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Positive Isotopies of Legendrian Submanifolds and Applications

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We give a simple proof that there is no positive loop inside the component of a fiber in the space of Legendrian embeddings in the contact manifold ST^*M, provided that the universal cover of M is $\mathbb{R}^n$. We consider some related more general results in the space of one-jet of functions on a compact manifold and we give an application dealing with positive isotopies in homogeneous neighbourhoods of surfaces in a tight contact three-manifold.

1 Introduction and formulation of the results

1.1 Presentation

On the Euclidean unit 2-sphere, the set of points which are at a given distance of the north pole is in general a circle. When the distance is π, this circle becomes trivial: it is reduced to the south pole. Such a focusing phenomenon cannot appear on a surface of constant, non-positive curvature. In this case, the image by the exponential map of a unit circle of vectors tangent to the surface at a given point is never reduced to one point.

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In this article, we generalize this remark in the context of contact topology (In particular, no Riemannian structure is involved). Our motivation comes from the theory of the orderability of the group of contactomorphisms of Eliashberg, Kim and Polterovitch [11, 12].

Before we come to the central result of this article, Theorem 4 below, we will formulate and prove independently some of its consequences (Theorems 1 and 2), which are more directly related to the remark above, and which will help us to introduce the subject in a natural way.

1.2 Positive isotopies

Consider a $(2n+1)$-dimensional manifold V endowed with a *cooriented* contact structure ξ. At each point of V, the contact hyperplane then separates the tangent space in *a positive and a negative side*.

Definition 1. A smooth path $L_t = \varphi_t(L)$, $t \in [0,1]$, in the space of Legendrian embeddings (resp. immersions) of a n-dimensional compact manifold L in (V,ξ) is called *a Legendrian isotopy (resp. homotopy)*. If, in addition, for every $x \in L$ and every $t \in [0,1]$, the velocity vector $\dot{\varphi}_t(x)$ lies in the *positive* side of ξ at $\varphi_t(x)$, then this Legendrian isotopy (resp. homotopy) will be called *positive*. □

Remark 1. This notion of positivity does not depend on the parametrization of the L_t's. □

If the cooriented contact structure ξ is induced by a globally defined contact form α, the above condition can be rephrased as $\alpha(\dot{\varphi}_t(x)) > 0$. In particular, a positive contact Hamiltonian induces positive isotopies.

A positive isotopy (resp. homotopy) will be also called a *positive path* in the space of Legendrian embeddings (resp. immersions). We will call a closed positive path a positive *loop*.

Example 1. The space $J^1(N) = T^*N \times \mathbb{R}$ of one-jet of functions on a n-dimensional manifold N has a natural contact one-form $\alpha = du - \lambda$, where λ is the Liouville one-form of T^*N and u is the $\mathbb{R}$-coordinate. The corresponding contact structure will be denoted by ζ. Given a smooth function $f\colon N \to \mathbb{R}$, the image of its one-jet extension is a Legendrian submanifold which will be denoted by $j^1 f$. A one-parameter family of functions gives then rise to an isotopy of Legendrian embeddings.

A path $f_t, t \in [0,1]$, of functions on N such that, for any fixed $q \in N$, $f_t(q)$ is an increasing function of t, gives rise to a positive Legendrian isotopy $j^1 f_t, t \in [0,1]$, in $J^1(N)$.

Conversely, one can check that a positive isotopy consisting only of one-jet extensions of functions is always of the above type. *In particular there are no positive loops consisting only of one-jet extensions of functions*. □

Remark 2. In particular a positive loop never contracts to a constant Legendrian L through positive loops since a small positive loop would yield a positive loop amongst 1-jet extensions of functions in the 1-jet space neighbourhood of L. □

Example 2. Consider a Riemannian manifold (N, g). Its unit tangent bundle $\pi : S_1N \to N$ has a natural contact one-form: If u is a unit tangent vector to N, and v a vector tangent to S_1N at u, then

$$\alpha(u) \cdot v = g(u, D\pi(u) \cdot v).$$

The corresponding contact structure will be denoted by ζ_1. The constant contact Hamiltonian $h = 1$ induces the geodesic flow.

Any fiber of $\pi : S_1N \to N$ is Legendrian. Moving a fiber by the geodesic flow is a typical example of a positive path. □

1.3 Formulation of the results

Let N be a closed manifold.

Theorem 1. There is no positive loop in the component of the space of Legendrian embeddings in $(J^1(N), \zeta)$ containing the one-jet extensions of functions. □

The Liouville one-form of T^*N induces a contact distribution on the fiber-wise spherization ST^*N. This contact structure is contactomorphic to the ζ_1 of Example 2. Our generalization of the introductory Paragraph 1.1 is as follows.

Theorem 2. There is no positive path of Legendrian embeddings between two distinct fibers of $\pi : ST^*N \to N$, provided that the universal cover of N is $\mathbb{R}^n$. □

Theorem 3. **o.** Any *compact* Legendrian submanifold of $J^1(\mathbb{R}^n)$ belongs to a positive loop of Legendrian embeddings, which is contractible amongst loops of Legendrian embeddings.

i. There exists a component of the space of Legendrian embeddings in $(J^1(S^1), \zeta)$ whose elements are homotopic through smooth embeddings to the zero section $j^1 0$ and which contains a positive loop.

ii. There exists a positive loop of Legendrian immersions in $(J^1(S^1), \zeta)$ based at the zero section.

iii. Given any connected surface N, there exists a positive path of Legendrian immersions between any two fibers of $\pi : ST^*N \to N$. □

Laudenbach [16] proved the following generalization of Theorem 3 **ii**: for any closed N, there exists a positive loop in the space of Legendrian immersions in $(J^1(N), \zeta)$, based at the zero section.

Theorem 3 **o** implies that for any contact manifold (V, ξ), there exists a contractible positive loop of Legendrian embeddings (just consider a Darboux ball and embed the example of Theorem 3 **o**).

A Legendrian submanifold $L \subset (J^1(N), \zeta)$ will be called *positive* if it is connected by a positive path to the one-jet extension of the zero function. The one-jet extension of a positive function is a positive Legendrian submanifold. But, in general, the value of the u-coordinate can be negative at some points of a positive Legendrian submanifold.

Consider a closed manifold N and fix a function $f : N \to \mathbb{R}$. Assume that 0 is a regular value of f. Denote by $\Lambda(f)$ the union for $\lambda \in \mathbb{R}$ of the $j^1(\lambda f)$. It is a *smooth embedding* of $\mathbb{R} \times N$ in $J^1(N)$, foliated by the $j^1(\lambda f)$. We denote by $\Lambda_+(f)$ the subset $\bigcup_{\lambda>0}(j^1(\lambda f)) \subset \Lambda(f)$.

Example 3. In $J^1(S^1)$, if $f : S^1 \to \mathbb{R}$ is $\theta \mapsto \sin\theta$, then $\Lambda(f)$ is the cylinder $\{(\theta, \lambda\cos\theta, \lambda\sin\theta),\ \theta \in \mathbb{R}/2\pi\mathbb{Z},\ \lambda \in \mathbb{R}\}$. □

Consider the manifold $M = f^{-1}([0, +\infty[) \subset N$. Its boundary ∂M is the set $f^{-1}(0)$. Fix some field $\mathbb{K}$ and denote by $b(f)$ the total dimension of the homology of M with coefficients in that field ($b(f) = \dim_{\mathbb{K}} H_*(M, \mathbb{K})$). We say that a point $x \in J^1(N)$ is *above* some subset of the manifold N if its image under the natural projection $J^1(N) \to N$ belongs to this subset.

Theorem 4. For any positive Legendrian submanifold $L \subset (J^1(N), \zeta)$ in general position with respect to $\Lambda(f)$, there exists at least $b(f)$ points of intersection of L with $\Lambda_+(f)$ lying above $M \setminus \partial M$.

More precisely, for a generic positive Legendrian submanifold L, there exists at least $b(f)$ different *positive* numbers $\lambda_1, \ldots, \lambda_{b(f)}$ such that L intersects each manifold $j^1(\lambda_i f)$ above $M \setminus \partial M$. □

Remark 3. Theorem 4 implies the Morse estimate for the number of critical points of a Morse function F on N. This can be seen as follows. By adding a sufficiently large constant to F, one can assume that $L = j^1(F)$ is a positive Legendrian submanifold. If f is a constant positive function, then $M = N$, and intersections of $L = j^1 F$ with Λ are in one to one correspondence with the critical points of F. Furthermore, F is Morse if and only if L is transverse to $\Lambda(f)$.

In fact, one can prove that Theorem 4 implies (a weak form of) Arnold's conjecture for Lagrangian intersection in cotangent bundles, proved by Laudenbach and Sikorav [17], and generalized in the Legendrian setting by Chekanov [6] and Chaperon [5]. This is no accident. Our proof relies on the main ingredient of those works: the technique of generating families (see Theorem 8).

One can also prove that Theorem 4 implies Theorem 1. Theorem 5 below, which in turn implies Theorem 2, is also a direct consequence of Theorem 4. □

Theorem 5. Consider a line in $\mathbb{R}^n$. Denote by Λ the union of all the fibers of $\pi : ST^*\mathbb{R}^n \to \mathbb{R}^n$ above this line. Consider one of these fibers and a positive path starting from this fiber. The end of this positive path is a Legendrian sphere. This sphere must intersect Λ in at least 2 points. □

1.4 A remark about stabilized knots and exotic contact structures

Consider a Legendrian knot L in a 3D contact manifold (V, ξ). By a relative Darboux Theorem, a neighbourhood of L in (V, ξ) is contactomorphic to a neighbourhood of the zero section in $(J^1(L), \zeta)$. A stabilization of L is a Legendrian knot L' obtained by modifying L inside this neighbourhood near one point, to the effect that L' has a small Z-shaped front in the $J^1(L)$ model (we recall the notion of front in Section 2.1).

Remark 4. Any stabilized knot belongs to a positive loop. Indeed, circulating the $Z-$shaped piece along the L direction (i.e., the $0-$section in the $J^1(L)$ model) can be done in a positive way. □

This remark provides a rough way, independent from the original work of Bennequin [3], to show that $J^1(S^1)$ is *tight*. Recall that an *overtwisted disc* in a contact manifold is an embedded disc which is everywhere transverse to the contact structure in its interior, except at one point, and tangent to the contact structure along its boundary. If a contact manifold contains such a disc, then it is called *overtwisted*, and *tight* otherwise.

A folklore result is that, if L is a Legendrian knot in an overtwisted contact manifold such that L does not intersect the overtwisted disc, then L is a stabilization. This can be seen in an elementary way by working in an explicit model neighbourhood of an overtwisted disc. Hence, if $J^1(S^1)$ were not tight, the above remark would contradict Theorem 1.

1.5 An application to positive isotopies in homogeneous neighbourhoods of a surface in a tight contact three-manifold

In Theorems 4 and 5, we observe the following feature: The submanifold Λ is foliated by Legendrian submanifolds. We pick one of them, and we conclude that we cannot disconnect it from Λ by a positive contact isotopy. In dimension 3, our Λ is a surface foliated by Legendrian curves (in a non-generic way).

Recall that, generically, a closed oriented surface S contained in a contact three-manifold (V,ξ) is *convex*: there exists a vector field tranversal to S and whose flow preserves ξ. Equivalently, a convex surface admits a *homogeneous neighbourhood* $U \simeq S \times \mathbb{R}$, $S \simeq S \times \{0\}$, where the restriction of ξ is $\mathbb{R}$-invariant. Given such a homogeneous neighbourhood, we obtain a smooth, canonically oriented, multicurve $\Gamma_U \subset S$, called the *dividing curve* of S, made of the points of S where ξ is tangent to the $\mathbb{R}$-direction. It is automatically transverse to ξ. According to Giroux [14], the dividing curve Γ_U does not depend on the choice of U up to an isotopy amongst the multicurves transverse to ξ in S. The *characteristic foliation* ξS of $S \subset (V,\xi)$ is the integral foliation of the singular line field $TS \cap \xi$.

Let S be a closed oriented surface of genus $g(S) \geq 1$ and (U,ξ), $U \simeq S \times \mathbb{R}$, be a homogeneous neighbourhood of $S \simeq S \times \{0\}$. The surface S is ξ-convex, and we denote by Γ_U its dividing multicurve. We assume that ξ is tight on U, which, after Giroux, is the same as to say that no component of Γ_U is contractible in S.

Theorem 6. In this setting, assume L is a Legendrian curve in S having minimal (in its smooth isotopy class in S) geometric intersection $2k > 0$ with Γ_U. If $(L_s)_{s\in[0,1]}$ is a positive Legendrian isotopy in U such that $L = L_0$ then $\sharp(L_1 \cap S) \geq 2k$. □

Remark 5. The positivity assumption is essential: if we push L in the homogeneous direction, we get an isotopy of Legendrian curves which becomes instantaneously disjoint from S. If $k = 0$, this is a positive isotopy of L that disjoints L from S. □

Remark 6. For a small positive isotopy, the result is obvious. Indeed, L is an integral curve of the characteristic foliation ξS of S, which contains at least one singularity in each component of $L \setminus \Gamma_U$. For two consecutive components, the singularities have opposite signs. Moreover, when one moves L by a small positive isotopy, the positive singularities are pushed in $S \times \mathbb{R}^+$ and the negative ones in $S \times \mathbb{R}^-$. Between two singularities of opposite signs, we will get one intersection with S. □

The relationship to the preceding results is given by the following corollary of Theorem 4, applied to $N = S^1$ and $f(\theta) = \cos(k\theta)$, for some fixed $k \in \mathbb{N}$. In this situation, the surface Λ of Theorem 4 will be called Λ_k. It is an infinite cylinder foliated by Legendrian circles. Its characteristic foliation $\xi \Lambda_k$ has $2k$ infinite lines of singularities which consist of all those $(\theta, p, 0) \in J^1(S^1), p \in \mathbb{R}$, where θ is such that $f(\theta) = 0$. The standard contact space $(J^1(S^1), \zeta)$ is itself a homogeneous neighbourhood of Λ_k, and the corresponding dividing curve consists in $2k$ infinite lines, alternating with the lines of singularities.

Let $L_0 = j^1 0 \subset \Lambda_k$. Theorem 4 gives:

Corollary 1. Let L_1 be a generic positive deformation of L_0. Then $\sharp\{L_1 \cap \Lambda_k\} \geq 2k$. □

Indeed, there are k intersections with $\Lambda_{k,+}$, and k other intersections which are obtained in a similar way by considering the function $-f$. ■

This corollary will be the building block to prove Theorem 6.

1.6 Organization of the article

This article is organized as follows. The proof of Theorem 3, which in a sense shows that the hypothesis of Theorems 1 and 2 are optimal, consists essentially in a collection of explicit constructions. It is done in the next section, and it might serve as an introduction to the main notions and objects discussed in this paper. The rest of the article is essentially devoted to the proof of Theorems 1, 2, 4, 5, and 6, but contains a few statements which are slightly more general than the theorems mentioned in this introduction.

2 Proof of theorem 3

2.1 A positive loop

In order to prove statement **i** of Theorem 3, we begin by the description of a positive loop in the space of Legendrian embeddings in $J^1(S^1)$. Due to Theorem 1, this cannot happen in the component of the zero section $j^1 0$.

Denote by (q, p) some canonical coordinates on T^*S^1, so that the contact form α of $J^1(S^1)$ is $\alpha = du - pdq$. Recall that the *front* of a Legendrian curve $L \subset J^1(S^1)$ is its projection on the 0−jet space $S^1 \times \mathbb{R}$, which is obtained by forgetting p. Given some function f on S^1, the front of $j^1 f$ is the graph of f. But, in general, the front of L is a singular curve. In a generic situation, the singularities of the front are transverse self-intersections and cusps. One may then recover L from its front, since the missing coordinate p is a slope which is well defined on the front and equal to $\frac{du}{dq}$ outside of the cusps (see [2]).

Consider for example the Legendrian curve L defined by the front drawn in Figure 1. The slope is everywhere positive, hence we can assume that it lies in the half-space $\{p > 2\epsilon\} \subset J^1(S^1)$. One can also prove (e.g., by showing an explicit path where we create two double points as in Figure 2 and eliminate successively pairs of opposite

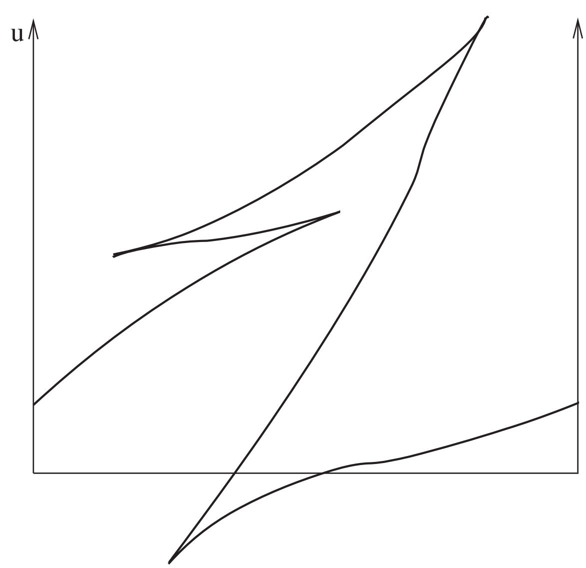

Fig. 1. The front projection of an $L \subset \{p > 2\epsilon\}$, which is homotopic to $j^1 0$ through Legendrian immersions.

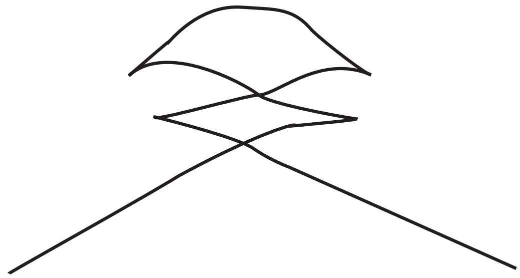

Fig. 2. An intermediate position in the deformation from L to $j^1 0$

cusps along a swallowtail singularity) that this L is homotopic to $j^1 0$ through Legendrian immersions. In addition, it is isotopic to $j^1 0$ through smooth embeddings.

Consider the contact flow $\varphi_t\colon (q,p,u) \to (q-t,p,u-t\epsilon)$, $t \in \mathbb{R}$. The corresponding contact Hamiltonian $h(q,p,u) = -\epsilon + p$ is positive near $\varphi_t(L)$, for all $t \in \mathbb{R}$, and hence $\varphi_t(L)$ gives rise to a positive path which will move the front downwards in a kind of helicoidal movement.

On the other hand, one can go from $\varphi_{2\pi}(L)$ back to L just by increasing the u-coordinate, which is also a positive path. This proves statement **i** of Theorem 3.

2.2 A positive loop of immersions based at the zero section

We now consider statement **ii**. Take $L \subset \{p > 2\epsilon\}$ as above. Recall that L is homotopic to $j^1 0$ through Legendrian immersions (However, like any other Legendrian submanifold in $\{p > 2\epsilon\} \subset J^1(S^1)$, it cannot be Legendrian isotopic to $j^1 0$, since, if it were the case, it would intersect $\{p = 0\}$, by [6]).

Step 1: The homotopy between $j^1 0$ and L can be transformed into a positive path of Legendrian immersions between $j^1 0$ and a vertical translate L' of L, by combining it with an upwards translation with respect to the u-coordinate.

Step 2: Then, using the flow φ_t (defined in Section 2.1) for $t \in [0, 2k\pi]$ with k large enough, one can reach another translate L'' of L, on which the u-coordinate can be arbitrarily low.
Step 3: Consider now a path of Legendrian immersions from L to $j^1 0$. It can be modified into a positive path between L'' and $j^1 0$, like in Step 1.

This proves statement **ii** of Theorem 3.

2.3 Positive loops for local Legendrians

The proof of statement **o** uses again the same idea. Given any compact Legendrian submanifold $L \subset J^1(\mathbb{R}^n)$, there exists L', which is Legendrian isotopic to L and which is contained into the half-space $p_1 > \epsilon > 0$, for some system $(p_1, \dots, p_n, q_1, \dots, q_n)$ of canonical coordinates on $T^*(\mathbb{R}^n)$. To find such a L', one can proceed? as follows. Consider

$$\Psi_t : (p_1, \dots, p_n, q_1, \dots, q_n, u) \mapsto (tp_1, \dots, tp_n, tq_1, \dots, tq_n, t^2u).$$

This defines a contact flow in $J^1(\mathbb{R}^n)$. Applying Ψ_t on L for t small enough results in a Legendrian submanifold Legendrian isotopic to L on which the p-coordinates are arbitrarily small. Then, one can tilt a little bit this Legendrian submanifold to achieve $p_1 > 0$. Tilting can be achieved by a contact flow of the form

$$(p_1, \dots, p_n, q_1, \dots, q_n, u) \mapsto (p_1 + t, p_2 \dots, p_n, q_1, \dots, q_n, u + tq_1).$$

This ensures the existence of L'. It is possible to find a positive path between L and a sufficiently high-vertical translate L'' of L', like in Step 1. Because $p_1 > \epsilon$, one can now slide down L'' by a positive path as low as we want with respect to the u-coordinate, as in Step 2. Hence we can assume that L is connected by a positive path of embeddings to some L''', which is a vertical translate of L', on which the u-coordinate is very negative. So we can close this path back to L in a positive way.

If we drop the positivity requirement, this path of fronts/Legendrians contracts trivially to the constant one. (Recall a positive loop never contracts to a constant one through positive loops by Remark 2.)

2.4 Positive paths

We now prove statement **iii**. Notice first that it is enough to consider the particular case $N = \mathbb{R}^2$. Indeed, given two points x and y on a surface N and an embedded path from x to y, we can consider an open neighbourhood $\mathcal{U}$ of this path diffeomorphic to $\mathbb{R}^2$.

2.4.1 The hodograph transform

We now recall the classical "hodograph" contactomorphism [1] which identifies $(ST^*\mathbb{R}^2, \zeta_1)$ and $(J^1(S^1), \zeta)$, and more generally $(ST^*\mathbb{R}^n, \zeta_1)$ and $(J^1(S^{n-1}), \zeta)$. The same trick will be used later to prove Theorems 2 and 5 (Sections 4.4 and 5.3).

Fix a scalar product $\langle ., . \rangle$ on $\mathbb{R}^n$ and identify the sphere S^{n-1} with the standard unit sphere in $\mathbb{R}^n$. Identify a covector at a point $q \in S^{n-1}$ with a vector in the hyperplane tangent to the sphere at q (perpendicular to q). Then to a point $(p, q, u) \in J^1(S^{n-1}) =$

$T^*S^{n-1} \times \mathbb{R}$ we associate the cooriented contact element at the point $uq + p \in \mathbb{R}^n$, which is parallel to $T_q S^{n-1}$, and cooriented by q.

The reverse map is given by the following formulas :

$$(ST^*\mathbb{R}^n, \zeta_1) \to (J^1(S^{n-1}), \zeta)$$
$$(x, q) \mapsto (q, p = \langle x, \cdot \rangle_{|T_q S^{n-1}}, u = \langle x, q \rangle)$$

Remark 7. One can check that the fiber of $\pi : ST^*\mathbb{R}^n \to \mathbb{R}^n$ over some point $x \in \mathbb{R}^n$ is the image by this contactomorphism of $j^1 l_x$, where $l_x : S^{n-1} \to \mathbb{R}$ is the function defined by $q \mapsto \langle x, q \rangle$. □

2.4.2 End of the proof of Theorem 3 iii

One can assume that $x = 0 \in \mathbb{R}^2$. The case when $x = y$ follows directly from Theorem 3 **ii** via the contactomorphism described above. The fiber $\pi^{-1}(x)$ corresponds to $j^1 0$.

Suppose now that $x \neq y$. We need to find a positive path of Legendrian immersions in $(J^1(S^1), \zeta)$ between $j^1 0$ and $j^1 l_y$.

To achieve this, it is enough to construct a positive path of Legendrian immersions between $j^1 0$ and a translate of $j^1 0$ that would lie entirely below $j^1 l_y$, with respect to the u-coordinate. This can be done as in Section 2.2, just by decreasing even more the u-coordinate, like in Step 2. This finishes the proof of Theorem 3. ■

3 Generating families and Morse theory

3.1 Generating families

We briefly recall the construction of a generating family for a Legendrian submanifold in the one-jet bundle $J^1(N)$ of a smooth manifold N (the details can be found in [2]). Let $\rho : \mathcal{E} \to N$ be a smooth fibration over N with fiber W. Let $F : \mathcal{E} \to \mathbb{R}$ be a smooth function. For a point q in N we consider the set $B_q \subset \rho^{-1}(q)$ whose points are the critical points of the restriction of F to the fiber $\rho^{-1}(q)$. Denote B_F the set $B_F = \bigcup_{q \in N} B_q \subset \mathcal{E}$. Assume that the rank of the matrix (F_{wq}, F_{ww}) (w, q are local coordinates on the fiber and base respectively) formed by second derivatives is maximal (i.e., equal to the dimension of N) at each point of B_F. This condition holds for a generic F and does not depend on the choice of the local coordinates w, q.

The set $B_F \subset \mathcal{E}$ is then a smooth submanifold of the same dimension as N, and the restriction of the map

$$(q, w) \overset{\ell_F}{\longmapsto} (q, d_N(F(q, w)), F(q, w)),$$

where d_N denotes the differential along N, to B_F defines a Legendrian immersion of B_F into $(J^1(N), \zeta)$. If this is an embedding (this is generically the case), then F is called a *generating family* of the Legendrian submanifold $L_F = \ell_F(B_F)$.

A point $x \in J^1(N)$ is by definition a triple consisting of a point $q(x)$ in the manifold N, a covector $p(x) \in T^*_{q(x)}N$ and a real number $u(x)$. A point $x \in L$ will be called a *critical point* of the Legendrian submanifold $L \subset (J^1(N), \zeta)$ if $p(x) = 0$. The value of the u-coordinate at a critical point of a Legendrian manifold L will be called a *critical value* of L. The set of all critical values will be denoted by $Crit(L)$.

Observe that, for a manifold $L = L_F$ given by a generating family F, the set $Crit(L_F)$ coincides with the set of critical values of the generating family F.

We call a critical point $x \in L$ *non-degenerate* if L intersects the manifold given by the equation $p = 0$ transversally at x. If an embedded Legendrian submanifold L_F is given by a generating family F, then the non-degenerate critical points of F are in one to one correspondence with the non-degenerate critical points of L_F.

3.2 Generating families quadratic at infinity and their generalization

We describe now the class of generating families we will be working with. Pick a closed manifold E which is a fibration over some closed manifold N, and some $K \in \mathbb{N}$. We will apply the preceding construction to the fibration $\rho : \mathcal{E} = E \times \mathbb{R}^K \mapsto N$. A function $F: E \times \mathbb{R}^K \to \mathbb{R}$ is called *quadratic at infinity* if it is a sum of a non-degenerate quadratic form Q on $\mathbb{R}^K$ and a function on $E \times \mathbb{R}^K$ with bounded differential (i.e., the norm of the differential is uniformly bounded for some Riemannian metric which is a product of a Riemannian metric on E and the Euclidean metric on $\mathbb{R}^K$). This definition does not depend on the choice of the metrics. If a function which is quadratic at infinity is a generating family (with respect to the fibration $\rho : E \times \mathbb{R}^K \to N$), then we call it a *generating family quadratic at infinity*.

In the case when $E = N$ (the fiber is a point), we recover the classical notion of a generating family quadratic at infinity [10, 20]. The proofs of Theorems 1,2,4,5, and 6, formulated in the introduction, are based on this classical case only. The case of a general bundle E is however very natural, and a great variety of Legendrian submanifolds can be produced in this way (see [13]). The fibration E is used here only to prove a generalization of Theorem 1, Theorem 7 below, which provides many isotopy classes of Legendrian embeddings in $J^1(N)$ not containing any positive loop.

3.3 Morse theory for generating families quadratic at infinity

We gather here some results from Morse theory which will be needed later. Let $E \to N$ be a fibration with E a closed manifold. Consider a function F quadratic at infinity as above. Denote by F^a the set $\{F \leq a\}$. For sufficiently large positive numbers $C_1 < C_2$, the set F^{-C_2} is a deformation retract of F^{-C_1}. Hence for C large enough, the homology groups $H_*(F^a, F^{-C}, \mathbb{K})$ depend only on a. We will denote them by $H_*(F, a)$. It is known (see [10]) that for sufficiently large a, $H_*(F, a)$ is isomorphic to $H_*(E, \mathbb{K})$.

For any function F which is quadratic at infinity, and any integer $k \in \{1, \ldots, \dim H_*(E, \mathbb{K})\}$, we define a *Viterbo number* $c_k(F)$ by

$$c_k(F) = \inf\{c | \dim i_*(H_*(F, c)) \geq k\},$$

where i_* is the map induced by the natural inclusion $F^c \to F^a$, when a is a sufficiently large number. Our definition is similar to Viterbo's construction [20] in the symplectic setting, which was adapted in the contact setting by Bhupal [4]. The following proposition is a straightforward generalization of this, using the language of generating families quadratic at infinity.

Proposition 1.

i. Each number $c_k(F)$, $k \in \{1, \ldots, \dim H_*(E, \mathbb{K})\}$ is a critical value of F, and if F is an excellent Morse function (i.e., all its critical points are non-degenerate and all critical values are different) then the numbers $c_k(F)$ are different.

ii. Consider a family F_t, $t \in [0, 1]$, of functions which are all quadratic at infinity. For any $k \in \{1, \ldots, \dim H_*(E, \mathbb{K})\}$ the number $c_k(F_t)$ depends continuously on t. If the family F_t, $t \in [0, 1]$, is generic (i.e., intersects the discriminant consisting of non excellent Morse functions transversally at its smooth points) then $t \mapsto c_k(F_t)$ is a continuous piecewise smooth function with a finite number of singular points. □

Remark 8. At this moment, it is unknown whether $c_k(L_F)$ depends only on L if $L = L_F$. □

4 Proof of Theorems 1 and 2

We will in fact prove Theorem 7 below, which is more general than Theorem 1. Fix a closed manifold N and a smooth fibration $E \to N$ such that E is closed. A Legendrian submanifold $L \subset (J^1(N), \zeta)$ will be called a E-*quasifunction* if it is Legendrian isotopic to a manifold given by some generating family quadratic at infinity along the lines of Section 3.2. We say that a connected component $\mathcal{L}$ of the space of Legendrian submanifolds in $(J^1(N), \zeta)$ is E-*quasifunctional* if $\mathcal{L}$ contains an E-quasifunction. For example, the component $\mathcal{L}$ containing the one-jet extensions of the smooth functions on M is E-quasifunctional, with E coinciding with N (the fiber is just a point). More complicated manifolds E will give rise to an interesting zoology of Legendrian submanifolds to which the following theorem applies :

Theorem 7. An E-quasifunctional component contains no positive loop. □

4.1

The proof of Theorem 7 will be given in Section 4.3. It will use the following generalization of Chekanov's theorem (see [18]), and Proposition 2.

Theorem 8. Consider a Legendrian isotopy $L_t, t \in [0, 1]$, such that L_0 is an E-quasifunction. Then there exists a number K and a smooth family of functions quadratic at infinity $F_t\colon E \times \mathbb{R}^K \to \mathbb{R}$, such that for any $t \in [0, 1]$, F_t is a generating family of L_t. □

Note that it follows from Theorem 8 that *any* Legendrian submanifold in some E-quasifunctional component is in fact an E-quasifunction.

Proposition 2. Consider a positive path $L_t, t \in [0, 1]$, given by a family $F_t, t \in [0, 1]$, of quadratic at infinity generating families. The Viterbo numbers of the family F_t are monotone increasing functions with respect to t: $c_i(F_0) < c_i(F_1)$ for any $i \in \{1, \ldots, \dim H_*(E)\}$. □

4.2 Proof of Proposition 2

Assume that the inequality is proved for a generic family. This, together with the continuity of the Viterbo numbers, gives us a weak inequality $c_i(F_0) \leq c_i(F_1)$, for any family. But positivity is a C^∞–open condition, so we can perturb the initial family F_t into some family $\widetilde{F}_t$ coinciding with F_t when t is sufficiently close to 0 and 1, such that $\widetilde{F}_t$ still generates a positive path of Legendrian submanifolds and such that the family $\widetilde{F}_{t,t\in[1/3,2/3]}$ is generic. We have

$$c_i(F_0) = c_i(\widetilde{F}_0) \le c_i(\widetilde{F}_{1/3}) < c_i(\widetilde{F}_{2/3}) \le c_i(\widetilde{F}_1) = c_i(F_1),$$

and hence the inequality is strong for all families.

We now prove the inequality for generic families. Excellent Morse functions form an open dense set in the space of all quadratic at infinity functions on $E \times \mathbb{R}^K$. The complement of the set of excellent Morse functions forms a discriminant, which is a singular hypersurface. A generic one-parameter family of quadratic at infinity functions F_t on $E \times \mathbb{R}^K$ has only a finite number of transverse intersections with the discriminant in its smooth points, and for every t except possibly finitely many, the Hessian $d_{ww}F_t$ is non-degenerate at every critical point of the function F_t.

We will use the notion of *Cerf diagram* of a family of functions $g_t, t \in [0,1]$, on a smooth manifold. The Cerf diagram is a subset in $[0,1] \times \mathbb{R}$ consisting of all the pairs of type (t,z), where z is a critical value of g_t. In the case of a generic family of functions on a closed manifold, the Cerf diagram is a curve with non-vertical tangents everywhere, with a finite number of transverse self-intersections and cuspidal points as singularities, since it is the front of a Legendrian curve in $J^1([0,1])$, obtained from g viewed as a generating family.

The graph of the Viterbo number $c_i(F_t)$ is a subset of the Cerf diagram of the family F_t. To prove the monotonicity of the Viterbo numbers, it is sufficient to show that the Cerf diagram of F_t has a positive slope at every point except for a finite set. The rest of the proof of Proposition 2 is devoted to that.

We say that a point x on a Legendrian submanifold $L \subset J^1(N)$ is non-vertical if the differential of the natural projection $L \to N$ is non-degenerate at x. Let L_t be a smooth family of Legendrian submanifolds and $x(t_0) = (q(t_0), p(t_0), u(t_0))$ be a non-vertical point. By the implicit function theorem, there exists a unique family $x(t) = (q(t), p(t), u(t))$, defined for t sufficiently close to t_0, such that $x(t) \in L_t$ and $q(t) = q(t_0)$. We call the number $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} u(t)$ the *vertical velocity* of the point $x(t_0)$.

Lemma 1. For a positive path of Legendrian submanifolds, the vertical velocity of every non-vertical point is positive. □

Consider a path L_t in the space of Legendrian submanifolds given by a generating family F_t defined on the total space of a fibration $\rho : \mathcal{E} = E \times \mathbb{R}^K \mapsto N$. Consider a point

$x \in L_{t_0}$ corresponding to a vertical (with respect to ρ) critical point $(e, v) \in E \times \mathbb{R}^K$ of F_{t_0}, and local coordinates (q, w) adapted to the fibration ρ in a neighbourhood of (e, v). Then x is equal to (q_0, p_0, u_0) such that $q_0 = \rho(e, v)$ and

$$d_w F_{t_0}(q_0, w_0) = 0, p = d_q F_{t_0}(q_0, w_0), u = F_{t_0}(q_0, w_0).$$

In that case x is non-vertical if and only if the Hessian $d_{ww} F_{t_0}(q_0, w_0)$ is non-degenerate. For such a point x, the following lemma holds:

Lemma 2. The vertical velocity at x is equal to $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} F_t(q_0, w_0)$. □

Let G_t be a family of smooth functions and assume that the point $z(t_0)$ is a Morse critical point for G_{t_0}. By the implicit function theorem, for each t sufficiently close to t_0, the function G_t has a unique critical point $z(t)$ close to z_0, and $z(t)$ is a smooth path.

Lemma 3. The vertical velocity of the critical value $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} G_t(z(t))$ does not depend of the path $z(t)$ and is equal to $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} G_t(z_0)$.

Indeed, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} G_t(z(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} G_t(z(t_0)) + \frac{\partial G_t}{\partial z}(z(t_0)) \cdot \frac{\mathrm{d}z}{\mathrm{d}t}(t_0)$.

At almost every point on the Cerf diagram, the slope of the Cerf diagram at this point is the velocity of a critical value of the function F_t. By Lemma 3 and Lemma 2, it is the vertical velocity at some non-vertical point. By Lemma 1, it is positive. This finishes the proof of Proposition 2. □

4.3 Proof of Theorem 7

Suppose now that there is a positive loop $L_t, t \in [0, 1]$, in some E-quasifunctional component $\mathcal{L}$. The condition of positivity is open. We slightly perturb the loop $L_t, t \in [0, 1]$, such that $Crit(L_0)$ is a finite set of cardinality A. Note that $A > 0$, since L_0 is a E-quasifunction. Consider the A-th multiple of the loop $L_t, t \in [0, 1]$. By Theorem 8, L_t has a generating family F_t, for all $t \in [0, A]$. By Proposition 2, we have that

$$c_1(F_0) < c_1(F_1) < \cdots < c_1(F_A).$$

All these $A+1$ numbers belong to the set $Crit(L_0)$. This is impossible due to the cardinality of this set. This finishes the proof of Theorem 7 and hence of Theorem 1. □

4.4 Proof of Theorem 2

Recall that we assume that the universal cover of N is $\mathbb{R}^n$. A positive path between two fibers of $ST^*N \to N$ could be lifted to a positive path between two fibers of $ST^*\mathbb{R}^n \to \mathbb{R}^n$, hence it is enough to consider the case when $N = \mathbb{R}^n$. We will use the contactomorphism between $(ST^*(\mathbb{R}^n), \zeta_1)$ and $(J^1(S^{n-1}), \zeta)$ defined in Section 2.4.1.

Without loss of generality, we assume that there exists a positive path which starts at the fiber $\pi^{-1}(0)$ and ends at $\pi^{-1}(x)$, for some non-zero $x \in \mathbb{R}^n$. The fiber $\pi^{-1}(x)$ corresponds to a Legendrian submanifold $j^1 l_x \subset J^1(S^{n-1})$, where l_x is the function defined by $q \mapsto \langle q, x \rangle$. It is a Morse function for $x \neq 0$, and has only two critical points and two critical values $\pm ||x||$. So at the end of the path, the Viterbo numbers must be $+||x||$ and $-||x||$. On the other hand, the starting point corresponds to the zero section $j^1 l_0$, where the Viterbo numbers must be equal to zero. Hence the existence a positive path between those two fibers would contradict the monotonicity (Proposition 2) of the Viterbo numbers. □

5 Morse theory for positive Legendrian submanifolds

In this section, we prove Theorem 4 and deduce Theorem 5 from it. We need first to generalize some of the previous constructions and results to the case of manifolds with boundary.

Let N be a closed manifold. Fix a function $f : N \to \mathbb{R}$ such that 0 is a regular value of f. Denote by M the set $f^{-1}([0, +\infty[)$. Denote by $b(f) = \dim_{\mathbb{K}} H_*(M)$ the dimension of $H_*(M)$ (all the homologies here and below are counted with coefficients in a fixed field $\mathbb{K}$).

5.1 Viterbo numbers for manifolds with boundary

The definition of the Viterbo numbers for a function quadratic at infinity on a manifold with boundary is the same as in the case of a closed manifold. We repeat it briefly. Given a function F which is quadratic at infinity, we define the Viterbo numbers $c_{1,M}(F), \ldots, c_{b(f),M}(F)$ as follows.

A *generalized critical value* of F is a real number which is a critical value for F or for the restriction $F|_{\partial M \times \mathbb{R}^K}$. Denote by F^a the set $\{(q, w) | F(q, w) \leq a\}$. The homotopy type of the set F^a is changed only if a passes through a generalized critical value. One can show that, for sufficiently large $C_1, C_2 > 0$, the homology of the pair (F^{C_1}, F^{-C_2}) is independent of C_1, C_2, and naturally isomorphic, by the Thom isomorphism, to $H_{*-\mathrm{ind}\,Q}(M)$. So, for any

$a \in \mathbb{R}$ and sufficiently large C_2 the inclusion $H_*(F^a, F^{-C_2}) \to H_{*-\mathrm{ind}\,Q}(M)$ is well defined and independent of C_2. Denote the image of this inclusion by $H_*(F, a)$.

Definition 2. The Viterbo numbers are

$$c_{k,M}(F) = \inf\{c \mid \dim H_*(F, c) \geq k\}, k \in \{1, \ldots, b(f)\}. \qquad \square$$

Any Viterbo number $c_{k,M}(F)$ is a generalized critical value of the function F. Obviously, $c_{1,M}(F) \leq \ldots \leq c_{b(f),M}(F)$. For any continuous family F_t of quadratic at infinity functions, $c_{k,M}(F_t)$ depends continuously on t.

5.2 Proof of Theorem 4

Consider a family of quadratic at infinity functions $F_t\colon N \times \mathbb{R}^K \to \mathbb{R}$, parametrized by $t \in [0, 1]$, such that F_t is a generating family for the Legendrian submanifold L_t and such that the path $L_t, t \in [0, 1]$, is positive. We will consider the restriction of the function F_t to $M \times \mathbb{R}^K$ and denote it by F_t also. The following proposition generalizes Proposition 2.

Proposition 3. The Viterbo numbers of the family F_t are monotone increasing: $c_{i,M}(F_0) < c_{i,M}(F_1)$ for any $i \in \{1, \ldots, b(f)\}$. $\square$

The difference with Proposition 2 is that the Cerf diagram of a generic family has one more possible singularity. This singularity corresponds to the case when a Morse critical point meets the boundary of the manifold. In this case, the Cerf diagram is locally diffeomorphic to a parabola with a tangent half-line (i.e., a neighbourhood of a branching point in a train-track). It remains to show that for a path $z(t)$ of critical points in ∂M, their critical values $F_t(z(t))$ has positive derivative. The formula $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} F_t(z(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} F_t(z(t_0)) + \frac{\partial F_t}{\partial z}(z(t_0)) \cdot \frac{\mathrm{d}z}{\mathrm{d}t}(t_0)$ from the proof of Lemma 3 still holds. To conclude, we remark that $\frac{\mathrm{d}z}{\mathrm{d}t}(t_0)$ is tangent to ∂M where $\frac{\partial F_t}{\partial z}(z(t_0))|_{T(\partial M)}$ vanishes. $\square$

We now prove Theorem 4. Consider the 1-parameter family of functions H_λ defined on $M \times \mathbb{R}^K$ for $\lambda \geq 0$ by

$$H_\lambda(q, w) = F_1(q, w) - \lambda f(q).$$

The manifold L_1 intersects $j^1\lambda_0 f$ at some point above $M \setminus \partial M$ if and only if the function H_{λ_0} has 0 as an ordinary critical value (not a critical value of the restriction to the boundary).

Consider the numbers $c_{k,M}(H_\lambda)$. By Proposition 3, $c_{k,M}(H_0) > 0$. For a sufficiently large value of λ, each of them is negative. To show that, consider a sufficiently small

$\varepsilon > 0$ belonging to the component of the regular values of f which contains 0. Denote by $M_1 \subset M$ the set $\{f \geq \varepsilon\}$. The manifold M_1 is diffeomorphic to the manifold M, and the inclusion map is a homotopy equivalence. Denote by G_λ the restriction of H_λ to the $M_1 \times \mathbb{R}^K$. Consider the following commutative diagram:

$$\begin{tikzcd} H_*(G_\lambda^a, G_\lambda^{-C_2}) \arrow[r, "i_1"] \arrow[d, "j_1"'] & H_*(G_\lambda^{C_1}, G_\lambda^{-C_2}) \arrow[r, "Th_1"] & H_{*-\mathrm{ind}(Q)}(M_1) \arrow[d, "j_2"'] \\ H_*(H_\lambda^a, H_\lambda^{-C_2}) \arrow[r, "i_2"] & H_*(H_\lambda^{C_1}, H_\lambda^{-C_2}) \arrow[r, "Th_2"] & H_{*-\mathrm{ind}(Q)}(M) \end{tikzcd}$$

where C_1, C_2 are sufficiently large numbers, Th_1, Th_2 denote Thom isomorphisms and i_1, i_2, j_1, j_2 are the maps induced by the natural inclusions. It follows from the commutativity of the diagram and from the fact that j_2 is an isomorphism that $c_{k,M_1}(G_\lambda) \geq c_{k,M}(H_\lambda)$ for every k.

For sufficiently large λ and for every $q \in M_1$, the critical values of the function G_λ restricted to $q \times \mathbb{R}^K$ are negative. Hence all generalized critical values of G_λ are negative. It follows that all the numbers $c_{k,M_1}(G_\lambda)$ are negative, and the same holds for $c_{k,M}(H_\lambda)$. We fix λ_0 such that $c_{k,M}(H_{\lambda_0}) < 0$ for every $k \in \{1, \dots, b(f)\}$.

Consider now $c_{k,M}(H_\lambda)$ as a function of $\lambda \in [0, \lambda_0]$. We are going to show that its zeroes correspond to the intersections above $M \setminus \partial M$. For a manifold L_1 in general position, all the generalized critical values of F_1 are non-zero. In particular all the critical values of the function $F_1|_{\partial M \times \mathbb{R}^K}$ are non-zero. The function $F_1|_{\partial M \times \mathbb{R}^K}$ coincides with $H_\lambda|_{\partial M \times \mathbb{R}^K}$ since $f = 0$ on ∂M. Hence, if zero is a critical value for H_λ, then it is an ordinary critical value at some inner point. This finishes the proof of Theorem 4. □

Remark 9. The function $c_{i,M}(H_\lambda)$ can be constant on some sub-intervals in $]0, \lambda_0[$, even for a generic function F_1. Indeed, the critical values of the restriction of H_λ to $\partial M \times \mathbb{R}^K$ do not depend on λ. It is possible that $c_{i,M}(H_\lambda)$ is equal to such a critical value for some λ's. □

The following proposition concerns the case of a general (non-necessarily generic) positive Legendrian submanifold. We suppose again that f is a function having 0 as regular value and that L is a positive Legendrian submanifold.

Proposition 4. For any connected component of the set $M = \{f \geq 0\}$ there exists a positive λ such that L intersects $j^1 \lambda f$ above this component. □

Consider a connected component M_0 of the manifold M. It is possible to replace f by some function $\tilde{f}$ such that 0 is a regular value for $\tilde{f}$, $\tilde{f}$ coincides with f on M_0 and $\tilde{f}$ is negative on $N \setminus M_0$. We consider $c_{1,M_0}(F_1 - \lambda\tilde{f})$ as a function of λ. It is a continuous function, positive in some neighbourhood of zero, and negative for the large values of λ.

Fix some α and β such that $c_{1,M_0}(F_1 - \alpha\tilde{f}) > 0$ and $c_{1,M_0}(F_1 - \beta\tilde{f}) < 0$. Assume that for any $\lambda \in [\alpha, \beta]$, L does not intersect $j^1\lambda\tilde{f}$ above M_0. Then this is also true for any small enough generic perturbation $L'$of L. Denote by F' a generating family for L'. Each zero λ_0 of $c_{1,M_0}(F' - \lambda_0\tilde{f})$ corresponds to an intersection of L' with $j^1\lambda_0\tilde{f}$ above M_0. Such a λ_0 exists by Theorem 4. This is a contradiction. □

5.3 Proof of Theorem 5

We can suppose that the origin of $\mathbb{R}^n$ belongs to the line considered in the statement of Theorem 5. Consider now again the contactomorphism which identifies $(J^1(S^{n-1}), \zeta)$ and $(ST^*(\mathbb{R}^n), \zeta_1)$ (see Section 2.4.1).

For such a choice of the origin, the union of all the fibers above the points on the line forms a manifold of type $\Lambda(f)$, where f is the restriction of a linear function to the sphere S^{n-1} (see Remark 7 in paragraph 2.4.1).

The manifold $M = \{f \geq 0\}$ has one connected component (it is a hemisphere). By Proposition 4, there is at least one intersection of the considered positive Legendrian sphere with $\Lambda_+(f)$. Another point of intersection comes from $\Lambda_+(-f)$. These two points are different because $\Lambda_+(-f)$ does not intersect $\Lambda_+(f)$. □

6 Positive isotopies in homogeneous neighbourhoods

The strategy for proving Theorem 6 is to link the general case to the case of $\Lambda_k \subset (J^1(S^1), \zeta)$.

Let $d = \sharp(L_1 \cap S)$. We first consider the infinite cyclic cover $\overline{S}$ of S associated with $[L] \in \pi_1(S)$. The surface $\overline{S}$ is an infinite cylinder. We call $\overline{U}$ the corresponding cover of U endowed with the pullback $\overline{\xi}$ of ξ. By construction, $\overline{U}$ is $\overline{\xi}$-homogeneous. We also call $\overline{L}_s$ a continuous compact lift of L_s in $\overline{U}$.

By compactness of the family $(\overline{L}_s)_{s\in[0,1]}$, we can find a large compact cylinder $\overline{C} \subset \overline{S}$ such that for all $s \in [0,1]$, $\overline{L}_s \subset int(\overline{C} \times \mathbb{R})$. We also assume that $\partial\overline{C} \pitchfork \Gamma_{\overline{U}}$.

The following lemma shows that in addition we can assume that the boundary of $\overline{C}$ is Legendrian.

Lemma 4. If we denote by $\pi : \overline{S} \times \mathbb{R} \to \overline{S}$ the projection forgetting the $\mathbb{R}$-factor, we can find a lift $\overline{C}_0$ of a C^0-small deformation of $\overline{C}$ in $\overline{S}$ which contains $\overline{L}_0$, whose geometric intersection with $\overline{L}_1$ is d and whose boundary is Legendrian. □

To prove this, we only have to find a Legendrian lift γ of a small deformation of $\partial\overline{C}$, and make a suitable slide of $\overline{C}$ near its boundary along the $\mathbb{R}$-factor to connect γ to a small retraction of $\overline{C} \times \{0\}$. The plane field $\overline{\xi}$ defines a connection for the fibration $\pi : \overline{S} \times \mathbb{R} \to \overline{S}$ outside any small neighbourhood $N(\Gamma_{\overline{S}})$ of $\Gamma_{\overline{S}}$. We thus can pick any $\overline{\xi}$-horizontal lift of $\partial\overline{C} - N(\Gamma_{\overline{S}})$.

We still have to connect the endpoints of these Legendrian arcs in $N(\Gamma_{\overline{S}}) \times \mathbb{R}$. These endpoints lie at different $\mathbb{R}$-coordinates, however this is possible to adjust since $\overline{\xi}$ is almost vertical in $N(\Gamma_{\overline{S}}) \times \mathbb{R}$ (and vertical along $\Gamma_{\overline{S}} \times \mathbb{R}$). To make it more precise, we first slightly modify $\overline{C}$ so that $\partial\overline{C}$ is tangent to $\overline{\xi S}$ near $\Gamma_{\overline{S}}$. Let δ be the metric closure of a component of $\partial\overline{C} \setminus \Gamma_{\overline{S}}$ contained in the metric closure R of a component of $\overline{S} \setminus \Gamma_{\overline{S}}$. On $int(R) \times \mathbb{R}$, the contact structure $\overline{\xi}$ is given by an equation of the form $\mathrm{d}z + \beta$ where z denotes the $\mathbb{R}$-coordinate and β is a 1-form on $int(R)$, such that $d\beta$ is an area form that goes to $+\infty$ as we approach ∂R. Now, let δ' be another arc properly embedded in R and which coincides with δ near its endpoints. If we take two lifts of δ and δ' by π starting at the same point (these two lifts are compact curves, since they coincide with the characteristic foliation near their endpoints, and thus lift to horizontal curves near $\Gamma_{\overline{S}}$ where β goes to infinity), the difference of altitude between the lifts of the two terminal points is given by the area enclosed between δ and δ', measured with $d\beta$. As $d\beta$ is going to infinity near $\partial\delta = \partial\delta'$, taking δ' to be a small deformation of δ sufficiently close to $\partial\delta$, we can give this difference any value we want. This proves Lemma 4. □

Let $\overline{U}_0 = \overline{C}_0 \times \mathbb{R}$.

Lemma 5. There exists an embedding of $(\overline{U}_0, \overline{\xi}, \overline{L}_0)$ in $(J^1(S^1), \zeta, \Lambda_k)$ such that the image of $\overline{L}_1$ intersects d times Λ_k. □

The surface $\overline{C}_0$ is $\overline{\xi}$-convex and its dividing set has exactly $2k$ components going from one boundary curve to the other. All the other components of $\Gamma_{\overline{C}_0}$ are boundary parallel. Moreover, the curve $\overline{L}_0$ intersects by assumption exactly once every non boundary parallel component and avoids the others. Then one can easily embed $\overline{C}_0$ in a larger annulus $\overline{C}_1$ and extend the system of arcs $\Gamma_{\overline{C}_0}(\overline{\xi})$ outside of $\overline{C}_0$ by gluing small arcs, in order to obtain a system Γ of $2k$ non boundary parallel arcs on $\overline{C}_1$. Simultaneously, we extend the contact structure $\overline{\xi}$ from $\overline{U}_0$, considered as a homogeneous neighbourhood of $\overline{C}_0$, to a neighbourhood $\overline{U}_1 \simeq \overline{C}_1 \times \mathbb{R}$ of $\overline{C}_1$. To achieve this one only has to extend the characteristic foliation, in a way compatible with Γ, and such that the boundary of $\overline{C}_1$ is also Legendrian. Note that the $\mathbb{R}$-factor is not changed above $\overline{C}_0$.

To summarize, $\overline{U}_1$ is a homogeneous neighbourhood of $\overline{C}_1$ for the extension $\overline{\xi}_1$, and $\overline{C}_1$ has Legendrian boundary with dividing curve $\Gamma_{\overline{C}_1}(\overline{\xi}_1) = \Gamma$. By genericity, we can assume that the characteristic foliation of $\overline{C}_1$ is Morse-Smale. Then, using Giroux's realization lemma [14], one can perform a C^0-small modification of $\overline{C}_1$ relative to $\overline{L}_0 \cup \partial\overline{C}_1$, leading to a surface $\overline{C}_2$, through annuli transverse to the $\mathbb{R}$-direction, and whose support is contained in an arbitrary small neighbourhood of saddle separatrices of $\overline{\xi}_1\overline{C}_1$, so that the characteristic foliation of $\overline{C}_2$ for $\overline{\xi}_1$ is conjugated to $\zeta\Lambda_k$. If this support is small enough and if we are in the generic case (which can always been achieved) where $\overline{L_1}$ doesn't meet the separatrices of singularities of $\overline{\xi}_1\overline{C}_1$, we get that $\sharp(\overline{L}_1 \cap \overline{C}_2) = \sharp(\overline{L}_1 \cap \overline{C}_1) = d$. As we are dealing with homogeneous neighbourhoods, we see that $(\overline{U}_1, \overline{\xi}_1, \overline{L}_0)$ is conjugated with $(J^1(S^1), \zeta, \Lambda_k)$. This proves Lemma 5. □

The combination of Lemma 5 and Corollary 1 ends the proof of Theorem 6 by showing that $d \geq 2k$. □

When S is a sphere the conclusion of Theorem 6 also holds since we are in the situation where $k = 0$. However in this case, we have a more precise disjunction result.

Theorem 9. Let (U, ξ) be a ξ-homogeneous neighbourhood of a sphere S. If ξ is tight (i.e., Γ_U is connected), then any Legendrian curve $L \subset S$ can be made disjoint from S by a positive isotopy. □

Consider $\mathbb{R}^3$ with coordinates (x, y, z) endowed with the contact structure $\zeta = \ker(\mathrm{d}z + x\mathrm{d}y)$. The radial vector field

$$R = 2z\frac{\partial}{\partial z} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

is contact. Due to Giroux's realization lemma, the germ of ξ near S is isomorphic to the germ given by ζ near a sphere S_0 transverse to R. Let L_0 be the image of L in S_0 by this map. By genericity, we can assume that L_0 avoids the vertical axis $\{x = 0, y = 0\}$. Now, if we push L_0 enough by the flow of $\frac{\partial}{\partial z}$, we have a positive isotopy of L_0 whose endpoint L_1 avoids S_0. This isotopy takes place in a ζ-homogeneous collar containing S_0 and obtained by flowing back and forth S_0 by the flow of R. This collar embeds in U by an embedding sending S_0 to S and the R-direction to the $\mathbb{R}$-direction. □

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