

Statistical Estimation of the Jump Activity for Time-changed Lévy Processes

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Abstract

This paper is devoted to studying the problem of the statistical inference on the activity of jumps for a class of the so-called time-changed Lévy processes, i.e., for the processes in the form $Y_s = X_{\mathcal{T}(s)}$, where X is a Lévy process and \mathcal{T} is a non-negative and non-decreasing stochastic process, which is referred to as time change. First, starting from some natural assumptions on the Lévy measure of X , we infer on the asymptotic behavior of the characteristic function of Y . Next, we present a new method, which allows to consistently estimate the activity of small jumps in the difficult case of low-frequency data.

1. Introduction

In modern statistics, Lévy processes are very often used to model financial time series. The popularity of such models is based on their simplicity on the one hand and the ability to reproduce many specific properties of the economic data on the other hand (see [4], [11], [18]). In this article, we draw our attention to one modification of this class of processes.

Consider two one-dimensional real-valued not necessarily independent stochastic processes - $X = (X_t)_{t \geq 0}$ and $\mathcal{T} = (\mathcal{T}(s))_{s \geq 0}$. Let X be a Lévy process and \mathcal{T} be a non-negative, non-decreasing stochastic process with $\mathcal{T}(0) = 0$. Then the time-changed Lévy process is defined as $Y = (Y_s)_{s \geq 0} = X_{\mathcal{T}(s)}$. The change of time follows the idea that some economical effects (e.g., nervousness of the market which is indicated by volatility) can be expressed by “business” time, which runs faster than physical time in some periods.

This underlined class of processes is very large; e.g. Monroe [16] shows that even in the case of the Brownian motion X , the class $\{Y\}$ basically coincides with the class of all semimartingales. The problem of statistical inference for time-changed processes have got peculiar attention in the past decade, see [5] for an overview.

In this paper, it is assumed that the discrete observations $Y_0, Y_\Delta, \dots, Y_{n\Delta}$ of the process $Y = (Y_s)$ are available for some fixed $\Delta > 0$. So, we consider a low-frequency setup, which is in fact more difficult, see the discussion below. Another difficulty related to this model is that neither the process X nor \mathcal{T} is directly observable; this leads to the problems related to the composite function estimation (see [5]).

The paper is organized as follows. In the next section, we present the aims of the research, and give a short overview of the considered problems. Next, we introduce the main object of our study, the time-changed Lévy processes, and discuss the additional assumptions on the processes X and \mathcal{T} . Section 4 is devoted to some preliminary remarks concerning the increments of the characteristic functions of Y . Section 5 contains the Abelian theorem for the underlined class of models. Estimation algorithm for the Blumenthal-Gettoor index of X is given in Section 6. Theoretical results showing the consistency of the proposed estimators and the rates of convergence are presented in Section 7. We provide the reader with an numerical example in Section 8.

2. Aims of the research

This research has two aims.

1. The first aim is to derive the so-called Abelian theorem describing the asymptotic behavior of the characteristic function of increments of Y in terms of the asymptotic behavior of the Lévy measure of X at zero. Such Abelian theorems (and closely connected to them the so-called Tauberian theorems) are known in the literature for series and integrals (see [15] and [20]); perhaps the most famous result of this type is the Karamata Tauberian theorem (see [7]). For the Lévy processes, the corresponding result was proven by Bismut [8], and for the affine stochastic volatility models - by Belomestny and Panov [6]. The aim of this research is to prove the Abelian theorem for a broad class of time-changed Lévy processes, which in particular involves many

volatility models (see [10]).

2. The second objective of this article is to estimate the characteristics of activity of small jumps. For Itô semimartingales, Aït-Sahalia and Jacod [1] proposed a method which is able to consistently estimate the so-called jump activity index in the case of high-frequency data (i.e., under the assumption that the frequency of observations tends to infinity). It is emphasized in the paper by Aït-Sahalia and Jacod [1] as well as in other publications on this topic, that the case of low-frequency data (observation horizon tends to infinity, when the frequency of observations is fixed) is much more difficult; one may wonder if any kind of statistical inference is possible in this situation at all. The first results showing that a consistent estimation of the BG index based on low-frequency data is possible, were obtained by Belomestny [4] for the case of Lévy processes. Later, this methodology was applied to a broad class of affine stochastic volatility models (see [6], [17]). In this respect, the second aim of the paper is to estimate the characteristics of small jumps for time-changed Lévy processes.

More precisely, we are interested in the estimation of the so-called Blumenthal -Gettoor index, which is in fact a quite reasonable characteristic of the activity of small jumps (see [17]). For a one-dimensional Lévy process $Z = (Z_t)_{t \geq 0}$ with a Lévy measure ν , the Blumenthal-Gettoor index of Z is defined as

$$\text{BG}(Z) = \inf \left\{ r > 0 : \int_{|x| \leq 1} |x|^r \nu(dx) < \infty \right\}.$$

From practical point of view, the importance of the Blumenthal-Gettoor index lies in the fact that it determines the smoothness properties of the marginal density of Z and has significant impact on the convergence of different approximation algorithms (see [12] and [13]).

3. Set-up

3.1. Lévy process X

In this paper, we assume that the process X_t is a one-dimensional Lévy process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This in particular means that the characteristic function of X allows the following representation

$$\phi(u) := \mathbb{E} \left[\exp \left\{ iu^\top X_t \right\} \right] = \exp \{ t \psi(u) \},$$

where the function $\psi(u)$ is called the characteristic exponent of X . The Lévy-Khintchine formula yields

$$\psi(u) = i\mu u - \frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R} \setminus \{0\}} \left(e^{iux} - 1 - iux \cdot \mathbf{1}_{\{|x| \leq 1\}} \right) \nu(dx),$$

where $\mu \in \mathbb{R}$, σ is positive and ν is a Lévy measure on $\mathbb{R} \setminus \{0\}$, which satisfies

$$\int_{\mathbb{R} \setminus \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty.$$

A triplet (μ, σ^2, ν) is usually called a characteristic triplet of the Lévy process X_t . In this paper, we assume that

(AM)

$$\int_{|x| > \varepsilon} \nu(dx) = \varepsilon^{-\gamma} (\beta_0 + \beta_1 \varepsilon^\chi (1 + O(\varepsilon))), \quad \varepsilon \rightarrow +0$$

with $0 < \chi < \gamma < 2$ and $\beta_0 > 0$. This assumption is mild; we refer to Panov [17] for the detailed discussion.

3.2. Time change

Let $\mathcal{T} = (\mathcal{T}(s))_{s \geq 0}$ be an increasing right-continuous process with left limits such that $\mathcal{T}(0) = 0$ and for each fixed s , the random variable $\mathcal{T}(s)$ is a stopping time with respect to the filtration \mathcal{F} . This setup is quite typical for the processes that are referred to as time change (see, e.g., the book by Barndorff-Nielsen and Shiryaev [3]).

In this paper, it is assumed that

(A1T) process \mathcal{T} has stationary and ergodic increments;

(A2T) processes X and \mathcal{T} are independent;

(A3T) the density function of \mathcal{T}_Δ allows the representation

$$p^\Delta(x) = x^{\alpha-1} f(x), \quad (1)$$

with positive α and infinitely smooth function $f(x)$.

4. Preliminary remarks

Our first remark is that the compound process Y_s is not a Lévy process, since the increments of it are not any longer independent. This yields several technical difficulties, see [5] for discussion.

As it was already mentioned in Section 2, the first objective of this paper is to infer on the asymptotic behavior of the characteristic function of $Y_{t+\Delta} - Y_t$, which is denoted by $\phi^\Delta(u)$. It is worth mentioning that under the assumptions (A1T) and (A2T),

$$\phi^\Delta(u) = \mathbb{E} \exp \{ \mathcal{T}_\Delta \psi(u) \},$$

see [5], [9] for details. Therefore,

$$\left| \phi^\Delta(u) \right| = \mathbb{E} \exp \{ \mathcal{T}_\Delta \text{Re}(\psi(u)) \}.$$

The expression in the right-hand side can be considered as the Laplace transform of the density function of \mathcal{T}_Δ at the point $-\text{Re}(\psi(u))$.

5. Asymptotic behavior of the characteristic function

The first theorem can be viewed as the Abelian theorem for time - changed Lévy processes.

Theorem 5.1 *Consider the process $Y_s := X_{\mathcal{T}(s)}$, where the processes X_t and \mathcal{T}_s satisfy the conditions (AM), (A1T)-(A3T). Then the characteristic function of the increments of Y_s allows the following representation*

$$\left| \phi^\Delta(u) \right| = \frac{c_\alpha}{u^{2\alpha}} \left(1 - \frac{\tau_1 \alpha d_\gamma}{u^{2-\gamma}} - \frac{\tau_2 \alpha d_{\gamma-\chi}}{u^{2-(\gamma-\chi)}} (1 + o(1)) \right), \quad u \rightarrow \infty, \quad (2)$$

with some positive constants τ_1, τ_2 , and

$$c_\alpha = \frac{\Gamma(\alpha) f(0)}{(1/2\sigma^2)^\alpha}, \quad d_\gamma = \Gamma(1-\gamma) \sin((1-\gamma)\pi/2).$$

The proof of Theorem 5.1 is based on analyzing the asymptotic behavior of the characteristic exponent under the assumption (AM) (see [17]) and the asymptotic behavior of the Laplace transform (the so-called Watson lemma, [14]).

In comparison with the representation for the affine stochastic volatility models (see [6] and [17]), this result has at least two important drawbacks. First, the main term in the asymptotic behavior of the characteristic function involves the parameter α , which is unknown. Second, the natural estimators of the Blumenthal-Gettoor index γ depend on the parameter α , and therefore the estimator of α is included in the estimator of γ ; such construction makes the proofs of the properties of the latter estimator complicated.

6. Estimation of the Blumenthal - Gettoor index

The main idea is close to the methodology introduced by Belomestny [4] related to the Lévy processes. Similar ideas were later applied by Belomestny and Panov [6] to the affine stochastic volatility models.

6.1. The first step: estimation of the characteristic function

First, estimate $\phi^\Delta(u)$ by its empirical counterpart $\phi_n(u)$ defined as

$$\phi_n(u) := \frac{1}{n} \sum_{k=1}^n e^{iu(Y_{\Delta k} - Y_{\Delta(k-1)})}. \quad (3)$$

Note that by virtue of the Birkhoff ergodic theorem (see [2]),

$$\frac{1}{n} \sum_{k=1}^n e^{iu(Y_{\Delta k} - Y_{\Delta(k-1)})} \longrightarrow \phi^\Delta(u), \quad n \rightarrow \infty,$$

almost surely and in $L1$. For further properties of the estimator $\phi_n(u)$, we refer to [19].

6.2. The second step: estimation of α

Introduce a weighting function $w^{U_n}(u) = U_n^{-1} w^1(u/U_n)$, where U_n is a sequence of positive numbers tending to infinity, the smooth function w^1 is supported on $[\varepsilon, 1]$ for some $\varepsilon > 0$ and satisfies

$$\int_\varepsilon^1 w^1(u) du = 0, \quad \int_\varepsilon^1 w^1(u) \log u du = 1. \quad (4)$$

The examples of such weighting functions are given in [17]. Define an estimate for the parameter α in (2) by

$$\alpha_n := -\frac{1}{2} \int_0^\infty w^{U_n}(u) \log |\phi_n(u)| du. \quad (5)$$

This estimate can be alternatively defined as the solution $\alpha_n = \gamma_{opt}$ of the following optimization problem:

$$\int_0^\infty \tilde{w}^{U_n}(u) \left(\frac{1}{2} \log |\phi_n(u)| + \gamma \log(u) + \beta \right)^2 du \rightarrow \min_{\beta, \gamma}, \quad (6)$$

where $\tilde{w}^{U_n}(u)$ is a smooth positive function on \mathbb{R} that allows the representation

$$\tilde{w}^{U_n}(u) = \frac{1}{U_n} \tilde{w}^1 \left(\frac{u}{U_n} \right)$$

with some function \tilde{w}^1 supported on the interval $[\varepsilon, 1]$.

6.3. The third step: estimation of the Blumenthal-Gettoor index

Similar to (5), we introduce the estimator of γ by

$$\gamma_n(\alpha_n) := 2 - \int_0^\infty w^{V_n}(u) \log \left(1 - \frac{|\phi_n(u)| u^{2\alpha_n}}{c_{\alpha_n}} \right) du, \quad (7)$$

where V_n is a sequence of positive numbers tending to infinity. This estimate also allows the representation as the solution of the optimization problem in the spirit of (6).

7. Properties of the estimators

7.1. Properties of α_n

In order to see that α_n is a reasonable estimate for α , we introduce the deterministic quantity

$$\bar{\alpha}_n = -\frac{1}{2} \int_0^\infty w^{U_n}(u) \log |\phi^\Delta(u)| du,$$

which can be seen as the theoretical counterpart of α_n . The next lemma shows that the quantity $\bar{\alpha}_n$ is close to the true value α .

Lemma 7.1 For any n large enough, it holds

$$|\alpha - \bar{\alpha}_n| \leq C^{(1)} \frac{\tau_1 \alpha d_\gamma}{(\varepsilon U_n)^{2-\gamma}}, \quad u \rightarrow \infty, \quad (8)$$

where $C^{(1)} > 0$ is not depending on the parameters of the underlined model.

Next step is to show that α_n converges to $\bar{\alpha}_n$ in probability. The exact formulation is given in the next theorem:

Theorem 7.2 Let the assumptions of Theorem 5.1 be fulfilled. Let also the sequence U_n be such that $U_n = O(n^q)$ with some positive q . Then

$$\mathbb{P} \left\{ \frac{\sqrt{n}}{\log n} |\bar{\alpha}_n - \alpha_n| > B_1^{(1)} \frac{U_n^{2\alpha}}{c_\alpha} \right\} \leq B_2 n^{-1-\delta}, \quad (9)$$

where $B_1^{(1)}$, B_2 , and δ are positive.

Lemma 7.1 and Theorem 7.2 give that with a probability larger than $1 - B_2 n^{-\delta-1}$,

$$\begin{aligned} |\alpha - \alpha_n| &\leq |\alpha - \bar{\alpha}_n| + |\bar{\alpha}_n - \alpha_n| \\ &\leq \lambda_1 (U_n)^{\gamma-2} + \lambda_2 \frac{\log n}{\sqrt{n}} U_n^{2\alpha}, \end{aligned}$$

where $\lambda_1, \lambda_2 > 0$.

Taken U_n such that the summands in this representation are equal to each other, we arrive at the following rate of convergence:

$$\mathbb{P} \left\{ |\alpha - \alpha_n| \leq \Lambda(\alpha, \gamma) \left(\frac{\log n}{\sqrt{n}} \right)^{(2-\gamma)/(2\alpha+2-\gamma)} \right\} > 1 - B_2 n^{-\delta-1},$$

where $\Lambda(\alpha, \gamma) > 0$.

7.2. Properties of γ_n : case of known α

In this subsection, we assume that the parameter α is known. Similar to Section 7.1, we introduce the deterministic quantity

$$\bar{\gamma}_n(\alpha) := 2 + \int_0^\infty w^{U_n}(u) \log \left(1 - \frac{|\phi^\Delta(u)| u^{2\alpha}}{c_\alpha} \right) du. \quad (10)$$

Lemma 7.3 For any n large enough, it holds

$$|\gamma - \bar{\gamma}_n(\alpha)| \leq C^{(2)} \frac{d_{\gamma-\chi} d_\gamma^{-1}}{(\varepsilon U_n)^\chi}, \quad u \rightarrow \infty, \quad (11)$$

where $C^{(2)} > 0$ is not depending on the parameters of the underlined model.

Theorem 7.4 Let the sequence V_n be such that $V_n = o\left((\sqrt{n}/\log n)^{1/(2-\gamma+2\alpha)}\right)$. Then

$$\begin{aligned} \mathbb{P} \left\{ \frac{\sqrt{n}}{\log n} |\bar{\gamma}_n(\alpha) - \gamma_n(\alpha)| > \frac{B_1^{(2)}}{\alpha c_\alpha d_\gamma} V_n^{2-\gamma+2\alpha} \right\} \\ \leq B_2 n^{-1-\delta}, \end{aligned} \quad (12)$$

where $B_1^{(2)}$, B_2 , and δ are positive.

Next, we combine Lemma 7.3 and Theorem 7.4, using the ideas given at the end of Section 7.1 with respect to the estimation of α . This methodology leads to the following rate of convergence:

$$\begin{aligned} \mathbb{P} \left\{ |\gamma_n(\alpha) - \gamma| \leq \Upsilon(\alpha, \gamma, \chi) \left(\frac{\log n}{\sqrt{n}} \right)^{\chi/(2-\gamma+2\alpha+\chi)} \right\} \\ > 1 - B_2 n^{-\delta-1}, \end{aligned} \quad (13)$$

where $\Upsilon(\alpha, \gamma, \chi) > 0$.

7.3. Properties of γ_n : case of unknown α

This subsection is devoted to the realistic case of unknown α . The next theorem is the main result of this study. This theorem shows that the proposed estimator of the parameter γ is consistent.

Theorem 7.5 There exist positive constants B and δ such that

$$\begin{aligned} \mathbb{P} \{ |\gamma_n(\alpha_n) - \gamma| \\ \leq \Psi_1 V_n^{-\chi} + \Psi_2 \frac{\log n}{\sqrt{n}} V_n^{2\alpha+(2-\gamma)} + \Psi_3 V_n^{2-\gamma} \log V_n |\alpha_n - \alpha| \} \\ > 1 - B n^{-1-\delta}, \end{aligned}$$

where Ψ_1, Ψ_2 are Ψ_3 are some known functions depending on α and γ .

Corollary 7.6 From this theorem, one can choose the sequences U_n and V_n such that $\gamma_n(\alpha_n)$ converges to γ in probability. Moreover, one can choose the sequences U_n and V_n such that the rates of convergence in the case of unknown α is the same as in the case of known α (see the previous section).

8. Numerical example

Let us take for a Lévy process X_t the Brownian motion W_t , and for a time change $\mathcal{T}(s)$ - the gamma process with shape parameter $a = 1$ and scale parameter $b = 1$. It is a worth mentioning that the parameter α is equal to a , and the function f is equal to $f(x) = b^a \exp\{bx\} I\{x \geq 0\} / \Gamma(a)$. The parameter γ is equal to the Blumenthal-Gettoor index of the Lévy process, which is 2 in the case of the Brownian motion.

The estimation procedure for the parameters α and γ was introduced in Section 6. Here we choose this parameters as the solutions of the optimization problems. More precisely, we solve the optimization problem

$$\int_{U_{low}}^{U_{up}} \left\{ \frac{1}{2} \log |\phi_n(uU_n)| - \gamma \log(uU_n) - \beta \right\}^2 du \rightarrow \min_{\beta, \gamma}$$

where U_{low} and U_{up} are truncation levels, U_n tends to infinity as $n \rightarrow \infty$, and define the estimate $\alpha_n =$

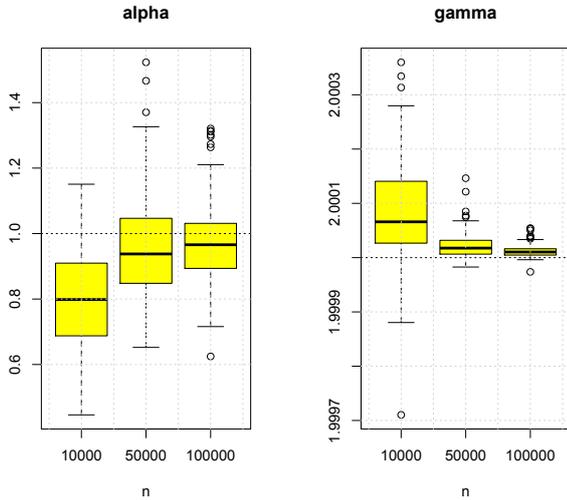


Figure 1. Boxplots of the estimate α_n and boxplots of the BG index estimate for different values of n based on 100 independent simulation runs for the time-changed Lévy model with Brownian motion as the Lévy process and Gamma process as the time change. True values are $\alpha = 1$ and $\gamma = 2$.

$-\gamma_{opt}/2$. In this numerical example, we take n equal to 10.000, 50.000 and 100.000, and U_n equal to 1, 2 and 3 resp. Truncation levels are equal to 0.1 and 0.5; this choice is based on the observation that the empirical characteristic function behaves (approximately) linearly on the segment $[0.1, 0.5]$.

Next, we optimize the expression

$$\int_{U_{low}}^{U_{up}} \left\{ \log \left(1 - \frac{|\phi_n(uU_n)| (uU_n)^{2\alpha_n}}{c_{\alpha_n}} \right) - \gamma \log(u) - \beta \right\}^2 du$$

with respect to β and γ . and define an estimate $\gamma_n(\alpha_n) = 2 - \beta_{opt}$.

Figure 1 shows the boxplots of the resulting estimates α_n and γ_n as a function of n based on 100 independent simulation runs.

9. Conclusion

In this article, we study the problem of estimating the Blumenthal-Gettoor index from a low-frequency observations of the time-changed Lévy process. We introduce an algorithm and show that the proposed estimator is able to consistently estimate the BG index.

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