\cup\mathcal{R}$ is finite then there is a natural number κ such that $f^\kappa(B) = B$ and $f^\kappa(V_B)\cap V_B \neq \varnothing$ for each connected component B of $\mathcal{A}\cup\mathcal{R}$. Denote by κ_0 the minimum from such κ. Set

$$f_0 = f^{\kappa_0}. \qquad (*)$$

Proposition 2. *For any diffeomorphism $f\in G$ the closure of each connected component from the set $M^3\setminus(\mathcal{A}\cup\mathcal{R})$ is homeomorphic to $\mathbb{T}^2\times[0,1]$.*

Proof. Let V be a connected component of $M^3\setminus(\mathcal{A}\cup\mathcal{R})$ with the boundary components A and R, where A is an attractor and R is a repeller of f_0. Without loss of generality we can assume that $V_A\subset V$ and $V_R\subset V$.

Let us show that there exists a natural number ν_* such that torus $T_A^* = f_0^{-\nu_*}(T_A)$ belongs to V_R. Indeed, each point $t\in T_A$ is wandering for f_0 and its negative iteration goes to repeller R. Then there is a closed neighborhood $U_t\subset T_A$ of the point t and natural number $\nu(t)$ such that $f_0^{-\nu}(U_t)\subset V_R$ for $\nu\geqslant\nu(t)$. As the set T_A is compact then there is a finite subcover for cover $\{U_t, t\in T_A\}$. Thus there is a natural number ν_* such that $f_0^{-\nu}(T_A)\subset V_R$ for $\nu\geqslant\nu_*$.

Let us show that R and T_R belong to the different connected components of the set $cl(V_R)\setminus T_A^*$. Assume the contrary. Then according to [9][8] (lemma 3.1), T_A^* is the boundary of some domain $D\subset V_R$. It follows from lemma 2, that if we denote $P = V_R$, $Q = f_0^{-\nu_*}(V_A)$ we get $A\subset V_R$. Since the surface A is f_0-invariant, we get a contradiction.

Thus the set $cl(V_R)\setminus T_A^*$ consists of two connected components. By theorem 3.3 in the paper [9], the closure of each component is homeomorphic to $\mathbb{T}^2\times[0,1]$. Then the surfaces R, T_A^* bound a closed domain in M^3, which is homeomorphic to $\mathbb{T}^2\times[0,1]$. Since the set $f_0^{-\nu_*}(cl(V_A))$ is also homeomorphic to $\mathbb{T}^2\times[0,1]$ and by the lemma 3, V is homeomorphic to the direct product $\mathbb{T}^2\times[0,1]$. □

The next statement is well known fact in topology (see, for example, [12]); we proved it for completeness.

Statement 4. *Each mapping torus M_τ is homeomorphic to mapping torus $M_{\widehat{J}}$ where $J\in GL(2,\mathbb{Z})$ is a matrix, which is defined by the action of the automorphism $\tau_* : \pi_1(\mathbb{T}^2)\to\pi_1(\mathbb{T}^2)$.*

Proof. Since the homeomorphisms τ and $\widehat{J}$ act the same way in the fundamental group they are isotopic (see, for example, theorem 4 in [26]), and therefore an isotopy $\xi_t : \mathbb{T}^2\to\mathbb{T}^2$, $t\in[0,1]$ from map $\xi_0 = \widehat{J}\tau^{-1}$ to identity map ξ_1 exists. Define a homeomorphism $E_J : \mathbb{T}^2\times[0,1]\to\mathbb{T}^2\times[0,1]$ by the formula $E_J(z,t) = (\xi_t(z),t)$. Then the homeomorphism $\check{E}_J : M_\tau\to M_{\widehat{J}}$, which maps equivalence class $[(z,t)]$ to equivalence class $[E_J(z,t)]$, is the desired. □

[8] In fact in paper [9] the objects are required to be smooth, but actually the results are true in the case when the objects are tame.

For $J \in \mathcal{J}$ let us represent the manifold $M_{\widehat{J}}$ as the orbit space $M_{\widehat{J}} = (\mathbb{T}^2 \times \mathbb{R})/\Gamma$, where $\Gamma = \{\gamma^i, i \in \mathbb{Z}\}$ is the group of powers of the diffeomorphism $\gamma : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$, defined by $\gamma(z, r) = (\widehat{J}(z), r - 1)$. Let

$$p_{_J} : \mathbb{T}^2 \times \mathbb{R} \to M_{\widehat{J}}$$

be the natural projection. Let us represent $\mathbb{S}^1$ as $\mathbb{S}^1 = \{(\cos 2\pi r,\ \sin 2\pi r) \in \mathbb{R}^2 : r \in \mathbb{R}\}$. Denote by

$$\pi : \mathbb{R} \to \mathbb{S}^1$$

the projection given by formula $\pi(r) = (\cos 2\pi r, \sin 2\pi r)$. Let us define a map

$$\pi_{_J} : M_{\widehat{J}} \to \mathbb{S}^1$$

by formula

$$\pi_{_J}(p_{_J}(z, r)) = \pi(r).$$

The following statements 5 and 6, were proved in [6].

Statement 5.

- *Let $J \in GL(2, \mathbb{Z})$. Then the fundamental group $\pi_1(M_{\widehat{J}})$ is a semi-direct product of the subgroup $R_J \cong \mathbb{Z}$ and of the normal subgroup $N_J = p_{_J*}(\mathbb{T}^2 \times \mathbb{R}) \cong \mathbb{Z}^2$, that is, any homotopy class $[c] \in \pi_1(M_{\widehat{J}})$ can be uniquely written as (a, b), $a \in R_J, b \in N_J$ and the group operation is $(a_1, b_1)(a_2, b_2) = (a_1 + a_2, J^{a_1}(b_2) + b_1)$.*
- *If the homeomorphism $h : M_{\widehat{J}} \to M_{\widehat{J}'}$ induces the isomorphism $h_* : \pi_1(M_{\widehat{J}}) \to \pi_1(M_{\widehat{J}'})$ such that $h_*(N_J) = N_{J'}$, then h_* is uniquely defined by matrix $H \in GL(2, \mathbb{Z})$ and by an element $\beta \in N_{J'}$ such that $h_*(0, b) = (0, H(b)), b \in \mathbb{Z}^2$ either $h_*(1, 0) = (1, \beta)$ and $HJ = J'H$, or $h_*(1, 0) = (-1, \beta)$ and $HJ^{-1} = J'H$. Herewith the homeomorphism h lifts to a homeomorphism $\tilde{h} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ such that $\tilde{h}_* : \pi_1(\mathbb{T}^2 \times \mathbb{R}) \to \pi_1(\mathbb{T}^2 \times \mathbb{R})$ is defined by the matrix H.*

Statement 6. *For $J, J' \in GL(2, \mathbb{Z})$ two mapping tori $M_{\widehat{J}}$ and $M_{\widehat{J}'}$ are homeomorphic if and only if there is a matrix $H \in GL(2, \mathbb{Z})$ such that one of the following assertions holds:*

- *$JH = HJ'$,*
- *$J^{-1}H = HJ'$.*

Denote by $\mathcal{C}$ the set of the hyperbolic matrices from $GL(2, \mathbb{Z})$. For $C \in \mathcal{C}$ denote by $Z(\widehat{C})$ the centralizer of $\widehat{C}$ in the group $\{\widehat{J} : J \in GL(2, \mathbb{Z})\}$, that is $Z(\widehat{C}) = \{\widehat{J} : J \in GL(2, \mathbb{Z}), \widehat{C}\widehat{J} = \widehat{J}\widehat{C}\}$.

The following result is due to [23].

Statement 7. *The group $Z(\widehat{C}), C \in \mathcal{C}$ is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z}_2$.*

Set $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $-Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathcal{J} = \mathcal{C} \cup Id \cup (-Id)$. As $\widehat{C}$ and $-\widehat{C}$ belong to $Z(\widehat{C})$ then the next fact follows from statement 7.

Corollary 1. *If $\widehat{J} \in Z(\widehat{C})$ for $C \in \mathcal{C}$ then $J \in \mathcal{J}$. Moreover, C and J have the same forms in the following sense: $C = (-Id)^{j_C}\xi^{k_C}$ and $J = (-Id)^{j_J}\xi^{k_J}$ where $\xi \in \mathcal{C}$, $k_C, k_J \in \mathbb{Z}$, $j_C, j_J \in \{0, 1\}$.*

Recall that we denote by M_τ a mapping torus that is a space derived from $\mathbb{T}^2 \times [0, 1]$ by the identification of points $(z, 1)$ and $(\tau(z), 0)$, where $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism. Thus each point of M_τ is equivalence class $[(z, t)]$, $(z, t) \in \mathbb{T}^2 \times [0, 1]$ with respect to τ. Let us prove that the mapping torus M_τ is homeomorphic to mapping torus $M_{\widehat{J}}$ where $J \in GL(2, \mathbb{Z})$ is a matrix which is defined by the action of the automorphism $\tau_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$.

2.1. Proof of theorem 1

Let us recall that $\mathcal{J}$ is a class of hyperbolic matrices from $GL(2, \mathbb{Z})$ together with Id and $-Id$. Let us prove that the manifold M^3, admitting a diffeomorphism f from the class G, is diffeomorphic to the manifold $M_{\widehat{J}}$, where $J \in \mathcal{J}$.

Proof. We fix a connected component B of nonwandering set $NW(f)$ of diffeomorphism $f \in G$. By lemmas 1 and 3 there is a continuous map $E_f : \mathbb{T}^2 \times [0, 1] \to M^3$ such that maps $E_f|_{\mathbb{T}^2 \times (0,1)} : \mathbb{T}^2 \times (0, 1) \to M^3 \setminus B$, $E_f|_{\mathbb{T}^2 \times \{0\}} : \mathbb{T}^2 \times \{0\} \to B$, $E_f|_{\mathbb{T}^2 \times \{1\}} : \mathbb{T}^2 \times \{1\} \to B$ are homeomorphisms. Set $E_f|_{\mathbb{T}^2 \times \{0\}} = E_{f,0}$ ($E_f|_{\mathbb{T}^2 \times \{1\}} = E_{f,1}$) and $\tau = E_{f,0}^{-1} E_{f,1} : \mathbb{T}^2 \to \mathbb{T}^2$. Then by construction the manifold M_τ is homeomorphic to the manifold M^3 by a homeomorphism $\check{E}_f$, which maps the equivalence class $[(z, t)]$, $(z, t) \in \mathbb{T}^2 \times [0, 1]$ to the point $E_f(z, t)$.

Denote by $J \in GL(2, \mathbb{Z})$ the matrix, which is defined by the action of the automorphism $\tau_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$. Due to statement 4, there is a homeomorphism $\zeta : M_{\widehat{J}} \to M^3$. Thus the smooth manifolds $M_{\widehat{J}}$ and M^3 are homeomorphic by the homeomorphism ζ. By the theorem on smoothing homeomorphisms[9] (see, for example, the corollary in [20]) they are diffeomorphic. Let us show that $J \in \mathcal{J}$.

Let f_0 be a diffeomorphism definded by (*). Set $\psi_0 = \zeta^{-1} f_0 \zeta : M_{\widehat{J}} \to M_{\widehat{J}}$. Since the homeomorphism ψ_0 is topologically conjugated to diffeomorphism f_0 and $f_0(B) = B$, then $\psi_{0*}(N_J) = N_J$, and by statement 2, the action of ψ_{0*} is defined by a hyperbolic matrix $C_0 \in \mathcal{C}$. By statement 5, the matrix J commutes with the matrix C_0, which means that the diffeomorphism $\widehat{J}$ belongs to the centralizer $Z(\widehat{C_0})$ of the hyperbolic automorphism of 2-torus $\widehat{C_0}$. By corollary 1, $J \in \mathcal{J}$. □

Remark 1. Let manifold M^3 admit a diffeomorphism $f \in G$ whose nonwandering set consists of $2m$ periodic components. It follows from lemmas 1 and 3 and the proof of theorem 1 that there is a matrix $J \in \mathcal{J}$ and a homeomorphism $\zeta : M_{\widehat{J}} \to M^3$ such that $\zeta\left(\bigcup_{i=0}^{2m} p_J\left(\mathbb{T}^2 \times \left\{\frac{i}{2m}\right\}\right)\right) = NW(f)$.

3. Locally direct product of a hyperbolic automorphism of the 2-torus and rough diffeomorphism of the circle

Let $MS(\mathbb{S}^1)$ be a class of structurally stable transformations of the circle, which coincides, due to Mayer's results [16], with the class of Morse–Smale diffeomorphisms on $\mathbb{S}^1$. Divide $MS(\mathbb{S}^1)$ into two subclasses $MS_+(\mathbb{S}^1)$ and $MS_-(\mathbb{S}^1)$, consisting of preserving orientation and reverse orientation diffeomorphisms, accordingly. Below we formulate Mayer's results on the topological classification of structurally stable transformations.

Statement 8.

1. *For each diffeomorphism $\varphi \in MS_+(\mathbb{S}^1)$ the set $NW(\varphi)$ consists of $2n$, $n \in \mathbb{N}$ periodic orbits, each of them of period k.*

[9] **Theorem on smoothing homeomorphisms.** Let X, Y be smooth 3-manifolds. Then $Diff^r(X, Y)$ is dense in $Diff^s(X, Y)$ while $0 \leqslant s < r$.

2. *For each diffeomorphism $\varphi \in MS_-(\mathbb{S}^1)$ the set $NW(\varphi)$ consists of $2q$, $q \in \mathbb{N}$ periodic points, two of them are fixed, others have period 2.*

Let $\varphi \in MS_+(\mathbb{S}^1)$. Enumerate the periodic points from $NW(\varphi)$: $p_0, p_1, \dots, p_{2nk-1}, p_{2nk} = p_0$ starting from an arbitrary periodic point p_0, clockwise, then $\varphi(p_0) = p_{2nl}$, where l is an integer such that for $k = 1$, $l = 0$ while for $k > 1$, $l \in \{1, \dots, k-1\}$ and (k, l) are coprime[10]. Notice that number l does not depend on the choice of the point p_0.

For $\varphi \in MS_-(\mathbb{S}^1)$ we set $\nu = -1$; $\nu = 0$; $\nu = +1$ if its fixed points are sources; sink and source; sinks, accordingly. Notice that $\nu = 0$ if q is odd and $\nu = \pm 1$ if q is even.

Statement 9.

1. *Two diffeomorphisms $\varphi; \varphi' \in MS_+(\mathbb{S}^1)$ with parameters $n, k, l; n', k', l'$ are topologically conjugated if and only if $n = n'$, $k = k'$ and at least one of the following assertions holds:*
   - *$l = l'$ (herewith, if $l \neq 0$ then the conjugating homeomorphism is preserving orientation),*
   - *$l = k' - l'$ (herewith, the conjugating homeomorphism is reversing orientation).*
2. *Two diffeomorphisms $\varphi; \varphi' \in MS_-(\mathbb{S}^1)$ with parameters q, ν; q', ν' are topologically conjugated if and only if $q = q'$ and $\nu = \nu'$.*

For $n, k \in \mathbb{N}$ and integer l such that for $k = 1$, $l = 0$ while for $k > 1$, $l \in \{1, \dots, k-1\}$, let us construct a standard representative φ_+ in $MS_+(\mathbb{S}^1)$ with parameters n, k, l. For $q \in \mathbb{N}$, $\nu \in \{-1, 0, +1\}$ let us construct a standard representative φ_- in $MS_-(\mathbb{S}^1)$ with parameter q.

Let us introduce the following maps:

$\psi_m : \mathbb{R} \to \mathbb{R}$ is the time-one map of the flow generated by $\dot{r} = \sin(2\pi m r)$ for $m \in \mathbb{N}$;
$\chi_{k,l} : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism given by the formula $\chi_{k,l}(r) = r - \frac{l}{k}$;
$\chi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism given by the formula $\chi(r) = -r$;
$\tilde{\varphi}_{n,k,l} = \psi_{n \cdot k} \chi_{k,l} : \mathbb{R} \to \mathbb{R}$;
$\tilde{\varphi}_{q,0} = \psi_q \chi : \mathbb{R} \to \mathbb{R}$ for odd q;
$\tilde{\varphi}_{q,+1} = \psi_q \chi : \mathbb{R} \to \mathbb{R}$ and $\tilde{\varphi}_{q,-1} = \tilde{\varphi}^{-1}_{q,+1} : \mathbb{R} \to \mathbb{R}$ for even q.
Set $\tilde{\Pi}_+ = \{\tilde{\varphi}_+ = \tilde{\varphi}_{n,k,l}\}$ and $\tilde{\Pi}_- = \{\tilde{\varphi}_- = \tilde{\varphi}_{q,\nu}\}$. It was verified directly that $\tilde{\varphi}_\sigma(r + \mu) = \tilde{\varphi}_\sigma(r)$ for $\sigma \in \{+, -\}$ and $\mu \in \mathbb{Z}$. Hence the following diffeomorphisms are well defined: $\varphi_\sigma = \pi \tilde{\varphi}_\sigma \pi^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$. Set $\Pi_+ = \{\varphi_+\}$, $\Pi_- = \{\varphi_-\}$ and $\Pi = \Pi_+ \cup \Pi_-$.

Denote by $\tilde{\phi}_\sigma : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ the product of the diffeomorphism $\tilde{\varphi}_\sigma \in \tilde{\Pi}_\sigma$ and automorphism $\widehat{C}$, $C \in \mathcal{C}$ that is $\tilde{\phi}_\sigma(z, r) = (\widehat{C}(z), \tilde{\varphi}_\sigma(r))$.

Statement 10. *The diffeomorphism $\tilde{\phi}_\sigma$ can be projected to diffeomorphism $\phi_\sigma : M_{\hat{J}} \to M_{\hat{J}}$ as $\phi_\sigma = p_{_J} \tilde{\phi}_\sigma p_{_J}^{-1}$ if and only if*

- *$CJ = JC$ for $\sigma = +$;*
- *$J \in \{Id, -Id\}$ for $\sigma = -$.*

Thus we get the descriptions of the models.

Let $J_+ \in \mathcal{J}$ and $C_+ \in \mathcal{C}$ such that $C_+ J_+ = J_+ C_+$. Let $J_- \in \{Id, -Id\}$ and $C_- \in \mathcal{C}$. Set $\tilde{\phi}_\sigma(z, r) = (\widehat{C}_\sigma(z), \tilde{\varphi}_\sigma(r))$. It is immediately verified that $\tilde{\phi}_\sigma \gamma_\sigma = \gamma_\sigma \tilde{\phi}_\sigma$ where $\gamma_\sigma(z, r) = (J_\sigma(z), r - 1)$ is the generator of the group $\Gamma_\sigma = \{\gamma^i_\sigma, i \in \mathbb{Z}\}$. Then the following concept is well defined.

[10] Indeed, instead of number l, Mayer used number r_1, which he called the *ordering number*, such that $l \cdot r_1 \equiv 1 (\mathrm{mod}\ k)$

Definition 1. We say that diffeomorphism $\phi_\sigma : M_{\widehat{J}_\sigma} \to M_{\widehat{J}_\sigma}, \sigma \in \{+, -\}$ is a locally direct product of $\widehat{C}_\sigma$ and φ_σ, if $\phi_\sigma = p_{J_\sigma} \tilde{\phi}_\sigma p_{J_\sigma}^{-1}$ and write $\phi_\sigma = \widehat{C}_\sigma \otimes \varphi_\sigma$.

Let us recall (see, for example, [3] and [13]) that a diffeomorphism g on M^3 is called *partially hyperbolic* if there exists a continuous splitting of the tangent bundle $T_{M^3} = E^s \oplus E^c \oplus E^u$ invariant under the derivative Dg, where $\dim E^s = \dim E^c = \dim E^u = 1$ and the strong expansion of the unstable bundle E^u and the strong contraction of the stable bundle E^s dominate any expansion or contraction on the center E^c[11]. Herewith g is *dynamically coherent* if there are g-invariant foliations tangent to $E^{cs} = E^s \oplus E^c$, $E^{cu} = E^c \oplus E^u$, (and consequently there is g-invariant foliation tangent to E^c).

Notice that if in the construction of $\phi_\sigma \in \Phi_\sigma$ above we can use instead $\dot{r} = \sin(2\pi m r)$ vector field $\dot{r} = ln(\mu) \cdot \sin(2\pi m r)$, where $\mu < |\lambda|$ and $|\lambda|, \frac{1}{|\lambda|}$ are absolute values of eigenvalues of C_σ. Then the constructed model will be dynamically coherent. Thus, by theorem 2, we get the following result.

Corollary 2. *Each diffeomorphism ϕ from the class Φ is topologically conjugate to a dynamically coherent diffeomorphism.*

4. Topological classification of model diffeomorphisms

This section is devoted to the proof of theorem 2.

In section 1 we described a construction of some model diffeomorphism $\phi_\sigma : M_{\widehat{J}} \to M_{\widehat{J}}$, $\sigma \in \{+, -\}$ from the class Φ_σ which is a locally direct product $\phi_\sigma = \widehat{C}_\sigma \otimes \varphi_\sigma$ of a hyperbolic automorphism $\widehat{C}_\sigma$ on $\mathbb{T}^2$ and a model structurally stable diffeomorphism $\varphi_\sigma \in \Pi_\sigma$ on $\mathbb{S}^1$. Let us prove the auxiliary facts.

4.1. Proof of statement 10

Proof. Since $M_{\widehat{J}} = (\mathbb{T}^2 \times \mathbb{R})/\Gamma$ and Γ is a cyclic group with generator $\gamma(z, r) = (\widehat{J}(z), r - 1)$ then either $\tilde{\phi}_\sigma \gamma = \gamma \tilde{\phi}_\sigma$ or $\tilde{\phi}_\sigma \gamma^{-1} = \gamma \tilde{\phi}_\sigma$ is a necessary and sufficient condition to project the diffeomorphism $\tilde{\phi}_\sigma$ to diffeomorphism $\phi_\sigma : M_{\widehat{J}} \to M_{\widehat{J}}$ as $\phi_\sigma = p_J \tilde{\phi}_\sigma p_J^{-1}$ (see, for example, [10]). From what follows, $CJ = JC$ for $\sigma = +$ and $CJ^{-1} = JC$ for $\sigma = -$. As $CJ^{-1} = JC$ implies $C^2 J = JC^2$ then $\widehat{J} \in Z(\widehat{C}^2)$, hence, due to corollary 1, $J \in \{Id, -Id\}$ for $\sigma = -$. □

Lemma 4. *If two diffeomorphisms $\phi_\sigma = \widehat{C}_\sigma \otimes \varphi_\sigma, \phi'_{\sigma'} = \widehat{C}'_{\sigma'} \otimes \varphi'_{\sigma'} \in \Phi$ are topologically conjugated then:*

(1) *there exists a matrix $H \in GL(2, \mathbb{Z})$ such that $C_\sigma H = HC'_{\sigma'}$;*

(2) *$\varphi_\sigma, \varphi'_{\sigma'}$ are topologically conjugated.*

Proof. If two diffeomorphisms $\phi_\sigma = \widehat{C}_\sigma \otimes \varphi_\sigma, \phi'_{\sigma'} = \widehat{C}'_{\sigma'} \otimes \varphi'_{\sigma'} \in \Phi$ are topologically conjugated by means of a homeomorphism $h : M_{\widehat{J}_\sigma} \to M_{\widehat{J}'_{\sigma'}}$, then, due to statement 5, h induces the isomorphism $h_* : \pi_1(M_{\widehat{J}_\sigma}) \to \pi_1(M_{\widehat{J}'_{\sigma'}})$ such that $h_*(N_{J_\sigma}) = N_{J'_{\sigma'}}$ and $h_*|_{N_{J_\sigma}}$ is defined by a matrix $H \in GL(2, \mathbb{Z})$ such that $C_\sigma H = HC'_{\sigma'}$.

[11] More exactly a diffeomorphism f is *partially hyperbolic* if there is $N \in \mathbb{N}$ and a Dg-invariant continuous splitting $TM^3 = E^s \oplus E^c \oplus E^u$ into one-dimensional subbundles such that $\|Dg^N|_{E^s_x}\| < \|Dg^N|_{E^c_x}\| < \|Dg^N|_{E^u_x}\|$ and $\|Dg^N|_{E^s_x}\| < 1 < \|Dg^N|_{E^u_x}\|$ for every $x \in M^3$.

Let $z_0 \in \mathbb{T}^2$ be a fixed point of $\widehat{C}_\sigma$. Set

$$\tilde{S}_{z_0} = \{z_0\} \times \mathbb{R}$$

and

$$S_{z_0} = p_{_{J_\sigma}}(\tilde{S}_{z_0}).$$

By the construction, S_{z_0} is a circle and the map φ_σ coincides with $\pi_{J_\sigma}\phi_\sigma(\pi_{J_\sigma}|_{S_{z_0}})^{-1}$. The situation is the same for $\phi'_{\sigma'}$. As h conjugates ϕ_σ with $\phi'_{\sigma'}$ then $h(S_{z_0}) = S_{z'_0}$ where $z'_0 \in \mathbb{T}^2$ is a fixed point of $\widehat{C}_\sigma$. Thus the diffeomorphisms $\varphi_\sigma, \varphi'_{\sigma'}$ are topologically conjugated by means of a homeomorphism

$$\eta_{_h} = \pi_{J'_{\sigma'}} h(\pi_{J_\sigma}|_{S_{z_0}})^{-1} : \mathbb{S}^1 \to \mathbb{S}^1.$$

□

4.2. Proof of theorem 2

By the construction each difffeomorphism $\phi_+ \in \Phi_+$ is uniquely defined by parameters $\{J_+, C_+, n, k, l\}$ and each difffeomorphism $\phi_- \in \Phi_-$ is uniquely defined by parameters $\{J_-, C_-, q, \nu\}$.

1. Two diffeomorphisms $\phi_+; \phi'_+ \in \Phi_+$ with parameters $\{J_+, C_+, n, k, l\}; \{J'_+, C'_+, n', k', l'\}$ are topologically conjugated if and only if $n = n', k = k'$, there exists a matrix $H \in GL(2, \mathbb{Z})$ such that $C_+H = HC'_+$ and at least one of the following assertions holds:

- $J_+H = HJ'_+$ and $l = l'$,
- $J_+^{-1}H = HJ'_+$ and either $l = l' = 0$ or $l = k' - l'$.

2. Two diffeomorphisms $\phi_-; \phi'_- \in \Phi_-$ with parameters $\{J_-, C_-, q, \nu\}; \{J'_-, C'_-, q', \nu'\}$ are topologically conjugated if and only if $q = q', \nu = \nu', J_- = J'_-$ and there exists a matrix $H \in GL(2, \mathbb{Z})$ such that $C_-H = HC'_-$.

Proof. Necessity. Let the diffeomorphisms $\phi_\sigma, \phi'_\sigma$ from the class Φ_σ be topologically conjugated by a homeomorphism $h : M_{\widehat{J}_\sigma} \to M_{\widehat{J}'_\sigma}$. Then h induces an isomorphism $h_* : \pi_1(M_{\widehat{J}_\sigma}) \to \pi_1(M_{\widehat{J}'_\sigma})$ such that $h_*(N_{J_\sigma}) = N_{J'_\sigma}$ and according to the statement 5, $h_*|_{N_{J_\sigma}}$ is defined by the matrix $H \in GL(2, \mathbb{Z})$ such that $h_*(0, b) = (0, H(b))$. From the condition of topological conjugacy, it follows that $HC_\sigma = C'_\sigma H$ and, by the statement 5, either $h_*(1, 0) = (1, \beta)$ and $HJ_\sigma = J'_\sigma H$, or $h_*(1, 0) = (-1, \beta)$ and $HJ_\sigma^{-1} = J'_\sigma H$.

It follows from lemma 4 that the diffeomorphisms φ_σ and φ'_σ are topologically conjugate by a homeomorphism $\eta_{_h} = \pi_{J'_\sigma} h(\pi_{J_\sigma}|_{S_{z_0}})^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$. Herewith $\eta_{_h}$ preserves orientation if $HJ_\sigma = J'_\sigma H$ and $\eta_{_h}$ reverses orientation if $HJ_\sigma^{-1} = J'_\sigma H$. It follows from statement 9 that at least one of the following assertions holds:

- $J_\sigma H = HJ'_\sigma; \sigma = +; n = n', k = k'; l = l'$,
- $J_\sigma^{-1}H = HJ'_\sigma; \sigma = +; n = n', k = k'$; either $l = l' = 0$ or $l = k' - l'$,
- either $J_\sigma H = HJ'_{\sigma'}$ or $J^{-1}H = HJ'; \sigma = -; q = q'$ and $\nu = \nu'$.

The necessity of the conditions of the theorem is proved in both cases 1 and 2.

Sufficiency. Let diffeomorphisms $\phi_\sigma, \phi'_\sigma \in \Phi_\sigma$ be such that the algebraic conditions of the theorem 2 hold. Let us construct a homeomorphism $h : M_{\widehat{J}_\sigma} \to M_{\widehat{J}'_\sigma}$ conjugating ϕ_σ with ϕ'_σ.

It follows from the conditions that there is a matrix $H \in GL(2, \mathbb{Z})$ such that $C_\sigma H = HC'_\sigma$ and either (i) $J_\sigma H = HJ'_\sigma$ or (ii) $J_\sigma^{-1}H = HJ'_\sigma$. In the case (i), let us define diffeomorphism $\tilde{h} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ such that $\tilde{h}(z, r) = (\widehat{H}(z), r)$. Then $\gamma_+\tilde{h} = \tilde{h}\gamma_+$. In the case (ii) let us define diffeomorphism $\tilde{h}(z, r) = (\widehat{H}(z), -r)$. In both cases the diffeomorphism $\tilde{h}$ conjugates the diffeomorphisms $\tilde{\phi}_\sigma, \tilde{\phi}'_\sigma$ and, therefore, it is projected onto homeomorphism $h = p_{J'_\sigma}\tilde{h}p_{J_\sigma}^{-1}$ which conjugates the diffeomorphisms ϕ_σ and ϕ'_σ. $\square$

5. Invariants of ambient Ω-conjugacy for diffeomorphisms from the class *G*

This section is devoted to a proof of theorem 3.

Let us prove that any diffeomorphism from the class G is ambient Ω-conjugated to a diffeomorphism from the class Φ.

Proof. Let $f \in G$ be a diffeomorphism with $2m$ connected components in $NW(f)$. Due to remark 1 there is a matrix $J \in \mathcal{J}$ and a homeomorphism $\zeta : M_{\widehat{J}} \to M^3$ such that $\zeta\left(\bigcup_{i=0}^{2m} p_J\left(\mathbb{T}^2 \times \left\{\frac{i}{2m}\right\}\right)\right) = NW(f)$. Set $g = \zeta^{-1}f\zeta : M_{\widehat{J}} \to M_{\widehat{J}}$ and $\mathcal{T} = \left\{\frac{i}{2m},\ i \in \mathbb{Z}\right\}$. Since the homeomorphism g is topologically conjugated with the diffeomorphism f and $NW(g) = p_J(\mathbb{T}^2 \times \mathcal{T})$, then $g(p_J(\mathbb{T}^2 \times \mathcal{T})) = p_J(\mathbb{T}^2 \times \mathcal{T})$. Then $g_*(N_J) = N_J$. Denote by C the matrix, which is defined by an isomorphism $g_*|_{N_J}$. According to statement 2, $C \in \mathcal{C}$. By statement 5 and theorem 1, $CJ = JC$ or $CJ^{-1} = JC$ and the diffeomorphism g is lifted up to a homeomorphism $\tilde{g} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$.

The homeomorphism g induces a map η_g on the set $N_g = \pi_J(NW(g))$ by the formula $\eta_g = \pi_J g \pi_J^{-1}$. Then there is a diffeomorphism $\varphi \in \Pi$ such that $NW(\varphi) = N_g$, $\varphi|_{NW(\varphi)} = g|_{N_g}$ and $\pi_J(\mathcal{A})$ is the set of sinks for φ. The diffeomorphism φ has parameters n, k, l if it preserves orientation and has parameter q, ν if it reverses orientation. Set $\phi = \widehat{C} \otimes \varphi$.

Without loss of generality, we assume that the lift $\tilde{g}$ has been chosen so that $\tilde{g}(z, \tau) = \left(\tilde{g}_\tau(z), \tau - \frac{l}{k}\right)$ $(\tilde{g}(z, \tau) = (\tilde{g}_\tau(z), -\tau))$ for $\tau \in \mathcal{T}$. Notice that the action of the isomorphism $\tilde{g}_{\tau*} : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ is determined by the matrix C.

We divide the proof into two cases: (I) φ preserves orientation; (II) φ reverses orientation.

In case (I), let us consider two subcases (Ia) $k = 1$; (Ib) $k > 1$.

In the subcase (Ia) we will construct a homeomorphism $X : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ such that $X\gamma = \gamma X$ and the homeomorphism $\tilde{\psi} = X\tilde{g}X^{-1}$ will coincide on the set $\mathbb{T}^2 \times \mathcal{T}$ with the diffeomorphism $\tilde{\phi}_+$, completing the proof of the theorem.

We define the set $\mathcal{T}^0 \subset \mathbb{R}$ by the formula $\mathcal{T}^0 = \left\{\frac{i}{2n},\ i = 0, \ldots, 2n-1\right\}$. Since for any $\tau \in \mathcal{T}^0$ the isomorphism $\tilde{g}_{\tau*} : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ is defined by the hyperbolic matrix C, then by statement 3 there is an isotopic to the identity homeomorphism $h_\tau : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\widehat{C} = h_\tau \tilde{g}_\tau h_\tau^{-1}$. Let $h_{\tau,s},\ s \in [0, 1]$ be an isotopy such that $h_{\tau,0} = h_\tau$ and $h_{\tau,1} = id$.

For $t \in \left[-\frac{1}{6n}, 1 - \frac{1}{6n}\right]$ we define a homeomorphism $x_t : \mathbb{T}^2 \to \mathbb{T}^2$ by the formula $x_t = h_{\tau, 6n|t-\tau|},\ |t - \tau| \leqslant \frac{1}{6n},\ \tau \in \mathcal{T}^0$ and $x_t = id$ for all other t. Let us define a homeomorphism $x : \mathbb{T}^2 \times \left[-\frac{1}{6n}, 1 - \frac{1}{6n}\right] \to \mathbb{T}^2 \times \left[-\frac{1}{6n}, 1 - \frac{1}{6n}\right]$ by the formula $x(z,\ t) = (x_t(z),\ t)$.

Notice that $x(z,\tau)=(h_\tau(z),\tau)$. For $r\in\mathbb{R}$ we denote by $m(r)\in\mathbb{Z}$ an integer number, such that $(r-m(r))\in\left[-\frac{1}{6n},1-\frac{1}{6n}\right)$. Let $X(z,r)=\gamma^{-m(r)}x\gamma^{m(r)}(z,r)$ for $(z,r)\in\mathbb{T}^2\times\mathbb{R}$.

Any point of set $\mathcal{T}$ has the form $\tau+m$, where $\tau\in\mathcal{T}^0,\ m\in\mathbb{Z}$. By construction $\tilde\phi_+(z,\tau+m)=(\widehat{C}(z),\mathcal{T}+\mathrm{m})$. Therefore, the verification of equality $\tilde\psi(z,\tau+m)=(\widehat{C}(z),\tau+m)$ will complete the proof in the case (Ia).

Indeed, $\tilde\psi(z,\tau+m)=X\tilde gX^{-1}(z,\tau+m)=\gamma^{-m}x\gamma^m\tilde g\gamma^{-m}x^{-1}\gamma^m(z,\tau+m)=\gamma^{-m}x\tilde gx^{-1}\gamma^m(z,\tau+m)$ $=\gamma^{-m}x\tilde gx^{-1}(\widehat{J}^m(z),\tau)=\gamma^{-m}x\tilde g(h_\tau^{-1}\widehat{J}^m(z),\tau)=\gamma^{-m}x(g_\tau h_\tau^{-1}\widehat{J}^m(z),\tau)=\gamma^{-m}(h_\tau g_\tau h_\tau^{-1}\widehat{J}^m(z),\tau)=(\widehat{J}^{-m}h_\tau g_\tau h_\tau^{-1}\widehat{J}^m(z),\tau+m)=(\widehat{J}^{-m}\widehat{C}\widehat{J}^m(z),\tau+m)=(\widehat{C}(z),\tau+m)$

In the case (Ib), using the results of the case (Ia), we can assume that the homeomorphism g^k coincides with the diffeomorphism ϕ_+ given by parameters $\{J,C^k,nk,0\}$ on their common nonwandering set $p_J(\mathcal{T})$.

For $j=0,\dots,k-1$ we define the sets $\mathcal{T}_j,\ \mathcal{T}_j^0\subset\mathbb{R}$ by the formulas $\mathcal{T}_j=\left\{\frac{i}{2nk}-\frac{jl}{k}+\kappa,\ i=0,\dots,2n-1,\ \kappa\in\mathbb{Z}\right\}$, $\mathcal{T}_j^0=\left\{\frac{i}{2nk}-\frac{jl}{k},\ i=0,\dots,2n-1\right\}$. Notice, that $\mathcal{T}=\mathcal{T}_0\cup\dots\cup\mathcal{T}_{k-1}$. Let $U(\mathcal{T}_j)=\bigcup_{\tau\in\mathcal{T}_j}\left[\tau-\frac{1}{6nk},\tau+\frac{1}{6nk}\right]$, $U(\mathcal{T}_j^0)=\bigcup_{\tau\in\mathcal{T}_j^0}\left[\tau-\frac{1}{6nk},\tau+\frac{1}{6nk}\right]$. Set $g_0=g,\ \tilde g_0=\tilde g$ and successively construct homeomorphisms $Y_0,\tilde g_1,Y_1,\tilde g_2,\dots,Y_{k-2},\tilde g_{k-1}:\mathbb{T}^2\times\mathbb{R}\to\mathbb{T}^2\times\mathbb{R}$ with the following properties for $j=1,\dots,k-1$:

(1) $\tilde g_j=Y_{j-1}\tilde g_{j-1}Y_{j-1}^{-1}$, where Y_{j-1} commutes with γ and Y_{j-1} is the identity out of $U(\mathcal{T}_j)$;

(2) for any $\tau\in\mathcal{T}$ on the set $\mathbb{T}^2\times\{\tau\}$ the homeomorphism $\tilde g_j$ has the form $\tilde g_j(z,\tau)=\left(\tilde g_{j,\tau}(z),\tau-\frac{l}{k}\right)$, where $\tilde g_{j,\tau}=\widehat{C}$ for $\tau\in\mathcal{T}_{j-1}$ and $\tilde g_{j,\tau}=\tilde g_{j-1,\tau}\tilde g_{j-1,\tau+\frac{l}{k}}\widehat{C}^{-1}$ for $\tau\in\mathcal{T}_j$.

From the properties (1) and (2) it follows that $\tilde g_{k-1,\tau}=\widehat{C}$ for $\tau\in(\mathcal{T}_0\cup\mathcal{T}_1\cup\dots\cup\mathcal{T}_{k-2})$ and $\tilde g_{k-1,\tau}=\tilde g_{k-2,\tau}\tilde g_{k-2,\tau+\frac{l}{k}}\widehat{C}^{-1}=\tilde g_{0,\tau}\tilde g_{k-3,\tau+\frac{l}{k}}\tilde g_{k-3,\tau+\frac{2l}{k}}\widehat{C}^{-2}=\dots=\tilde g_{0,\tau}\tilde g_{0,\tau+\frac{l}{k}}\cdots\tilde g_{0,\tau+\frac{(k-2)l}{k}}\tilde g_{0,\tau+\frac{(k-1)l}{k}}\widehat{C}^{-(k-1)}$ for $\tau\in\mathcal{T}_{k-1}$. Since the homeomorphism g_0^k coincides with the diffeomorphism from Φ_+ defined by parameters $\{C^k,\ nk,\ 1,\ 0\}$ on their common nonwandering set $p_J(\mathcal{T})$ then $\tilde g_{0,\tau}\tilde g_{0,\tau+\frac{l}{k}}\cdots\tilde g_{0,\tau+\frac{(k-2)l}{k}}\tilde g_{0,\tau+\frac{(k-1)l}{k}}=\widehat{C}^k$ and, therefore, $\tilde g_{k-1,\tau}=\widehat{C}$ for $\tau\in\mathcal{T}_{k-1}$. Since $\tilde\phi_+(z,\tau)=(\widehat{C}(z),\tau-\frac{l}{k})$ for any $\tau\in\mathcal{T}$ then the construction of homeomorphisms with the described properties will complete the proof, since the homeomorphism g_{k-1} coincides with the diffeomorphism ϕ_+ on their common nonwandering set $p_J(\mathcal{T})$ and is topologically conjugated with the diffeomorphism g.

Let us show how to construct a homeomorphism Y_{j-1} for $j=1,\dots,k-1$, assuming that the homeomorphism $\tilde g_{j-1}$ is already constructed.

Let $\tau\in\mathcal{T}_{j-1}^0$. Since the homeomorphism $\tilde g_{j-1,\tau}$ is isotopic to the diffeomorphism $\widehat{C}$, then the homeomorphism $\widehat{C}\tilde g_{j-1,\tau}^{-1}$ is isotopic to the identity map. Let $h_{j-1,\tau,s}:\mathbb{T}^2\to\mathbb{T}^2$, $s\in[0,1]$ be an isotopy, such that $h_{j-1,\tau,0}=\widehat{C}\tilde g_{j-1,\tau}^{-1}$ and $h_{j-1,\tau,1}=id$. For $t\in\left[-\frac{jl}{k}-\frac{1}{6nk},1-\frac{jl}{k}+\frac{1}{6nk}\right]$ we define a homeomorphism $y_{j-1,t}:\mathbb{T}^2\to\mathbb{T}^2$ by the formula $y_{j-1,t}=h_{j-1,i,6nk|t+\frac{l}{k}-\tau|}$, $|t+\frac{l}{k}-\tau|\leqslant\frac{1}{6nk}$ and $y_{j-1,\ t}=id$ for all other t. Let us define a homeomorphism $y_{j-1}:\mathbb{T}^2\times\left[-\frac{jl}{k}-\frac{1}{6nk},1-\frac{jl}{k}+\frac{1}{6nk}\right]\to\mathbb{T}^2\times\left[-\frac{jl}{k}-\frac{1}{6nk},1-\frac{jl}{k}+\frac{1}{6nk}\right]$ by the formula $y_{j-1}(z,t)=(y_{j-1,t}(z),t)$. Notice that $y_{j-1}\left(z,\tau-\frac{l}{k}\right)=\left(\widehat{C}\,\tilde g^{-1}_{j-1,\tau}(z),\tau-\frac{l}{k}\right)$.

For $r\in\mathbb{R}$ we denote by $m(r)\in\mathbb{Z}$ an integer number, such that $(r-m(r))\in\left[-\frac{jl}{k}-\frac{1}{6nk},\right.$ $\left.1-\frac{jl}{k}+\frac{1}{6nk}\right)$. For $(z,r)\in\mathbb{T}^2\times\mathbb{R}$ let $Y_{j-1}(z,r)=\gamma^{-m(r)}y_{j-1}\gamma^{m(r)}(z,r)$. Set $\tilde g_j=Y_{j-1}\tilde g_{j-1}Y^{-1}_{j-1}$.

Since any point of the set $\mathcal{T}_{j-1}$ ($\mathcal{T}_j$) has the form $\tau+m$ $\left(\tau+m-\frac{l}{k}\right)$, where $\tau\in\mathcal{T}^0_{j-1}$, $m\in\mathbb{Z}$, then the checking equalities $\tilde g_j(z,\tau+m)=\left(\widehat{C}(z),\tau+m-\frac{l}{k}\right)$ and $\tilde g_j\left(z,\tau+m-\frac{l}{k}\right)=\left(\tilde g_{j-1,\tau+m-\frac{l}{k}}\tilde g_{j-1,\tau+m}\widehat{C}^{-1}(z),\tau+m-\frac{2l}{k}\right)$ will complete the proof in the case (b).

Indeed, $\tilde g_j(z,\tau+m)=Y_{j-1}\tilde g_{j-1}Y^{-1}_{j-1}(z,\tau+m)=Y_{j-1}\tilde g_{j-1}(z,\tau+m)=\gamma^{-m}y_{j-1}\gamma^m\tilde g_{j-1}(z,\tau+m)$ $=\gamma^{-m}y_{j-1}\tilde g_{j-1}\gamma^m(z,\tau+m)=\gamma^{-m}y_{j-1}\tilde g_{j-1}(J^m(z),\tau)=\gamma^{-m}y_{j-1}\left(\tilde g_{j-1,\tau}J^m(z),\tau-\frac{l}{k}\right)=\gamma^{-m}\left(\widehat{C}\right.$ $\left.\tilde g^{-1}_{j-1,\tau}\tilde g_{j-1,\tau}J^m(z),\tau-\frac{l}{k}\right)=\left(\widehat{J}^{-m}\widehat{C}\widehat{J}^m(z),\tau+m-\frac{l}{k}\right)=\left(\widehat{C}(z),\tau+m-\frac{l}{k}\right)$.

Further, $\tilde g_j\left(z,\tau+m-\frac{l}{k}\right)=Y_{j-1}\tilde g_{j-1}Y^{-1}_{j-1}\left(z,\tau+m-\frac{l}{k}\right)=\tilde g_{j-1}Y^{-1}_{j-1}\left(z,\tau+m-\frac{l}{k}\right)=\tilde g_{j-1}\gamma^{-m}$ $y^{-1}_{j-1}\gamma^m\left(z,\tau+m-\frac{l}{k}\right)=\tilde g_{j-1}\gamma^{-m}y^{-1}_{j-1}\left(\widehat{J}^m(z),\tau-\frac{l}{k}\right)=\tilde g_{j-1}\gamma^{-m}\left(\tilde g^{-1}_{j-1,\tau}\widehat{C}^{-1}\widehat{J}^m(z),\tau-\frac{l}{k}\right)=\tilde g_{j-1}$ $\left(J^{-m}\tilde g^{-1}_{j-1,\tau}\widehat{C}^{-1}\widehat{J}^m(z),\tau+m-\frac{l}{k}\right)=\left(\tilde g_{j-1,\tau+m-\frac{l}{k}}\widehat{J}^{-m}\tilde g^{-1}_{j-1,\tau}\widehat{C}^{-1}\widehat{J}^m(z),\tau+m-\frac{2l}{k}\right)=\left(\tilde g_{j-1,\tau+m-\frac{l}{k}}\right.$ $\left.\widehat{J}^{-m}\tilde g^{-1}_{j-1,\tau}\widehat{J}^m\widehat{C}^{-1}(z),\tau+m-\frac{2l}{k}\right)=\left(\tilde g_{j-1,\tau+m-\frac{l}{k}}\widehat{J}^{-m}\widehat{J}^m\tilde g^{-1}_{j-1,\tau+m}\widehat{C}^{-1}(z),\tau+m-\frac{2l}{k}\right)=\left(\tilde g_{j-1,\tau+m-\frac{l}{k}}\right.$ $\left.\tilde g_{j-1,\tau+m}\widehat{C}^{-1}(z),\tau+m-\frac{2l}{k}\right)$.

In the case (II), due to theorem 1, $J=\pm Id$. Using the results of the case (Ia), we can assume that the homeomorphism g^2 coincides with the diffeomorphism ϕ_+ given by parameters $\{J, C^2, q, 0\}$ on their common nonwandering set $p_J(\mathcal{T})$.

Set $\mathcal{T}^0=\left\{\frac{i}{2q},\ i=0,\dots,2q-1\right\}$ and $\mathcal{T}=\left\{\frac{i}{2q},\ i\in\mathbb{Z}\right\}$. Let us construct homeomorphism $Y:\mathbb{T}^2\times\mathbb{R}\to\mathbb{T}^2\times\mathbb{R}$ such that $Y\gamma=\gamma Y$ and $\tilde\psi=Y\tilde gY^{-1}_{j-1}$ coincides on the set $\mathbb{T}^2\times\mathcal{T}$ with the diffeomorphism $\tilde\phi_-$.

Let $\tau\in\mathcal{T}^0$. Since the homeomorphism $\tilde g_\tau$ is isotopic to the diffeomorphism $\widehat{C}$, then, due to statement 3, there is an isotopic to identity homeomorphism $h_\tau:\mathbb{T}^2\to\mathbb{T}^2$ such that $\widehat{C}=h_\tau\tilde g_\tau h_\tau^{-1}$. Let $h_{\tau,s}$, $s\in[0,1]$ be an isotopy such that $h_{\tau,0}=h_\tau$ and $h_{\tau,1}=id$.

For $\tau\in\mathcal{T}^0$ and $t\in\left[-\frac{1}{6q},1-\frac{1}{6q}\right]$ we define a homeomorphism $y_t:\mathbb{T}^2\to\mathbb{T}^2$ by the formula $y_t=h_{i,6q|t-\tau|}$, $|t-\tau|\leqslant\frac{1}{6q}$ and $y_t=id$ for all other t. Let us define a homeomorphism $y:\mathbb{T}^2\times\left[-\frac{1}{6q},1-\frac{1}{6q}\right]\to\mathbb{T}^2\times\left[-\frac{1}{6q},1-\frac{1}{6q}\right]$ by the formula $y(z,\,t)=(y_t(z),\,t)$. Notice, that $y(z,\tau)=(h_\tau,\tau)$.

For $r\in\mathbb{R}$ we denote by $m(r)\in\mathbb{Z}$ an integer number, such that $(r-m(r))\in\left[-\frac{1}{6q},1-\frac{1}{6q}\right)$. For $(z,r)\in\mathbb{T}^2\times\mathbb{R}$ let $Y(z,r)=\gamma^{-m(r)}y\gamma^{m(r)}(z,r)$.

Any point of set $\mathcal{T}$ has the form $\tau+m$, where $\tau\in\mathcal{T}^0$, $m\in\mathbb{Z}$. By construction $\tilde\phi_-(z,\tau+m)=(\widehat{C}(z),-\tau-m)$. Therefore, the verification of equality $\tilde\psi(z,\tau+m)=(\widehat{C}(z),-\tau-m)$ will complete the proof in the case (II).

Indeed, $\tilde\psi(z,\tau+m)=Y\tilde g Y^{-1}(z,\tau+m)=Y\tilde g\gamma^{-m}y^{-1}\gamma^m(z,\tau+m)=Y\tilde g\gamma^{-m}y^{-1}(J^m(z),\tau)=Y\tilde g\gamma^{-m}(h_\tau^{-1}J^m(z),\tau)=Y\tilde g(J^{-m}h_\tau^{-1}J^m(z),\tau+m)=Y(\tilde g_\tau J^{-m}h_\tau J^m(z),-\tau-m)=\gamma^{m+1}y\gamma^{-m-1}(\tilde g_\tau J^{-m}h_\tau^{-1}J^m(z),-\tau-m)=\gamma^{m+1}y(J^{-m-1}\tilde g_\tau J^{-m}h_\tau^{-1}J^m(z),-\tau)=\gamma^{m+1}(h_{1-\tau}J^{-m-1}\tilde g_\tau J^{-m}h_\tau^{-1}J^m(z),-\tau)=(J^{m+1}h_{1-\tau}J^{-m-1}\tilde g_\tau J^{-m}h_\tau^{-1}J^m(z),-\tau-m)=(J^mJh_{1-\tau}J^{-1}\tilde g_\tau h_\tau^{-1}J^m(z),-\tau-m)=(J^mh_\tau\tilde g_\tau h_\tau^{-1}J^m(z),-\tau-m)=(J^m\widehat CJ^m(z),-\tau-m)=(\widehat C(z),-\tau-m)$ □

6. Asymptotic behaviour of two-dimensional invariant manifolds of nonwandering points of structurally stable diffeomorphism from *G*

Let $f\in G$ be a structurally stable diffeomorphism. By theorem 3, f is ambient Ω-conjugated with some diffeomorphism $\phi: M_{\hat J}\to M_{\hat J}$ from the class Φ by means of a homeomorphism $h: M^3\to M_{\hat J}, J\in\mathcal J$. Set $\psi=hfh^{-1}: M_{\hat J}\to M_{\hat J}$. Then we will apply to homeomorphism ψ the notions and denotations of (local) stable and (local) unstable manifolds of nonwandering points, understanding this as pre-image under h of the similar objects for diffeomorphism f. Using the construction, ψ and ϕ coincide on nonwandering sets and, by proof of theorem 3, there is a lift $\tilde\psi:\mathbb T^2\times\mathbb R\to\mathbb T^2\times\mathbb R$ of ψ coinciding with lift $\tilde\phi$ of ϕ on the set $\mathbb T^2\times\left(\bigcup_{i\in\mathbb Z}\frac{i}{2nk}\right)$.

Due to structural stability, two-dimensional invariant manifolds of different stabilities have transversal intersection which forms one-dimensional foliation $\dot{\mathcal I}_\psi$ on $M_{\hat J}\setminus NW(\psi)$.

Lemma 5. *The closures of the leaves of the foliation $\dot{\mathcal I}_\psi$ form one-dimensional foliation $\mathcal I_\psi$ on $M_{\hat J}$ such that each connected component $\tilde I$ of the pre-image with respect to p_J of each leaf $I\in\mathcal I_\psi$ has the property $\tilde I\cap\left(\mathbb T^2\times\frac{i}{2nk}\right)=z_0\times\frac{i}{2nk}$ for some $z_0\in\mathbb T^2$ and all $i\in\mathbb Z$.*

Proof. Let us divide the proof into steps.

Step 1. To prove lemma 5 it is enough to assume that homeomorphism ψ is such that ϕ belongs to Φ_+ and is defined by parameters $C\in\mathcal C, n\in\mathbb N, k=1, l=0$ (in the opposite case, we can take some power of ψ).

Denote by $p:\mathbb R^2\to\mathbb T^2$ universal cover such that $p(x,y)=(x(\text{mod }1),y(\text{mod }1))$. Let map $\widehat C:\mathbb T^2\to\mathbb T^2$ given by matrix $C=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\eta:\mathbb R^3\to\mathbb T^2\times\mathbb R$ be a cover given by formula $\eta(x,y,z)=(p(x,y),z)$. Denote by $\check\psi:\mathbb R^3\to\mathbb R^3$ a lift of $\tilde\psi$ with respect to η. As $\check\psi$ is a lift of $\tilde\psi$ and $\tilde\psi_*:\mathbb Z^2\to\mathbb Z^2$ given by matrix C then $b'=\check\psi b\check\psi^{-1}$, where $b:\mathbb R^3\to\mathbb R^3$ given by formula $b(x,y,z)=(x+\nu_1,y+\nu_2,z)$, $(\nu_1,\nu_2)\in\mathbb Z^2$ and $b'(x,y,z)=(x+a\nu_1+b\nu_2,y+c\nu_1+d\nu_2,z)$. Thus diffeomorphism $\check\psi(x,y,z)=(\check\psi_1(x,y,z),\check\psi_2(x,y,z),\check\psi_3(x,y,z))$ has the form

$$\begin{aligned}\check\psi_1(x,y,z)&=ax+by+h_1(x,y,z),\\ \check\psi_2(x,y,z)&=cx+dy+h_2(x,y,z),\qquad (*)\\ \check\psi_3(x,y,z)&=h_3(x,y,z),\end{aligned}$$

where $h_j(x+\nu_1,y+\nu_2,z)=h_j(x,y,z),\ j=1,2,3$ for each $\nu_1,\nu_2\in\mathbb Z$.

As any lift $\check C:\mathbb R^2\to\mathbb R^2$ of the diffeomorphism $\widehat C$ has the form $\check C(x,y)=(ax+by+\alpha,cx+dy+\beta)$ for some $\alpha,\beta\in\mathbb Z$ then the homeomorphism $\check\psi$ has exactly one fixed saddle point P_i

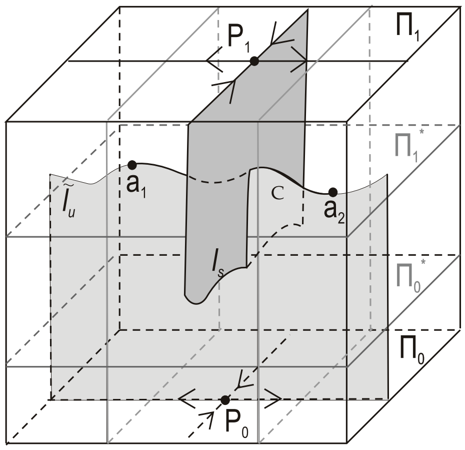

Figure 1. Illustration of the proof of lemma 5.

belonging to plane $\Pi_i = \mathbb{R}^2 \times \left\{\frac{i}{2n}\right\}$ for each $i \in \mathbb{Z}$. It can be directly verified that $P_i = \left(\check{z}_0, \frac{i}{2n}\right)$ for some $\check{z}_0 \in \mathbb{R}^2$.

Step 2. Set $\eta_{\hat{\jmath}} = p_{\hat{\jmath}}\eta : \mathbb{R}^3 \to M_{\hat{\jmath}}$. For stable (unstable) manifold $W^s(x)$ $(W^u(x))$ of non-wandering point $x \in NW(\psi)$ of homeomorphism ψ we will denote by $w^s(\check{x})$ $(w^u(\check{x}))$ the connected component of set $\eta_{\hat{\jmath}}^{-1}(W^s(x))$ $\eta_{\hat{\jmath}}^{-1}(W^u(x))$ passing through the point $\check{x} \in \eta_{\hat{\jmath}}^{-1}(x)$. For local stable (unstable) manifold $W^s_\gamma(x)$ $(W^u_\gamma(x))$, $\gamma > 0$ denote $w^s_\gamma(\check{x})$ $(w^u_\gamma(\check{x}))$ the connected component of set $\eta_{\hat{\jmath}}^{-1}(W^s_\gamma(x))$ $\eta_{\hat{\jmath}}^{-1}(W^u_\gamma(x))$ passing through the $\check{x}$. Notice that homeomorphism $\check{\psi}|_{\Pi_i}$ possesses two transversal one-dimensional $\check{\psi}$-invariant foliations F^s_i, F^u_i on Π_i consisting of parallel straight lines with different irrational slopes μ_s and μ_u. Let $L^s_i(\check{x}) = \left\{(x, y, z) \in \Pi_i : y = \mu_s x + b^s_i(\check{x}), L^u_i(\check{x}) = \{(x, y, z) \in \Pi_i : y = \mu_u x + b^u_i(\check{x})\right\}$ be leaves of foliations F^s_i, F^u_i passing through the point $\check{x} \in \Pi_i$. Further, it is useful to look at figure 1.

Set $N^u_\gamma(P_0) = \bigcup_{\check{x} \in L^u_0(P_0)} w^u_\gamma(\check{x})$ $(N^s_\gamma(P_1) = \bigcup_{\check{x} \in L^s_1(P_1)} w^s_\gamma(\check{x}))$ for some fixed $\gamma > 0$. In this step we show that there are numbers $b^u_1, b^u_2, b^s_1, b^s_2$ such that the closed box B^u (B^s) bounded by planes $\Pi_{-1}, \Pi_1, Q^u_1 = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b^u_1\}$, $Q^u_2 = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b^u_2\}$ $(\Pi_0, \Pi_2, Q^s_1 = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_s x + b^s_1\}$, $Q^s_2 = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_s x + b^s_2\})$ contains $N^u_\gamma(P_0)$ $(N^s_\gamma(P_1))$ in its interior. Let us construct the box B^u, for the box B^s construction is similar.

Let us fix $\varepsilon > 0$ and for each point $\check{x} \in L^u_0(P_0)$ construct a box $D^u(\check{x})$ bounded by planes $\Pi_{-1}, \Pi_1, Q^u_1(\check{x}) = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b^u_0(\check{x}) - \varepsilon(\check{x})\}$, $Q^u_2(\check{x}) = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b^u_0(\check{x}) + \varepsilon(\check{x})\}$, $Q^s_1(\check{x}) = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_s x + b^s_0(\check{x}) - \varepsilon(\check{x})\}$, $Q^s_2(\check{x}) = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_s x + b^s_0(\check{x}) + \varepsilon(\check{x})\}$ such that $\mathrm{dist}(w^u_\gamma(\check{x}), \partial D^u(\check{x})) > \varepsilon$. Due to the C^1-closeness of unstable manifolds on the compact set (for the diffeomorphism f) there is $\delta > 0$ such that the set $U(\check{x}) = \bigcup_{\check{y} \in \Pi_0 : \mathrm{dist}(\check{x}, \check{y}) < \delta} w^u_\gamma(\check{y})$ is a subset of $D^u(\check{x})$. Set $T_0 = \eta_{\hat{\jmath}}(\Pi_0)$ and $N(T_0) = \bigcup_{x \in T_0} W^u_\gamma(x)$. As $\eta_{\hat{\jmath}}(L^u_0(P_0))$ is dense everywhere on the torus T_0 then $\{\eta_{\hat{\jmath}}(U(\check{x})), \check{x} \in L^u_0(P_0)\}$ is a cover of a neighbourhood $N(T_0)$ of the torus T_0. Thus it has a finite subcover $\{\eta_{\hat{\jmath}}(U(\check{x}_1)), \ldots, \eta_{\hat{\jmath}}(U(\check{x}_n))\}$. Hence $N^u_\gamma(P_0)$ is obtained by integer shifts along the x-axis and the y-axis from discs $w^u_\gamma(\check{x})$ belonging to

$U(\check{x}_1)\cup\cdots\cup U(\check{x}_n)$. Set $\varepsilon_*=\max\{\varepsilon(\check{x}_1),\ldots,\varepsilon(\check{x}_n)\}$. Then the box B^u bounded by planes $\Pi_{-1},\Pi_1,Q_1^u=\{(x,y,z)\in\mathbb{R}^3:y=\mu_u x+b_0^u(P_0)-2\varepsilon_*\}$, $Q_2^u=\{(x,y,z)\in\mathbb{R}^3:y=\mu_u x+b_0^u(P_0)+2\varepsilon_*\}$ is required.

Step 3. In this step we show that $w^u(P_0)\cap w^s(P_1)\neq\varnothing$.

By the construction there is a homeomorphism $h_u:\mathbb{R}\times[-1,1]\to N_\gamma^u(P_0)$ ($h_s:\mathbb{R}\times[-1,1]\to N_\gamma^s(P_1)$) such that $h_u(\mathbb{R}\times\{0\})=L_0^u(P_0)$ ($h_s(\mathbb{R}\times\{0\})=L_1^s(P_1)$) and $h_u(\mathbb{R}\times(0,1))\subset\left(\mathbb{R}^2\times\left(0,\frac{1}{2n}\right)\right)$ $\left(h_s(\mathbb{R}\times(-1,0))\subset\left(\mathbb{R}^2\times\left(0,\frac{1}{2n}\right)\right)\right)$. Moreover, for curves $l_u=h_u(\mathbb{R}\times\{1\})$ and $l_s=h_s(\mathbb{R}\times\{-1\})$ there are numbers $z_0^*,z_1^*\in\left(0,\frac{1}{2n}\right)$ such that planes $\Pi_0^*=\{(x,y,z)\in\mathbb{R}^3:z=z_0^*\},\Pi_1^*=\{(x,y,z)\in\mathbb{R}^3:z=z_1^*\}$ divide domain $\mathbb{R}^2\times\left(0,\frac{1}{2n}\right)$ on three open sets V_0,V,V_1 with properties $(cl\ V_0)\setminus V_0=\Pi_0\cup\Pi_0^*,(cl\ V_1)\setminus V_1=\Pi_1\cup\Pi_1^*$ and $l_u,l_s\subset V$. Let us show that there is a number $k_*\in\mathbb{N}$ such that curve $\tilde{l}_u=\check{\psi}^{k_*}(l_u)$ is a subset of V_1.

Indeed, $\eta(cl\ V)=\mathbb{T}^2\times[z_0^*,z_1^*]$ and each point $t\in\eta(cl\ V)$ is wandering for $\tilde{\psi}$ and its positive iterations go to attractor $T_1=\eta(\Pi_1)$. Then there is neighbourhood $U_t\subset\mathbb{T}^2\times\left(0,\frac{1}{2n}\right)$ of the point t and natural number $k(t)$ such that $\tilde{\psi}^k(U_t)\subset\eta(V_1)$ for $k\geqslant k(t)$. As set $\eta(cl\ V)$ is compact then there is a finite subcover for cover $\{U_t,t\in\eta(cl\ V)\}$. Thus there is natural number k_* such that $\tilde{\psi}^k(\eta(cl\ V))\subset\eta(V_1)$ for $k\geqslant k_*$. Hence, $\check{\psi}^k(cl\ V)\subset V_1$ for $k\geqslant k_*$ and also $\tilde{l}_u\subset V_1$.

Due to the form (*), the set $\check{\psi}^{k_*}(N_\gamma^u(P_0))$ belongs to box $\tilde{B}^u$ bounded by planes $\Pi_{-1},\Pi_1,\{(x,y,z)\in\mathbb{R}^3:y=\mu_u x+\tilde{b}_1^u\},\{(x,y,z)\in\mathbb{R}^3:y=\mu_u x+\tilde{b}_2^u\}$ for some numbers $\tilde{b}_1^u,\tilde{b}_2^u$. As $\tilde{l}_u$ is situated in $\tilde{B}^u\cap V_1$ and intersects any plane of the form $\{(x,y,z)\in\mathbb{R}^3:y=\mu_s x+b,b\in\mathbb{R}\}$ then $\tilde{l}_u\cap B^s\neq\varnothing$ and, moreover, there is an arc $c\subset(\tilde{l}_u\cap B^s)$ which has end points $a_1\in Q_1^s,a_2\in Q_2^s$. Since $l_s\subset V$ then $c\cap N_\gamma^s(P_1)\neq\varnothing$. Thus $w^u(P_0)\cap w^s(P_1)\neq\varnothing$. As the original diffeomorphism f is structurally stable then the intersection $w^u(P_0)\cap w^s(P_1)$ is topologically transversal. So c is topologically transversal to $w^s(P_1)$. Since in a neighbourhood of P_1 homeomorphism $\check{\psi}$ is topologically conjugated with a hyperbolic saddle point then, due to λ-lemma, $w^u(P_1)\subset cl\left(\bigcup_{n\geqslant1}\check{\psi}^n(c)\right)$. Hence $w^u(P_1)\subset(cl\ w^u(P_0))$.

Step 4. Let us show that $(cl\ w^u(P_0))\cap\Pi_1=w^u(P_1)$.

Due to the hyperbolicity of the basic sets of the diffeomorphism f there is a γ such that $\psi(cl\ W_\gamma^s(x))\subset W_\gamma^s(\psi(x))$ for any $x\in\eta_j(\Pi_1)$ and diameter of $\psi^k(cl\ W_\gamma^s(x))$ tends to 0 as $k\to+\infty$. Set $N_\gamma(w^u(P_1))=\bigcup_{\check{x}\in w^u(P_1)}w_\gamma^s(\check{x})$. Then $\check{\psi}(cl\ N_\gamma(w^u(P_1)))\subset N_\gamma(w^u(P_1))$ and $\bigcap_{k\in\mathbb{N}}\check{\psi}^k(N_\gamma(w^u(P_1)))=w^u(P_1)$. For positive number $d<\gamma$ let us denote by B_d^s a box bounded by planes $\Pi_d^-=\left\{(x,y,z)\in\mathbb{R}^3:z=\frac{1}{2n}-d\right\},\Pi_d^+=\left\{(x,y,z)\in\mathbb{R}^3:z=\frac{1}{2n}+d\right\}$, $Q_d^-=\{(x,y,z)\in\mathbb{R}^3:y=\mu_u x+b_1^u(P_1)-d\},Q_d^+=\{(x,y,z)\in\mathbb{R}^3:y=\mu_u x+b_1^u(P_1)+d\}$. By the construction $B_d^s\subset N_\gamma(w^u(P_1))$ and hence $\check{\psi}^i(B_d^s)\subset B_d^s$ for some $i\in\mathbb{N}$. Without loss of generality we can assume that $i=1$.

Set $V_d=\left\{(x,y,z)\in\mathbb{R}^3:\frac{1}{2n}-d\leqslant z\leqslant\frac{1}{2n}\right\}$. Similarly to step 3, we can find a number $k^*\in\mathbb{N}$ such that the set $\check{\psi}^{k^*}(N_\gamma^u(P_0))$ belongs to box $\check{B}^u$ bounded by planes Π_{-1}, Π_1 {(x, y, z) $\in\mathbb{R}^3:y=\mu_u x+\check{b}_1^u\},\{(x,y,z)\in\mathbb{R}^3:y=\mu_u x+\check{b}_2^u\}$ for some numbers $\check{b}_1^u,\check{b}_2^u$ and the set

$K_u = (\check{\psi}^{k^*}(N^u_\gamma(P_0)) \setminus \check{\psi}^{k^*-1}(N^u_\gamma(P_0))) \cap \left(\mathbb{R}^2 \times \left(0, \frac{1}{2n}\right)\right)$ belongs to $\check{V}_d = \check{B}^u \cap V_d$. Let us choose a box $\check{V}$ containing K^u and bounded by planes $\Pi_d^-, \Pi_d^+, Q^- = \{(x,y,z) \in \mathbb{R}^3 : y = \mu_u x + b_1^u(P_1) - d_-\}, Q^+ = \{(x,y,z) \in \mathbb{R}^3 : y = \mu_u x + b_1^u(P_1) + d_+\}$ where Q^- (Q^+) is obtained from Q_d^- (Q_d^+) by integer shifts along the x-axis and the y-axis and the line $\check{\psi}(Q^-) \cap \Pi_1$ $(\check{\psi}(Q^+) \cap \Pi_1)$ has form $\left\{\left(x, y, \frac{1}{2n}\right) \in \mathbb{R}^3 : y = \mu_u x + b_1^u(P_1) - D_-\right\}$ $\left(\left\{\left(x, y, \frac{1}{2n}\right) \in \mathbb{R}^3 : y = \mu_u x + b_1^u(P_1) + D_+\right\}\right)$ with $|D_- - d_-| > d$ $(|D_+ - d_+| > d)$. As $\check{\psi}(Q_d^- \cap B_d^s) \cap (Q_d^- \cap B_d^s) = \varnothing$ $(\check{\psi}(Q_d^+ \cap B_d^s) \cap (Q_d^+ \cap B_d^s) = \varnothing)$ then $\check{\psi}(Q^- \cap \check{V}) \cap (Q_d^- \cap \check{V}) = \varnothing$ $(\check{\psi}(Q^+ \cap \check{V}) \cap (Q^+ \cap \check{V}) = \varnothing)$. Hence $\check{\psi}(cl\ \check{V}) \subset \check{V}$.

Set $w^u_+(P_0) = w^u(P_0) \cap \left(\mathbb{R}^2 \times \left[0, \frac{1}{2n}\right)\right)$. Let us represent the set $w^u_+(P_0)$ as the union $w^u_+(P_0) = \left(\check{\psi}^{k^*}(N^u_\gamma(P_0)) \cap \left(\mathbb{R}^2 \times \left[0, \frac{1}{2n}\right)\right)\right) \cup \bigcup_{i \in \mathbb{N}} \check{\psi}^i(K^u)$. Herewith $\check{\psi}^{k^*}(N^u_\gamma(P_0)) \subset \check{B}^u$ and $\bigcup_{i \in \mathbb{N}} \check{\psi}^i(K^u) \subset \check{V}$. Thus $(cl\ w^u_+(P_0) \setminus w^u_+(P_0)) \subset \left(\check{B}^u \cap \Pi_1\right)$. Since $cl\ w^u_+(P_0) \setminus w^u_+(P_0)$ is $\check{\psi}$-invariant and $w^u(P_1)$ is a unique $\check{\psi}$-invariant subset of $\check{B}^u \cap \Pi_1$ then $(cl\ w^u(P_0)) \cap \Pi_1 = w^u(P_1)$.

Step 5. Let us show that for any point $x \in \Pi_0$ there is a point $y \in \Pi_1$ such that $(cl\ w^u(x)) \cap \Pi_1 = w^u(y)$ and inversely, for any point $y \in \Pi_1$ there is a point $x \in \Pi_0$ such that $(cl\ w^s(y)) \cap \Pi_0 = w^s(x)$.

On the set $\mathbb{R}^2 \times \left[0, \frac{1}{2n}\right)$ there is a $\check{\psi}$-invariant two-dimensional foliation R_0, each leaf of which is homeomorphic to the semi-plane and coincides with $w^u(x) \cap \left(\mathbb{R}^2 \times \left[0, \frac{1}{2n}\right)\right)$ for some point $x \in \Pi_0$. Since $\psi|_{\eta_{\hat{\jmath}}(\Pi_0)}$ is hyperbolic automorphism then projection with respect to $\eta_{\hat{\jmath}}$ of an arbitrary point with the rational coordinates on Π_0 is a periodic point of $\psi|_{\eta_{\hat{\jmath}}(\Pi_0)}$. As each periodic point of ψ is the fixed point for some power of ψ then for arbitrary leaf G_0 of foliation R_0 passing through a point with the rational coordinates on Π_0, due to step 4, we have that $cl\ (G_0) \cap \Pi_1 = \{(x,y,z) \in \Pi_1 : y = \mu_u x + b_{G_0}\}$ for some $b_{G_0} \in \mathbb{R}$.

For $k \in \mathbb{Z}$ let us set $\check{\psi}^k(x,y,z) = (\check{\psi}_{1,k}(x,y,z), \check{\psi}_{2,k}(x,y,z), \check{\psi}_{3,k}(x,y,z))$. Notice that $\check{\psi}^k(x,y,z)$ is a lift of $\tilde{\psi}^k(x,y,z)$ with respect to η and it has exactly one fixed saddle point P_i belonging to the plane Π_i. As any lift $\check{\Psi} : \mathbb{R}^3 \to \mathbb{R}^3$ of the diffeomorphism $\tilde{\psi}^k$ has the form $\check{\Psi}(x,y,z) = (\check{\psi}_{1,k}(x,y,z) + \alpha, \check{\psi}_{2,k}(x,y,z) + \beta, \check{\psi}_{3,k}(x,y,z))$ for some $\alpha, \beta \in \mathbb{Z}$ then $b_{G_0} \neq b_{G_0'}$ if G_0 and G_0' are different leaves of the foliation R_0 passing through a point with the rational coordinates on Π_0. By continuity, we have that $cl\ (G_0) \cap \Pi_1 = \{(x,y,z) \in \Pi_1 : y = \mu_u x + b_{G_0}\}$ for some $b_{G_0} \in \mathbb{R}$ for arbitrary leaf G_0 of foliation R_0.

Analogously, on the set $\mathbb{R}^2 \times \left(0, \frac{1}{2n}\right]$ there is the $\check{\psi}$-invariant two-dimensional foliation, R_1, each leaf of which is homeomorphic to the semi-plane and coincides with $w^s(x) \cap \left(\mathbb{R}^2 \times \left[0, \frac{1}{2n}\right)\right)$ for some point $x \in \mathbb{T}^2 \times \left\{\frac{1}{2n}\right\}$. Similarly, for arbitrary leaf G_1 of foliation R_1 we have that $cl\ (G_1) \cap \Pi_0 = \{(x,y,z) \in \Pi_0 : y = \mu_s x + b_{G_1}\}$ for some $b_{G_1} \in \mathbb{R}$.

Thus the intersection $Y = G_0 \cap G_1$ is not empty for each of the leaves $G_0 \in R_0, G_1 \in R_1$ and $cl\ (Y) \setminus Y$ consists of two points $P^0_{G_0,G_1} \in \Pi_0, P^1_{G_0,G_1} \in \Pi_1$.

Step 6. Let us show that Y consists of one open curve z such that $cl\ z \cap (\Pi_0 \cup \Pi_1) = P^0_{G_0,G_1} \cup P^1_{G_0,G_1}$.

Notice that $cl\ Y$ is compact as G_0 and G_1 belong to boxes which have compact intersection. As f is structurally stable and it is topologically conjugated with ψ then the set $Z = cl(Y)\setminus\left(P^0_{G_0,G_1}\cup P^1_{G_0,G_1}\right)$ consists of curves. Any curve z from Z has one of four types: (1) $cl\ z\cap(\Pi_0\cup\Pi_1)=\varnothing$; (2) $cl\ z\cap\left(\Pi_0\cup\Pi_1\right)=P^0_{G_0,G_1}$; (3) $cl\ z\cap\left(\Pi_0\cup\Pi_1\right)=P^1_{G_0,G_1}$; (4) $cl\ z\cap\left(\Pi_0\cup\Pi_1\right)=P^0_{G_0,G_1}\cup P^1_{G_0,G_1}$. Let us show that cases (1)–(3) are impossible.

Indeed, in case (1) the curve z bounds closed 2-disc $D\subset G_0$ whose each point $d\in D$ is an intersection point of G_0 with some leaf from R_1. As origin diffeomorphism f is structurally stable then the intersection is topologically transversal. So D is foliated by closed curves, which are the intersection of the leaves of foliation R_1 with D. That is impossible.

In case (2) the curve z bounds closed 2-disc $D\subset G_0$ whose each point $d\in D$ is an intersection point of G_0 with some leaf from R_1. As origin diffeomorphism f is structurally stable then the intersection is topologically transversal. So D is foliated by closed curves, which are the intersection of the leaves of foliation R_1 with D. Due to (1), int D is foliated by open curves $\tilde{z}$ such that the closure of $\tilde{z}$ in D is $\tilde{z}\cup P^0_{G_0,G_1}$. This means that $D\subset(G_0\cap G_1)$. This is a contradiction of the transversality condition.

Case (3) is similar to case (2).

Thus, each curve in Z is an open arc with two boundary points $P^0_{G_0,G_1}$ and $P^1_{G_0,G_1}$. Let us show that for each pair of leaves G_0, G_1 the set Z consists of a unique curve. Suppose the contrary: there is more than one curve in Z. That there are two curves $z_1, z_2\subset Z$ such that they bound an open 2-disc. Here we get contradiction as in case (2). □

7. Topological conjugacy of a structurally stable diffeomorphism from G to a model

This part is devoted to the proof of theorem 4.

Let $f\in G$ be a structurally stable diffeomorphism. By theorem 3, f is ambient Ω-conjugated with some diffeomorphism $\phi: M_{\hat{J}}\to M_{\hat{J}}$ from the class Φ by means of a homeomorphism $h: M^3\to M_{\hat{J}}, J\in\mathcal{J}$. Set $\psi = hfh^{-1}: M_{\hat{J}}\to M_{\hat{J}}$. In lemma 6 we construct one-dimensional foliation $\mathcal{I}_\psi$. Set $\mathcal{I}_f = h^{-1}(\mathcal{I}_\psi)$.

Lemma 6. *Let V be a connected component of the set $M^3\setminus(\mathcal{A}\cup\mathcal{R})$, such that $\partial V = A\cup R$, where $A\in\mathcal{A}$, $R\in\mathcal{R}$. Then on the set $cl\ V$ there is a two-dimensional f_0-invariant foliation $\mathcal{P}_V$, each of whose leaves is a torus, which intersects each leaf of the foliation $\mathcal{I}_f$ at exactly one point.*

Proof. We denote by $\mathcal{I}_V$ one-dimensional foliation on $cl\ V$, each of whose leaves is the closure of a connected component of the intersection of a leaf of foliation $\mathcal{I}_f$ with V. Since $\mathcal{I}_V$ is a one-dimensional foliation, each of whose leaves intersects A at one point, then there is a 2-torus $T\subset V$, which also intersects each leaf of the foliation $\mathcal{I}_V$ at one point. Denote by U a closed subset of the set $cl\ V$, which is bounded by tori A and T. Two cases are possible: (a) $f_0(T)\cap T=\varnothing$; (b) $f_0(T)\cap T\neq\varnothing$.

In case (a), $f_0(T)\subset \text{int}\ U$ and the set $K = U\setminus f_0(U)$ is a fundamental domain of the restriction of f_0 on V, which means that $\bigcup_{i\in\mathbb{Z},t\in[0,1]} f_0^i(K) = V$ and $f_0^i(K)\cap f_0^j(K)=\varnothing$ for $i\neq j$. We introduce a parameterization for each leaf ℓ of the foliation $\mathcal{I}_V\cap cl\ K$, associating a parameter $t\in[0,1]$ to a point $x\in\ell$, where t is the ratio of the length of the arc $\ell_x\subset\ell$, which is bounded by points x and $\partial U\cap\ell$, to the length of the arc ℓ. Denote the parameterization by $\rho: cl\ K\to[0,1]$. For $t\in[0,1]$ let $T_t=\rho^{-1}(t)$. Then $\bigcup_{i\in\mathbb{Z},t\in[0,1]} f_0^i(T_t)\cup A\cup R$ is a required foliation $\mathcal{P}_V$.

In case (b) we show that a modification of the torus T to the torus $\tilde{T}$ which intersects each leaf of the foliation $\mathcal{I}_V$ at one point and has the property $f_0(\tilde{T}) \cap \tilde{T} = \varnothing$, exists.

Since A is an attractor of the diffeomorphism f_0, then there is a number $n > 0$, such that $f_0^n(U) \subset \operatorname{int} U$. Let m be the smallest positive integer, for which $f_0^n(T) \cap T = \varnothing$ for any $n > m$. We divide the construction of the desired torus into steps.

Step 1. Let us prove the lemma in the case $m = 1$, that is, in the case when $f_0(T) \cap T \neq \varnothing$, $f_0^n(T) \cap T = \varnothing$ for $n > 1$.

Let $\tilde{U} = U \cup f_0(U)$. Then $f_0(\tilde{U}) \subset \tilde{U}$ since, by construction, $f_0(U) \subset \tilde{U}$ and by assumption of step 1, $f_0^2(U) \subset \tilde{U}$. Therefore the required torus $\tilde{T}$ is obtained by a small perturbation of the boundary component of the set $\tilde{U}$, which is different from A. A small perturbation is a pushing up of points in the set $f_0(T)$ from the set $\tilde{U}$ along the leaves of the foliation $\mathcal{I}_V$.

Step 2. Let us construct a required torus in the case $m > 1$. We choose a natural number r, such that $2^r \leqslant m < 2^{r+1}$. Let $g_r = f_0^{2^r}$. Then $g_r^n(T) \cap T = \varnothing$ for all $n > 1$. Using the technique of step 1, we construct a required torus for the diffeomorphism g_r. Continuing the process, we construct a required torus for the diffeomorphism f_0. □

7.1. Proof of theorem 4

Let us prove that if a diffeomorphism from the class G is structurally stable then it is topologically conjugate with a diffeomorphism from the class Φ.

Proof. Let $f : M^3 \to M^3$ be a diffeomorphism from the class G. By theorem 3, f is topologically conjugated with a homeomorphism $\psi : M_{\hat{J}} \to M_{\hat{J}}, J \in \mathcal{J}$, which coincides with some diffeomorphism $\phi : M_{\hat{J}} \to M_{\hat{J}}$ from the class Φ on their common nonwandering set and its lift $\tilde{\psi} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ coincides with $\tilde{\phi}$ on the set $p_J^{-1}(NW(\psi)) = \left(\bigcup_{i\in\mathbb{Z}} \left\{\frac{i}{2m}\right\}\right) \times \mathbb{T}^2$, where $m = nk$ in the case $\phi \in \Phi_+$ and $m = q$ in the case $\phi \in \Phi_-$. Let us construct a homeomorphism $\tilde{h} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$, which conjugates the maps $\tilde{\psi}, \tilde{\phi}$ and commutes with γ.

By lemma 6, the homeomorphism $\tilde{\psi}$ has a pair of transversal $\tilde{\psi}$-invariant foliations $\mathcal{I}_{\tilde{\psi}}$, $\mathcal{P}_{\tilde{\psi}}$. The foliation $\mathcal{I}_{\tilde{\psi}}$ is one-dimensional and each of its leaves is homeomorphic to a straight line. The foliation $\mathcal{P}_{\tilde{\psi}}$ is two-dimensional and each leaf is homeomorphic to 2-torus.

We will index a leaf of the foliation $\mathcal{I}_{\tilde{\psi}}$ by its intersection point with $\mathbb{T}^2 \times \{0\}$ in the following way: $\tilde{I}_{z_0}$ is a leaf of the foliation $\mathcal{I}_{\tilde{\psi}}$ passing through the point $z_0 \times \{0\}$. Let O be the neutral element of the group $\mathbb{T}^2$ and $\tilde{S} = \{O\} \times \mathbb{R}$. We will index a leaf of the foliation $\mathcal{P}_{\tilde{\psi}}$ by its intersection point with $\tilde{I}_O$ in the following way: $\tilde{P}_{a_0}$ is a leaf of the foliation $\mathcal{P}_{\tilde{\psi}}$ passing through point $a_0 \in \tilde{I}_O$. Due to statements 8 and 9, there is a homeomorphism $\tilde{h}_1 : \tilde{S} \to \tilde{I}_O$ such that $\tilde{h}_1(O \times \{0\}) = O \times \{0\}$ and $\tilde{\psi}_1\tilde{h}_1 = \tilde{h}_1\tilde{\phi}_1$, where $\tilde{\psi}_1 = \tilde{\psi}|_{\tilde{I}_O}$ and $\tilde{\phi}_1 = \tilde{\phi}|_{\tilde{S}}$.

Set $\bar{I}_{z_0} = \{(z, r) \in \mathbb{T}^2 \times \mathbb{R} : z = z_0\}$, $\bar{P}_{r_0} = \{(z, r) \in \mathbb{T}^2 \times \mathbb{R} : r = r_0\}$ and denote by $\bar{\mathcal{I}}$ foliation consisting of leaves $\bar{I}_{z_0}, z_0 \in \mathbb{T}^2$ and by $\bar{\mathcal{P}}$ foliation consisting of leaves $\bar{P}_{r_0}, r_0 \in \mathbb{R}^1$. By the construction, the foliations $\bar{\mathcal{I}}$ and $\bar{\mathcal{P}}$ are $\tilde{\phi}$-invariant. Let us define the homeomorphism $\tilde{h} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ by the formula

$$\tilde{h}(z, r) = \tilde{I}_z \cap \tilde{P}_{\tilde{h}_1(r)}.$$

Let us check that (a) $\tilde{\psi}\tilde{h} = \tilde{h}\tilde{\phi}$ and (b) $\gamma\tilde{h} = \tilde{h}\gamma$.

(a) is verified in the following way:

$\tilde{\psi}(\tilde{h}(z,r)) = \tilde{\psi}\left(\tilde{I}_z \cap \tilde{P}_{\tilde{h}_1(r)}\right) = \tilde{\psi}(\tilde{I}_z) \cap \tilde{\psi}(\tilde{P}_{\tilde{h}_1(r)}) = \tilde{\psi}(\tilde{I}_z) \cap \tilde{P}_{\tilde{\psi}_1(\tilde{h}_1(r))}$. In the other side $\tilde{h}(\tilde{\phi}(z,r)) = \tilde{h}(\widehat{C}(z), \tilde{\phi}_1(r)) = \tilde{I}_{\widehat{C}(z)} \cap \tilde{P}_{\tilde{h}_1(\tilde{\phi}_1(r))}$. Since $\tilde{\psi}(\tilde{I}_z) = \tilde{I}_{\widehat{C}(z)}$ by lemma 6 and $\tilde{\psi}_1\tilde{h}_1 = \tilde{h}_1\tilde{\phi}_1$ by the construction, then $\tilde{\psi}\tilde{h} = \tilde{h}\tilde{\phi}$.

(b) is verified in the following way:

$\gamma(\tilde{h}(z,r)) = \gamma\left(\tilde{I}_z \cap \tilde{P}_{\tilde{h}_1(r)}\right) = \gamma(\tilde{I}_z) \cap \gamma(\tilde{P}_{\tilde{h}_1(r)}) = \gamma(\tilde{I}_z) \cap \tilde{P}_{\tilde{h}_1(r)-1}$. In the other side $\tilde{h}(\gamma(z,r)) = \tilde{h}(\widehat{J}(z), r-1) = \tilde{I}_{J(z)} \cap \tilde{P}_{\tilde{h}_1(r-1)}$. Since $\gamma(\tilde{I}_z) = \tilde{I}_{\widehat{J}(z)}$ by lemma 6 and $\tilde{h}_1(r) - 1 = \tilde{h}_1(r-1)$ by the construction, then $\gamma\tilde{h} = \tilde{h}\gamma$. □

8. Construction of a diffeomorphism from *G* which is not structurally stable

Theorem 5. *There is a diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ from class G whose nonwandering set consists of two tori and f is not structurally stable.*

Proof. Set $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It can be verified directly that $\widehat{C}$ has a unique fixed point which we denote by O. Let us define diffeomorphism d on $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ by formula $\varphi(x,y) = \left(\frac{4x}{5-3y}, \frac{5y-3}{5-3y}\right)$. By the construction, φ is a Morse–Smale diffeomorphism whose nonwandering set consists of one source (0, 1) and one sink (0,−1). Define diffeomorphism $\phi : \mathbb{T}^3 \to \mathbb{T}^3$ by formula $\phi(z,x,y) = (\widehat{C}(z), \varphi(x,y))$. Then $\phi \in \Phi$ and the nonwandering set of diffeomorphism ϕ consists of one repeller $R = \mathbb{T}^2 \times \{(0,1)\}$ and one attractor $A = \mathbb{T}^2 \times \{(0,-1)\}$. Set $O_R = O \times (0,1)$ and $O_A = O \times (0,-1)$.

Denote by $l \subset \mathbb{S}^1$ a closed arc on a circle bounded by points $\left(\frac{3}{5}, \frac{4}{5}\right)$ and $\left(\frac{3}{4}, \frac{5}{13}\right)$ and such that $(0,1) \notin l$. Set $K = \mathbb{T}^2 \times l$, $\mathbb{S}^1_+ = \{(x,y) \in \mathbb{S}^1 : x > 0\}$ and $V^+ = \mathbb{T}^2 \times \mathbb{S}^1_+$. By the construction, K is the fundamental domain for diffeomorphism $\phi|_{V^+}$. Set $O_* = O \times \left(\frac{7}{10}, \frac{\sqrt{51}}{10}\right)$. By the construction $O_* \in \operatorname{int} K$ and $O_* \in (W^s(O_A) \cap W^u(O_R))$. Let $B \subset \operatorname{int} K$ be a 3-ball containing O_*. Denote by D_A and D_R connected components of $W^s(O_A) \cap B$ and $W^u(O_R) \cap B$, respectively, containing O_*. By the construction, the discs D_A and D_R have transversal intersection. Let $\theta : \mathbb{T}^3 \to \mathbb{T}^3$ be a diffeomorphism which is an identity out of B and such that discs D_A and $\theta(D_R)$ have a tangency. Then $f = \theta\phi : \mathbb{T}^3 \to \mathbb{T}^3$ is required of the diffeomorphism.

Indeed, by the construction, $NW(f) = NW(\phi)$ and $f|_{NW(f)} = \phi|_{NW(\phi)}$. Thus, diffeomorphism f satisfies axiom A and, hence, $f \in G$. Denote by O'_A and O'_R the fixed points of f. Denote by V^+_A, V^+_R the connected components of $V^+ \setminus K$ such that $A \subset cl\ V^+_A$ and $R \subset cl\ V^+_R$. Then $W^u(O'_R) \cap V^+_R = W^u(O_R) \cap V^+_R$ and $W^u(O'_R) \cap K = \theta(W^u(O_R) \cap K)$. On the other hand $W^s(O'_A) \cap (V^+_A \cup K) = W^u(O_R) \cap (V^+_A \cup K)$. It follows that the invariant manifolds $W^s(O'_A)$ and $W^u(O'_R)$ are not transversal. □

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