ISSN 1064-5624, Doklady Mathematics, 2012, Vol. 86, No. 2, pp. 707–710. © Pleiades Publishing, Ltd., 2012.
Published in Russian in Doklady Akademii Nauk, 2012, Vol. 446, No. 1, pp. 21–24.

COMPUTER SCIENCE

Global Tolerances in the Problems of Combinatorial Optimization with an Additive Objective Function[1]

V. V. Chistyakov[a], B. I. Goldengorin[a], and P. M. Pardalos[b]

Received March 30, 2012

Abstract—It is known that by means of minimal values of tolerances one can obtain necessary and sufficient conditions for the uniqueness of the optimal solution of a combinatorial optimization problem (*COP*) with an additive objective function and the set of nonembedded feasible solutions. Moreover, the notion of a tolerance is defined locally, i.e., with respect to a chosen optimal solution. In this paper we introduce the notion of a global tolerance with respect to the whole set of optimal solutions and prove that the nonembeddedness assumption on the set of feasible solutions of the *COP* can be relaxed, which generalizes the well known relations for the extremal values of the tolerances. In particular, we formulate a new criterion for the uniqueness of the optimal solution of the *COP* with an additive objective function, which is based on certain equalities between locally and globally defined tolerances.

DOI: 10.1134/S106456241205002X

After an optimal solution to a Combinatorial Optimization Problem (COP) has been found, the next natural step is to apply the sensitivity analysis, i.e., to determine how the optimality of the solution depends on a change of the input data. There are several reasons for performing the sensitivity analysis. (1) The input data of the COP may be inexact or have a natural uncertainty. In this case the sensitivity analysis verifies the credibility of the optimal solution and conclusions based on this solution. (2) Rather significant properties of the desired optimal solution in terms of the COP have not been built into the model due to the difficulty in formulating them. Having solved the simplified model, the decision maker wants to know how well the optimal solution fits in with the other considerations, which were not taken into account during the reduction procedure. The simplest sensitivity analysis studies the special case when the value of a single element in the optimal solution is subject to change. The goal of these perturbations is to determine the tolerances being defined as the maximum changes of the given individual cost (weight, distance, time, etc.), preserving the optimality of the given optimal solution under the assumption that the other data of the COP remain unchanged.

The interest to the notion of a tolerance is connected with the fact that the maximal value of tolerances of elements of the problem (called the bottleneck tolerance), which is a bound for the stability radius of the optimal solution, has been applied for design tolerance-based enumeration algorithms to solve different COPs. The first implicit application of tolerances goes back to Vogel's approximation method for finding the closest basic solution to the optimal one in the simplex method for the transportation problem [10] and to construction of a heuristic for solving the three-index assignment problem [1]. Among successful instances of application of tolerances [4], let us mention the algorithms for the exact [2, 12] and approximate [3, 7] solutions of the asymmetric traveling salesman problem.

It is known that by means of minimal values of tolerances one can obtain necessary and sufficient conditions for the uniqueness of the optimal solution of a COP with an additive objective function and the set of nonembedded feasible solutions. Moreover, the notion of a tolerance is defined locally, i.e., with respect to a chosen optimal solution. In this paper we introduce the notion of a global tolerance with respect to the whole set of optimal solutions (Section 2) and prove that the nonembeddedness assumption on the set of feasible solutions of the COP can be relaxed (Section 3), which generalizes the well known relations for the extremal values of the tolerances. In particular, we formulate a new criterion for the uniqueness of the optimal solution of the COP with an additive objective function, which is based on certain

[1]The article was translated by the authors.

[a] National Research University Higher School of Economics at Nizhni Novgorod, Nizhni Novgorod, Russia
[b] University of Florida, USA
e-mail: bgoldengorin@hse.ru; b.goldengorin@rug.nl; pardalos@ufl.edu; vchistyakov@hse.ru; czeslaw@mail.ru

equalities between locally and globally defined tolerances (Lemmas 2(d) and 3(a)).

1. THE PROBLEM OF COMBINATORIAL OPTIMIZATION

Let X be a finite set of cardinality $|X| \geq 2$, called the ground set, and $\mathcal{S} \subset 2^X$ be a collection of nonempty subsets of X. Given a function $C: X \to [0, \infty)$, called the cost function, we define the additive objective function $f = f_C$ on $\mathcal{S}$ by the rule: $f_C(S) = \sum_{x \in S} C(x)$ for all $S \in \mathcal{S}$. A Combinatorial Optimization Problem, determined by the quadruple of the input data $(X, C, \mathcal{S}, f_C)$, abbreviated as COP $(X, C, \mathcal{S}, f)$, is to minimize or maximize the function f on $\mathcal{S}$. To be more specific, in what follows we consider the minimization problem: find $S^* \in \mathcal{S}$ such that $f(S^*) \leq f(S)$ for all $S \in \mathcal{S}$. Any such set S^* is said to be an optimal solution and $f^* = f(S^*) = \min_{S \in \mathcal{S}} f(S)$ —the optimal value of the *COP* $(X, C, \mathcal{S}, f)$. In this respect it is convenient to call the collection $\mathcal{S}$ the set of feasible solutions and denote by $\mathcal{S}^*$ the set of all optimal solutions of the *COP* under consideration. Clearly $\mathcal{S}^* \subset \mathcal{S}$ and $|\mathcal{S}^*| \geq 1$. Since $f(S^*) = f^*$ for all $S^* \in \mathcal{S}^*$, it is also convenient to set $f(\mathcal{S}^*) = f^*$.

The union of the collection $\mathcal{S}$ and its intersection are denoted by $\cup\mathcal{S} = \{x \in X: (\exists S \in \mathcal{S})\, x \in S\}$ and $\cap\mathcal{S} = \{x \in X: (\forall S \in \mathcal{S})\, x \in S\}$, respectively. Clearly, $\cup\mathcal{S}^* \subset \cup\mathcal{S}$ and $\cap\mathcal{S}^* \supset \cap\mathcal{S}$. The equality $\cup\mathcal{S} = \cap\mathcal{S}$ holds if and only if $|\mathcal{S}| = 1$, and the inequality $\cup\mathcal{S} \neq \cap\mathcal{S}$ is equivalent to $|\mathcal{S}| \geq 2$. The *COP* is degenerated if $\cup\mathcal{S} = \cap\mathcal{S}$ (only one feasible solution is available) or $\cap\mathcal{S} \in \mathcal{S}$ ($\cap\mathcal{S}$ is always an optimal solution), and so, in what follows we assume that $|\mathcal{S}| \geq 2$ and $\cap\mathcal{S} \notin \mathcal{S}$.

Let us exhibit examples of *COP*s defined on a simple weighted graph $G = (V, E, C)$ with the set of vertices $V = \{1, 2, \ldots, n\}$ $(n \geq 3)$ and the set of edges $E \subset V \times V$ (or arcs $E = A \subset V \times V$ if the graph is directed), where $C: E \to [0, \infty)$ is a cost function of edges (arcs).

1. If $X = E$ and $\mathcal{S}$ is the set of Hamiltonian cycles S, i.e., cycles of the form $S = \{(i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n), (i_n, i_1)\} \subset E$ (where all pairs from S are different), then the objective function f is of the form $f(S) = \sum_{(i,j) \in S} C(i, j)$ for $S \in \mathcal{S}$, and the corresponding *COP* $(X, C, \mathcal{S}, f)$ is the symmetric traveling salesman problem (in the asymmetric case $X = A$).

2. If $X = A$ and $\mathcal{S}$ is a collection of sets $S_\pi \subset A$ of the form $S_\pi = \{(1, \pi(1)), (2, \pi(2)), \ldots, (n, \pi(n))\}$, corresponding to all permutations $\pi: V \to V$ of V, then the objective function is given by $f(S_\pi) = \sum_{i=1}^{n} C(i, \pi(i))$ for $S_\pi \in \mathcal{S}$, and the resulting *COP* $(X, C, \mathcal{S}, f)$ is the assignment problem.

2. GLOBAL TOLERANCES OF ELEMENTS FROM X

In this section we define numerical characteristics of elements $x \in X$, which express the degree of invariance of optimal solutions to the *COP* with respect to a perturbation of the single cost $C(x)$.

Let the *COP* $(X, C, \mathcal{S}, f)$ be given.

Given $x \in X$ and a number $\alpha \in \mathbb{R}$, we denote by $C_{x,\alpha}: X \to \mathbb{R}$ the perturbed cost function at the element x: $C_{x,\alpha}(y) = C(y)$ if $y \in X$ and $y \neq x$, and $C_{x,\alpha}(x) = C(x) + \alpha$. The global upper tolerance $u(x)$ of x is the least upper bound of those $\alpha \geq 0$, for which any optimal solution S^* of the initial *COP* $(X, C, \mathcal{S}, f)$ with $f = f_C$ is also an optimal solution of the perturbed *COP* $(X, C_{x,\alpha}, \mathcal{S}, f_{C_{x,\alpha}})$. The global lower tolerance $\ell(x)$ of an element x is defined similarly if we replace the perturbed *COP* above by the perturbed *COP* $(X, C_{x,-\alpha}, \mathcal{S}, f_{C_{x,-\alpha}})$. Clearly, $0 \leq u(x), \ell(x) \leq +\infty$, and these values do not depend upon the concrete optimal solution S^* of the original *COP* $(X, C, \mathcal{S}, f)$.

In order to be able to evaluate global tolerances of elements $x \in X$ efficiently, we set $\chi_x(y) = 0$ if $y \in X$ and $y \neq x$, and $\chi_x(x) = 1$, and denote by $\delta_x: 2^X \to \{0, 1\}$ the Dirac measure (point mass) concentrated at x (i.e., given $S \subset X$, we have: $\delta_x(S) = 1$ if $x \in S$, and $\delta_x(S) = 0$ if $x \notin S$). Note that $\delta_x(S) = \sum_{y \in S} \chi_x(y)$ for all $S \subset X$. Since the perturbed cost function is of the form $C_{x,\alpha} = C + \alpha\chi_x$ on X, the perturbed objective function is given by $f_{C_{x,\alpha}} = f + \alpha\delta_x$ on $\mathcal{S}$. Then the global tolerances of the element $x \in X$ are expressed by the formulas:

$$u(x) = \sup\{\alpha \geq 0: (f + \alpha\delta_x)(S^*) = \min_{S \in \mathcal{S}}(f + \alpha\delta_x)(S) \text{ for all } S^* \in \mathcal{S}^*\};$$

$$\ell(x) = \sup\{\alpha \geq 0: (f - \alpha\delta_x)(S^*) = \min_{S \in \mathcal{S}}(f - \alpha\delta_x)(S) \text{ for all } S^* \in \mathcal{S}^*\}.$$

The following lemmas are preparatory for the main results from Section 3. If $x \in X$, we set $\mathcal{S}_+(x) = \{S \in \mathcal{S}: x \in S\}$ and $\mathcal{S}_-(x) = \{S \in \mathcal{S}: x \notin S\}$; these subcollections of $\mathcal{S}$ are disjoint and their union is all of $\mathcal{S}$. In the first lemma we generalize a result from [9].

Lemma 1. *Given a COP* $(X, C, \mathcal{S}, f)$ *and* $x \in X$, *we have*:

(a) $x \in (\cup\mathcal{S}^*)\backslash(\cap\mathcal{S})$ *if and only if* $u(x) < +\infty$, *and if this is the case*, *then* $u(x) = \min_{S \in \mathcal{S}_-(x)} f(S) - f^*$;

(b) $x \in (\cup\mathcal{S})\backslash(\cap\mathcal{S}^*)$ *if and only if* $\ell(x) < +\infty$, *and if this is the case, then* $\ell(x) = \min\limits_{S \in \mathcal{S}_+(x)} f(S) - f^*$.

It follows that $u(x) = +\infty$ is equivalent to $x \in (X\backslash(\cup\mathcal{S}^*)) \cup (\cap\mathcal{S})$, and $\ell(x) = +\infty$ is equivalent to $x \in (X\backslash(\cup\mathcal{S})) \cup (\cap\mathcal{S}^*)$. In particular, $u(x) = +\infty = \ell(x)$ for all $x \in (X\backslash(\cup\mathcal{S})) \cup (\cap\mathcal{S})$. Elements from the set $X\backslash(\cup\mathcal{S})$ do not belong to any feasible solution, and elements from $\cap\mathcal{S}$ belong to all feasible solutions, and so, taking them into account may result in wasting efforts and time during the optimization procedure aiming at solving the *COP* $(X, C, \mathcal{S}, f)$. The following definition excludes this situation.

The *COP* $(X, C, \mathcal{S}, f)$ is said to be canonical if $\cup\mathcal{S} = X$ and $\cap\mathcal{S} = \phi$.

It can be shown that any *COP* can be reduced to a canonical *COP* with the preservation of the values of global upper and lower tolerances. If the *COP* is canonical, then the subcollections $\mathcal{S}_+(x)$ and $\mathcal{S}_-(x)$ are nonempty for all $x \in X$. Let us denote by $[\mathcal{S}_+(x)]^*$ the set of all optimal solutions to the *COP* $(X, C, \mathcal{S}_+(x), f)$ and by $f([\mathcal{S}_+(x)]^*) = \min\limits_{\mathcal{S}_+(x)} f$ —its optimal value; a similar meaning applies to the notations $[\mathcal{S}_-(x)]^*$ and $f([\mathcal{S}_-(x)]^*)$. Assertions of Lemma 1 assume the form:

(a) $u(x) < +\infty$ if and only if $x \in \cup\mathcal{S}^*$, and $u(x) = f([\mathcal{S}_-(x)]^*) - f(\mathcal{S}^*)$;

(b) $\ell(x) < +\infty$ if and only if $x \notin \cap\mathcal{S}^*$, and $\ell(x) = f([\mathcal{S}_+(x)]^*) - f(\mathcal{S}^*)$.

Consequently, $u(x) = +\infty$ if $x \notin \cup\mathcal{S}^*$, and $\ell(x) = +\infty$ if $x \in \cap\mathcal{S}^*$.

Lemma 2. *Given a canonical COP* $(X, C, \mathcal{S}, f)$ *and* $x \in X$, *we have*:

(a) $u(x) = 0$ *if and only if* $x \in (\cup\mathcal{S}^*)\backslash(\cap\mathcal{S}^*)$, *which is equivalent to the condition* $\ell(x) = 0$ (*in this case* $|\mathcal{S}^*| \geq 2$);

(b) $x \in \cap\mathcal{S}^*$ *if and only if* $0 < u(x) < +\infty$;

(c) $x \notin \cup\mathcal{S}^*$ *if and only if* $0 < \ell(x) < +\infty$.

The uniqueness of the optimal solution is characterized as follows:

(d) $|\mathcal{S}^*| = 1$ *if and only if* $0 < u(x) < +\infty$ *for all* $x \in \cup\mathcal{S}^*$;

(e) $|\mathcal{S}^*| = 1$ *if and only if* $0 < \ell(x) < +\infty$ *for all* $x \in X\backslash(\cap\mathcal{S}^*)$.

In [5, 6, 8, 11] upper tolerances $u_{S^*}(x)$ and lower tolerances $\ell_{S^*}(x)$ of $x \in X$ with respect to a given optimal solution S^* of the *COP* $(X, C, \mathcal{S}, f)$ under consideration have been studied and applied for different purposes. At the end of this section we establish their relationship with global tolerances $u(x)$ and $\ell(x)$. In the notation above the tolerances with respect to S^* are expressed by the relations:

$$u_{S^*}(x) = \sup\{\alpha \geq 0\colon (f + \alpha\delta_x)(S^*) = \min_{S \in \mathcal{S}}(f + \alpha\delta_x)(S)\},$$

$$\ell_{S^*}(x) = \sup\{\alpha \geq 0\colon (f - \alpha\delta_x)(S^*) = \min_{S \in \mathcal{S}}(f - \alpha\delta_x)(S)\},$$

and in the case of a canonical *COP* we have the following equalities:

$$u(x) = \min_{S^* \in \mathcal{S}^*} u_{S^*}(x) \text{ and } \ell(x) = \min_{S^* \in \mathcal{S}^*} \ell_{S^*}(x) \quad \text{for all } x \in X.$$

The uniqueness of the optimal solution can be characterized by certain relations between tolerances and global tolerances.

Lemma 3. *Given a canonical COP* $(X, C, \mathcal{S}, f)$, *we have*:

(a) $|\mathcal{S}^*| = 1$ *if and only if there exists an* $S^* \in \mathcal{S}^*$ *such that* $u = u_{S^*}$ *and* $\ell = \ell_{S^*}$ *on* X;

(b) $|\mathcal{S}^*| \geq 2$ *if and only if for all* $S^* \in \mathcal{S}^*$, *we find* $u \neq u_{S^*}$ *or* $\ell \neq \ell_S$ *on* X.

The above is illustrated by the following simple example.

Example 1. Let $X = \{x_1, x_2, x_3\}$, $C(x_1) = 0$, $C(x_2) = 1$, $C(x_3) = 2$ and $\mathcal{S} = \{S_1, S_2, S_3\}$, where $S_1 = \{x_1, x_2\}$, $S_2 = \{x_2\}$ and $S_3 = \{x_3\}$. Since $f(S_1) = C(x_1) + C(x_2) = 1$, $f(S_2) = C(x_2) = 1$ and $f(S_3) = 2$, the set of optimal solutions is $\mathcal{S}^* = \{S_1^*, S_2^*\}$, where $S_1^* = S_1$, $S_2^* = S_2$ and $f^* = 1$. The values of all types of tolerances are exposed in the following two tables:

x	x_1	x_2	x_3
$u(x)$	0	1	$+\infty$
$u_{S_1^*}(x)$	0	1	$+\infty$
$u_{S_2^*}(x)$	$+\infty$	1	$+\infty$

x	x_1	x_2	x_3
$\ell(x)$	0	$+\infty$	1
$\ell_{S_1^*}(x)$	$+\infty$	$+\infty$	1
$\ell_{S_2^*}(x)$	0	$+\infty$	1

3. EXTREMAL VALUES OF TOLERANCES

In this section we gather the main results of the paper concerning the extremal values of global upper and lower tolerances.

Assume that the *COP* $(X, C, \mathcal{S}, f)$ is canonical.

We say that the collection of feasible solutions $\mathcal{S}$ consists of nonembedded (into each other) sets provided $S_1\backslash S_2 \neq \phi$ for all $S_1, S_2 \in \mathcal{S}$, $S_1 \neq S_2$.

Theorem 1. (a) *If $\mathcal{S}$ consists of nonembedded sets and $S^* \in \mathcal{S}^*$, then*

$$\min_{y \in X \backslash S^*} \ell(y) = \min_{y \in X \backslash (\cap \mathcal{S}^*)} \ell(y)$$
$$= \min_{x \in \cup \mathcal{S}^*} u(x) = \min_{x \in \mathcal{S}^*} u(x).$$

(b) *If the collection of feasible solutions $\mathcal{S}$ is arbitrary, then*

$$\min_{y \in X \backslash (\cap \mathcal{S}^*)} \ell(y) \le \min_{x \in \cup \mathcal{S}^*} u(x)$$
$$\le \min_{y \in X \backslash [(\cap \mathcal{S}^*) \cup (\cup \mathcal{S}_0)]} \ell(y),$$

where $\mathcal{S}_0 = \{S_0 \in \mathcal{S}: \cup \mathcal{S}^* \subset S_0\}$ $(\min \phi = +\infty)$. *In particular, if sets from $\mathcal{S}$ are nonembedded, then $\mathcal{S}_0 = \phi$ and* $\min_{y \in X \backslash (\cap \mathcal{S}^*)} \ell(y) = \min_{x \in \cup \mathcal{S}^*} u(x)$.

(c) *If the cost function C is positive and $S^* \in \mathcal{S}^*$ then*

$$\left(\min_{y \in X \backslash S^*} \ell(y) \le \min_{x \in \cup \mathcal{S}^*} u(x) \le \min_{x \in S^*} u(x) \right)$$
$$\le \min_{y \in X \backslash [(\cap \mathcal{S}^*) \cup (\cup \mathcal{S}_0)]} \ell(y) \le \min_{y \in X \backslash [S^* \cup (\cup \mathcal{S}_0)]} \ell(y),$$

where $\mathcal{S}_0 = \mathcal{S}_0(S^*) = \{S_0 \in \mathcal{S}: S^* \subset S_0\}$.

Appropriate examples show that the assumptions in Theorem 1 are essential for its validity and all inequalities may be strict.

The case of maximal values of global upper and global lower tolerances seems to be more difficult. We have the following partial result, in which notations introduced before Lemma 2 are applied.

Theorem 2. *Suppose that the collection of feasible solutions $\mathcal{S}$ of the COP $(X, C, \mathcal{S}, f)$ consists of nonembedded sets and $S^* \in \mathcal{S}^*$ is the unique optimal solution of the COP. We have*:

(a) *if* $\left(\bigcup_{x \in \mathcal{S}^*} (\cup [\mathcal{S}_-(x)]^*) \right) \backslash S^* = X \backslash S^*$, *then* $\max_{y \in X \backslash S^*} \ell(y) \le \max_{x \in S^*} u(x)$;

(b) *if* $S^* \subset \bigcup_{y \in X \backslash S^*} (X \backslash (\cap [\mathcal{S}_+(y)]^*))$, *then* $\max_{x \in S^*} u(x) \le \max_{y \in X \backslash S^*} \ell(y)$.

ACKNOWLEDGMENTS

This work has been carried out at the Laboratory of Algorithms and Technologies for Networks Analysis, National Research University Higher School of Economics (NRU HSE), and supported by the Ministry of Education and Science of Russian Federation, Grant no. 11.G34.31.0057. The first two authors were also supported by the Scientific Foundation of the NRU HSE, Project "Teachers–Students" no. 11-04-0008.

REFERENCES

1. E. Balas and M. J. Saltzman, Oper. Res. **39**, 150–161 (1991).
2. R. Germs, B. Goldengorin, and M. Turkensteen, Comput. Oper. Res. **39**, 291–298 (2012).
3. D. Ghosh, B. Goldengorin, G. Gutin, and G. Jäger, in *Mathematical Programming and Game Theory for Decision Making* (World Scientific, Singapore, 2008), pp. 47–59.
4. B. Goldengorin, G. Jäger, and P. Molitor, J. Comput. Sci. **2**, 716–734 (2006).
5. B. Goldengorin and G. Sierksma, *SOM Research Report 03a30* (Univ. of Groningen, Groningen, The Netherlands, 2003), pp. 1–6.
6. D. Gusfield, Networks **13**, 191–196 (1983).
7. G. Gutin, B. Goldengorin, and J. Huang, J. Heuristics **14**, 169–181 (2008).
8. G. Jäger, *The Theory of Tolerances with Applications to the Traveling Salesman Problem* (Habilitationsschrift, Kiel, Germany, 2010).
9. M. Libura, Discret. Appl. Math. **30** (2–3), 197–211 (1991).
10. N. V. Reinfeld and W. R. Vogel, *Mathematical Programming* (Englewood Cliffs, N.J., Prentice-Hall, NJ, 1958).
11. D. R. Shier and C. Witzgall, Networks **10**, 277–291 (1980).
12. M. Turkensteen, D. Ghosh, B. Goldengorin, and G. Sierksma, Eur. J. Oper. Res. **189**, 775–788 (2008).