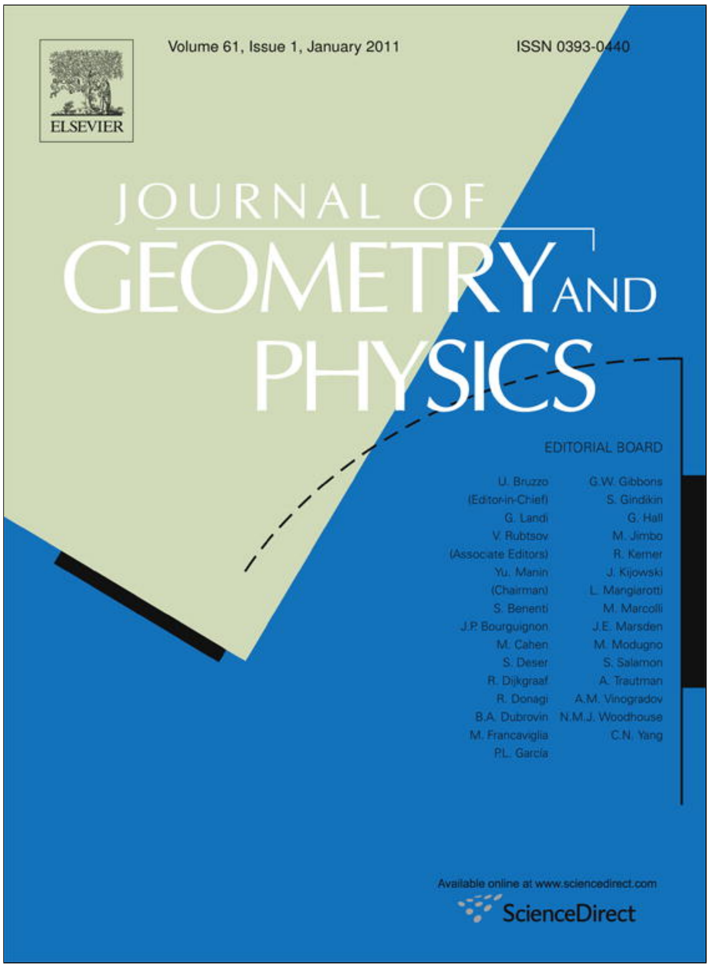

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Algebra of differential operators associated with Young diagrams

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To the memory of Vladimir Igorevich Arnold

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ABSTRACT

We establish a correspondence between Young diagrams and differential operators of infinitely many variables. These operators form a commutative associative algebra isomorphic to the algebra of the conjugated classes of finite permutations of the set of natural numbers. The Schur functions form a complete system of common eigenfunctions of these differential operators, and their eigenvalues are expressed through the characters of symmetric groups. The structure constants of the algebra are expressed through the Hurwitz numbers.

1. Introduction

The center of the group algebra A_n of the symmetric group S_n plays the main role in describing representations both of the symmetric group and of the matrix group $Gl(n)$. Its counterpart for the infinite symmetric group is the algebra A_∞ of the conjugated classes of finite permutations of an infinite set [1]. Its natural generators are the Young diagrams of arbitrary degree.

In the present paper, which is a continuation of [2], we construct an exact representation of algebra A_∞ in the algebra of differential operators of infinitely many variables. The differential operators $\mathsf{W}(\Delta)$, corresponding to the Young diagrams Δ, are closely related to the Hurwitz numbers, matrix integrals and integrable systems [3–7]. We prove that the Schur functions form a complete set of the common eigenfunctions of the operators $\mathsf{W}(\Delta)$, and find the corresponding eigenvalues. A key role in the construction is played by the Miwa variables, which naturally emerge in matrix models [8,9].

In Section 2, we define the algebra of Young diagrams, which is isomorphic to the algebra of conjugated classes of finite permutations of an infinite set, and express its structure constants through the structure constants of the algebra A_n. In Section 3, we construct a representation of the universal enveloping algebra $U(gl(\infty))$ in the algebra of differential operators of Miwa variables. Using this representation, in Section 4, we associate with any Young diagram a differential operator $\mathcal{W}(\Delta)$ of Miwa variables, which has a very simple form. This correspondence gives rise to an exact representation of the algebra A_∞.

The operators $\mathcal{W}(\Delta)$ preserve the subspace P of all symmetric polynomials of the Miwa variables. We study further the differential operators $\mathsf{W}(\Delta) = \mathcal{W}(\Delta)|_{\mathsf{P}}$ of the variables $p = \{p_i\}$, which form a natural basis in the space P. In Section 5, we prove that the Schur functions $s_R(p)$ form a complete system of eigenfunctions for $\mathsf{W}(\Delta)$ and find the corresponding

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eigenvalues. In Section 6, we explain an algorithm of calculating the operators $W(\Delta)$, the simplest non-trivial operator $W([2])$ being nothing but the "cut-and-join" operator [10], which plays an important role in the theory of Hurwitz numbers and moduli spaces.

In the last Section 7, we interpret the operators $W(\Delta)$ as counterparts of the "cut-and-join" operator for the arbitrary Young diagram. In particular, we prove that a special generating function of Hurwitz numbers satisfies a simple differential equation, which allows one to construct all the Hurwitz numbers successively.

2. Algebra A_∞ of Young diagrams

1. First we recall the standard facts that we need below. Denote through $|\mathfrak{M}|$ the number of elements in a finite set $\mathfrak{M}$ and through S_n the symmetric group which acts by permutations on the set $\mathfrak{M}$, where $|\mathfrak{M}| = n$. A permutation $g \in S_n$ gives rise to a subgroup $\langle g\rangle$, whose action divides $\mathfrak{M}$ into orbits $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$. The set of numbers $|\mathfrak{M}_1|, \ldots, |\mathfrak{M}_k|$ is called the *cyclic type of the permutation* g. It produces the Young diagram $\Delta(g) = [|\mathfrak{M}_1|, \ldots, |\mathfrak{M}_k|]$ of degree n. The permutations are conjugated in S_n if and only if they are of the same cyclic type.

Linear combinations of the permutations from S_n form the group algebra $G_n = G(S_n)$. Multiplication in this algebra is denoted as "$\circ$". Associate with each Young diagram Δ the sum $G_n(\Delta) \in G_n$ of all permutations of the cyclic type Δ. These sums form the basis *of the algebra of the conjugated classes* $A_n^\circ \subset G_n$, which coincides with the center G_n.

Denote through $C^{\Delta}_{\Delta_1,\Delta_2}$ the structure constants of the algebra A_n in this basis. In other words,

$$G_n(\Delta_1) \circ G_n(\Delta_2) = \sum_{\Delta\in\mathcal{A}_n} C^{\Delta}_{\Delta_1,\Delta_2} G_n(\Delta),$$

where $\mathcal{A}_n$ is the set of all Young diagrams Δ of degree $|\Delta| = n$.

The construction of algebra A_n° is continued in [1] to the algebra A_∞ of the conjugated classes of finite permutations of the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. The algebra A_∞ is generated by $G_\infty(\Delta)$, which are a formal sum of all finite permutations of the set $\mathbb{N}$ of the cyclic type Δ. Multiplication in the algebra is generated by the multiplication of permutations. According to [1], this algebra is naturally isomorphic to the algebra of the shifted Schur functions [11].

2. We express now the structure constants of the algebra A_∞ through the structure constants of the algebra A_n°. First, we represent the algebra A_n° as an algebra generated by Young diagrams. In other words, we consider A_n° as a vector space with the basis $\mathcal{A}_n$ and multiplication

$$\Delta_1 \circ \Delta_2 = \sum_{\Delta\in\mathcal{A}_n} C^{\Delta}_{\Delta_1,\Delta_2} \Delta.$$

Consider also the monomorphism of the vector space $\rho_k : A_n^\circ \to A_{n+k}^\circ$, where $\rho_k(\Delta) = \frac{(r+k)!}{r!k!}\Delta^k$. Here Δ^k is the Young diagram obtained from the Young diagram Δ by adding k unit length rows and r is the number of unit length rows originally present in the diagram Δ.

Let us define a multiplication of diagrams of arbitrary degree by the formula

$$\Delta_1\Delta_2 = \sum_{n=\max\{|\Delta_1|,|\Delta_2|\}}^{|\Delta_1|+|\Delta_2|} \{\Delta_1\Delta_2\}_n$$

where

$$\{\Delta_1\Delta_2\}_n = \begin{cases} \rho_{(n-|\Delta_1|)}(\Delta_1) \circ \rho_{(n-|\Delta_2|)}(\Delta_2) & \text{for } n = \max\{|\Delta_1|, |\Delta_2|\} \\ \rho_{(n-|\Delta_1|)}(\Delta_1) \circ \rho_{(n-|\Delta_2|)}(\Delta_2) - \displaystyle\sum_{k=\max\{|\Delta_1|,|\Delta_2|\}}^{n-1} \rho_{(n-k)}(\{\Delta_1\Delta_2\}_k) & \text{for } n > \max\{|\Delta_1|, |\Delta_2|\}. \end{cases}$$

Example 2.1. Put $\Delta_1 = [1]$ and $\Delta_2 = [2]$. Then, $\rho_1([1]) = 2[1, 1]$, $\rho_2([1]) = 3[1, 1, 1]$, $\rho_1([2]) = [2, 1]$. Therefore, $\{\Delta_1\Delta_2\}_2 = 2[1, 1] \circ [2] = 2[2]$, $\{\Delta_1\Delta_2\}_3 = 3[1, 1, 1] \circ [2, 1] - 2[2, 1] = [2, 1]$.

Example 2.2. Put $\Delta_1 = [2]$ and $\Delta_2 = [2]$. Then, $\rho_1([2]) = [2, 1]$, $\rho_2([2]) = [2, 1, 1]$, $\rho_1([1, 1]) = 3[1, 1, 1]$, $\rho_2([1, 1]) = 6[1, 1, 1, 1]$, $\rho_1([3]) = [3, 1]$. Therefore, $\{\Delta_1\Delta_2\}_2 = [2] \circ [2] = [1, 1]$, $\{\Delta_1\Delta_2\}_3 = [2, 1] \circ [2, 1] - 3[1, 1, 1] = 3[3]$ and $\{\Delta_1\Delta_2\}_4 = [2, 1, 1] \circ [2, 1, 1] - 3[3, 1] - 6[1, 1, 1, 1] = 2[2, 2]$.

Theorem 2.1. *The operation* $(\Delta_1, \Delta_2) \mapsto \Delta_1\Delta_2$ *gives rise on* $A_\infty^\circ = \bigoplus_n A_n^\circ$ *to the structure of a commutative associative algebra.*

Proof. Commutativity follows from the commutativity of the algebras A_n°. Associativity follows from the associativity of the algebras A_n° and the equality

$$\Delta_1\Delta_2\Delta_3 = \sum_{n=\max\{|\Delta_1|,|\Delta_2|,|\Delta_3|\}}^{|\Delta_1|+|\Delta_2|} \{\Delta_1\Delta_2\Delta_3\}_n$$

where

$$\{\Delta_1\Delta_2\Delta_3\}_n = \begin{cases} \rho_{(n-|\Delta_1|)}(\Delta_1)\circ\rho_{(n-|\Delta_2|)}(\Delta_2)\circ\rho_{(n-|\Delta_3|)}(\Delta_3) & \text{for } n=\max\{|\Delta_1|,|\Delta_2|,|\Delta_3|\} \\ \rho_{(n-|\Delta_1|)}(\Delta_1)\circ\rho_{(n-|\Delta_2|)}(\Delta_2)\circ\rho_{(n-|\Delta_3|)}(\Delta_3) & \\ \quad -\displaystyle\sum_{k=\max\{|\Delta_1|,|\Delta_2|,|\Delta_3|\}}^{n-1}\rho_{(n-k)}(\{\Delta_1\Delta_2\Delta_3\}_k) & \text{for } n>\max\{|\Delta_1|,|\Delta_2|,|\Delta_3|\}. \end{cases} \qquad \square$$

Theorem 2.2. *The algebras A_∞ and A°_∞ are naturally isomorphic.*

Proof. Product of the formal sums $G_\infty(\Delta_1)$ and $G_\infty(\Delta_2)$ is a finite sum of the formal sums of the type $G_\infty(\Delta)$, where $\max\{|\Delta_1|,|\Delta_2|\}\le|\Delta|\le|\Delta_1|+|\Delta_2|$. If $|\Delta_1|=|\Delta_2|=|\Delta|=n$, then $G_\infty(\Delta)=\sum_g C^{\Delta}_{\Delta_1,\Delta_2} gG_n(\Delta)g^{-1}$, where the sum goes over all finite permutations g of the set $\mathbb{N}$, which do not preserve $\{1,2,\dots,n\}$. By the same reason, the formal sum $G_\infty(\Delta)$ for $|\Delta_1|+1=|\Delta_2|+1=|\Delta|\in\mathcal{A}_n$ is equal to $\sum_g C^{\Delta}_{\rho_1(\Delta_1),\rho_1(\Delta_2)} gG_n(\Delta)g^{-1}$ minus $\rho_1(G_\infty(\hat\Delta))$, where $\Delta=\rho_1(\hat\Delta)$. Similar arguments prove the statement of the theorem for all Δ_1, Δ_2 of coinciding degrees. If, however, $|\Delta_1|<|\Delta_2|=|\Delta|$, then the term $G_\infty(\Delta)$, for the product of Δ_1 and Δ_2, coincides with the term $G_\infty(\Delta)$, for the product of $\tilde\Delta_1$ and Δ_2, where $\tilde\Delta_1=\rho_{|\Delta_2|-|\Delta_1|}(\Delta_1)$. $\square$

3. Differential representation of the algebra $U(gl(\infty))$

Consider the set of formal differential operators

$$D_{ab}=\sum_{e\in\{1,\dots,N\}} X_{ae}\frac{\partial}{\partial X_{be}}$$

of the Miwa variables $\{X_{ij}|i,j\le N\}$. Multiplication of operators is given by the rule

$$D_{ab}D_{cd}=\sum_{e_1,e_2\in\mathbb{N}} X_{ae_1}X_{ce_2}\frac{\partial}{\partial X_{be_1}}\frac{\partial}{\partial X_{de_2}}+\delta_{bc}\sum_{e\in\mathbb{N}} X_{ae}\frac{\partial}{\partial X_{de}}.$$

Commutation relations for the operators D_{ab} coincide with the commutation relations for the generators of the matrix algebra. Hence, the operators D_{ab} give rise to the algebra $U(N)$ naturally isomorphic to the universal enveloping algebra $U(gl(N))$.

In the limit $N\to\infty$ there emerges the algebra U_∞ of the formal differential operators which are finite or countable sums of operators of the form

$$:D_{a_1b_1}\cdots D_{a_nb_n}:=\sum_{e_1,\dots,e_n\in\mathbb{N}} X_{a_1e_1}\cdots X_{a_ne_n}\frac{\partial}{\partial X_{b_1e_1}}\cdots\frac{\partial}{\partial X_{b_ne_n}}.$$

We call the number $|\mathcal{U}|=n$ the *degree of the operator* $\mathcal{U}$. Linear combinations of the operators of the same degree are called *homogeneous operators*.

Thus, the vector space U_∞ is decomposed into the direct sum $U_\infty=\sum_{n\in\mathbb{N}} U_n$ of the subspaces of homogeneous operators of degree n. Consider the projection $pr_n : U_\infty\to U_n$, preserving the operators of degree n and mapping to zero all other homogeneous operators.

Introduce on U_n a multiplication "$\circ$" by the formula $\mathcal{U}_1\circ\mathcal{U}_2=pr_n(\mathcal{U}_1\mathcal{U}_2)$. This multiplication turns U_n into an associative algebra of the differential operators U°_n.

Consider an embedding of the vector spaces

$$\varrho_k : U_n\to U_{n+k}$$

where

$$\varrho_k(:D_{a_1b_1}\cdots D_{a_nb_n}:)=\frac{1}{k!}\sum_{c_1,\dots,c_k\in\mathbb{N}} :D_{c_1c_1}\cdots D_{c_kc_k}D_{a_1b_1}\cdots D_{a_nb_n}:$$

The operators $\mathcal{U}$ and $\varrho_k(\mathcal{U})$ act similarly on the monomials X of degree $n+k$ of the Miwa variables $\{X_{i,j}\}$.

One immediately checks the following claim:

Theorem 3.1. *There is an equality*

$$\mathcal{U}_1\mathcal{U}_2=\sum_{n=\max\{|\mathcal{U}_1|,|\mathcal{U}_2|\}}^{|\mathcal{U}_1|+|\mathcal{U}_2|}\{\mathcal{U}_1\mathcal{U}_2\}_n$$

where

$$\{\mathcal{U}_1\mathcal{U}_2\}_n = \begin{cases} \varrho_{(n-|\mathcal{U}_1|)}(\mathcal{U}_1)\circ\varrho_{(n-|\mathcal{U}_2|)}(\mathcal{U}_2) & \textit{for } n=\max\{|\mathcal{U}_1|,|\mathcal{U}_2|\} \\ \varrho_{(n-|\mathcal{U}_1|)}(\mathcal{U}_1)\circ\varrho_{(n-|\mathcal{U}_2|)}(\mathcal{U}_2) - \sum\limits_{k=\max\{|\mathcal{U}_1|,|\mathcal{U}_2|\}}^{n-1}\varrho_{(n-k)}(\{\mathcal{U}_1\mathcal{U}_2\}_k) & \textit{for } n>\max\{|\mathcal{U}_1|,|\mathcal{U}_2|\}. \end{cases}$$

Example 3.1. Put $\mathcal{U}_1 = \sum_{e\in\mathbb{N}} X_{a_1e}\frac{\partial}{\partial X_{b_1e}}$, $\mathcal{U}_2 = \sum_{e\in\mathbb{N}} X_{a_2e}\frac{\partial}{\partial X_{b_2e}}$. Then $\mathcal{U}_1\mathcal{U}_2 = \delta_{b_1,a_2}\sum_{e\in\mathbb{N}} X_{a_1e}\frac{\partial}{\partial X_{b_2e}} + \sum_{e_1,e_2\in\mathbb{N}} X_{a_1e_1}X_{a_2e_2}\frac{\partial}{\partial X_{b_1e_1}}\frac{\partial}{\partial X_{b_2e_2}}$. On the other hand, $\{\mathcal{U}_1\mathcal{U}_2\}_1 = \mathcal{U}_1\circ\mathcal{U}_2 = \delta_{b_1,a_2}\sum_{e\in\mathbb{N}} X_{a_1e}\frac{\partial}{\partial X_{b_2e}}$ and $\varrho_1(\{\mathcal{U}_1\mathcal{U}_2\}_1) = \delta_{b_1,a_2}\sum_{c\in\mathbb{N}}\sum_{e,f\in\mathbb{N}} X_{cf}X_{a_1e}\frac{\partial}{\partial X_{cf}}\frac{\partial}{\partial X_{b_2e}}$. Besides, $\varrho_1(\mathcal{U}_1) = \sum_{c_1\in\mathbb{N}}\sum_{e_1,e\in\mathbb{N}} X_{c_1f_1}X_{a_1e_1}\frac{\partial}{\partial X_{c_1f_1}}\frac{\partial}{\partial X_{b_1e_1}}$ and $\varrho_1(\mathcal{U}_2) = \sum_{c_2\in\mathbb{N}}\sum_{e_2,e\in\mathbb{N}} X_{c_2f_2}X_{a_2e_2}\frac{\partial}{\partial X_{c_2f_2}}\frac{\partial}{\partial X_{b_2e_1}}$. Hence, $\varrho_1(\mathcal{U}_1)\circ\varrho_1(\mathcal{U}_2) = \sum_{e_1,e_2\in\mathbb{N}} X_{a_1e_1}X_{a_2e_2}\frac{\partial}{\partial X_{b_1e_1}}\frac{\partial}{\partial X_{b_2e_2}} + \delta_{b_1,a_2}\sum_{c\in\mathbb{N}}\sum_{e,f\in\mathbb{N}} X_{cf}X_{a_1e}\frac{\partial}{\partial X_{cf}}\frac{\partial}{\partial X_{b_2e}}$. Thus, $\{\mathcal{U}_1\mathcal{U}_2\}_2 = \sum_{e_1,e_2\in\mathbb{N}} X_{a_1e_1}X_{a_2e_2}\frac{\partial}{\partial X_{b_1e_1}}\frac{\partial}{\partial X_{b_2e_2}}$.

Example 3.2. Put $\mathcal{U}_1 = \sum_{e^1_1\in\mathbb{N}} X_{a^1_1e^1_1}\frac{\partial}{\partial X_{b^1_1e^1_1}}$, $\mathcal{U}_2 = \sum_{e^1_2,e^2_2\in\mathbb{N}} X_{a^1_2e^1_2}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}}$. Then $\mathcal{U}_1\mathcal{U}_2 = \delta_{b^1_1,a^1_2}\sum_{e^1_1,e^2_2\in\mathbb{N}} X_{a^1_1e^1_1}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{b^1_2e^1_1}}\frac{\partial}{\partial X_{b^2_2e^2_2}} + \delta_{b^1_1,a^2_2}\sum_{e^1_1,e^1_2\in\mathbb{N}} X_{a^1_1e^1_1}X_{a^1_2e^1_2}\frac{\partial}{\partial X_{b^2_2e^1_1}}\frac{\partial}{\partial X_{b^1_2e^1_2}} + \sum_{e^1_1,e^1_2,e^2_2\in\mathbb{N}} X_{a^1_1e^1_1}X_{a^1_2e^1_2}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{b^1_1e^1_1}}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}}$. On the other hand, $\{\mathcal{U}_1\mathcal{U}_2\}_2 = \varrho_1(\mathcal{U}_1)\circ\mathcal{U}_2 = \left(\sum_{c^1_1\in\mathbb{N}}\sum_{f^1_1,e^1_1\in\mathbb{N}} X_{c^1_1f^1_1}X_{a^1_1e^1_1}\frac{\partial}{\partial X_{c^1_1f^1_1}}\frac{\partial}{\partial X_{b^1_1e^1_1}}\right)\circ\left(\sum_{e^1_2,e^2_2\in\mathbb{N}} X_{a^1_2e^1_2}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}}\right) = \delta_{b^1_1,a^1_2}\sum_{e^1_2,e^2_2\in\mathbb{N}} X_{a^1_1e^1_2}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}} + \delta_{b^1_1,a^2_2}\sum_{e^1_2,e^2_2\in\mathbb{N}} X_{a^1_1e^2_2}X_{a^1_2e^1_2}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}}$ and $\varrho_1(\{\mathcal{U}_1\mathcal{U}_2\}_2) = \sum_{c^1_1\in\mathbb{N}}\sum_{f^1_1,e^1_1\in\mathbb{N}}\delta_{b^1_1,a^1_2}\sum_{e^1_2,e^2_2\in\mathbb{N}} X_{c^1_1f^1_1}X_{a^1_1e^1_2}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{c^1_1f^1_1}}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}} + \delta_{b^1_1,a^2_2}\sum_{c^1_1\in\mathbb{N}}\sum_{e^1_2,e^2_2\in\mathbb{N}} X_{c^1_1f^1_1}X_{a^1_1e^2_2}X_{a^1_2e^1_2}\frac{\partial}{\partial X_{c^1_1f^1_1}}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}}$. Besides, $\varrho_2(\mathcal{U}_1)\circ\varrho_1(\mathcal{U}_2) = \sum_{e^1_1,e^1_2,e^2_2\in\mathbb{N}} X_{a^1_1e^1_1}X_{a^1_2e^1_2}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{b^1_1e^1_1}}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}} + \sum_{c^1_1\in\mathbb{N}}\sum_{f^1_1,e^1_1\in\mathbb{N}}\delta_{b^1_1,a^1_2}\sum_{e^1_2,e^2_2\in\mathbb{N}} X_{c^1_1f^1_1}X_{a^1_1e^1_2}X_{a^2_2e^2_2}\frac{\partial}{\partial X_{c^1_1f^1_1}}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}} + \delta_{b^1_1,a^2_2}\sum_{c^1_1\in\mathbb{N}}\sum_{e^1_2,e^2_2\in\mathbb{N}} X_{c^1_1f^1_1}X_{a^1_1e^2_2}X_{a^1_2e^1_2}\frac{\partial}{\partial X_{c^1_1f^1_1}}\frac{\partial}{\partial X_{b^1_2e^1_2}}\frac{\partial}{\partial X_{b^2_2e^2_2}}$.

Thus, $\mathcal{U}_1\mathcal{U}_2 = \{\mathcal{U}_1\mathcal{U}_2\}_1 + \{\mathcal{U}_1\mathcal{U}_2\}_2$.

4. Algebra $\mathcal{W}_\infty$ of the differential operators

Associate with the Young diagram $\Delta = [\mu_1,\mu_2,\ldots,\mu_l]$ with the ordered row lengths $\mu_1\geq\mu_2\geq\cdots\geq\mu_l$ the numbers $m_k = m_k(\Delta) = |\{i|\mu_i = k\}|$ and $\kappa(\Delta) = \left(\prod_k m_k!k^{m_k}\right)^{-1}$.

Associate with the Young diagram Δ the operator $\mathcal{W}(\Delta) = \kappa(\Delta)\prod_k : D_k^{m_k} :\in U_\infty$.

Example 4.1. $\mathcal{W}([1]) = \sum_{a\in\mathbb{N}} : D_{aa} : \mathcal{W}([2]) = \frac{1}{2}\sum_{a,b\in\mathbb{N}} : D_{ab}D_{ba} :$

Denote through $\mathcal{W}_n^\circ$ the vector space generated by the operators of the form $\mathcal{W}(\Delta)$, where $|\Delta| = n$.

Lemma 4.1. *The operation "$\circ$" provides the structure of algebra on $\mathcal{W}_n^\circ$. The correspondence $\Delta\mapsto\mathcal{W}(\Delta)$ gives rise to the isomorphism of algebras $\psi_n : A_n^\circ\to\mathcal{W}_n^\circ$.*

Proof. Associate with each permutation $g\in S_n$ the operator $\mathcal{W}(g) = \kappa(\Delta(g))\sum_{a_1,\ldots,a_n\in\mathbb{N}} : D_{a_1a_{g(1)}}\cdots D_{a_na_{g(n)}}$: Then $\mathcal{W}(\Delta(g)) = \mathcal{W}(g)$. Hence, the claim of the lemma follows from the equality $\mathcal{W}(\Delta(g_1)\circ\Delta(g_2)) = \mathcal{W}(g_1)\circ\mathcal{W}(g_2)$ $g_1,g_2\in S_n$. □

Lemma 4.2. *The embedding $\varrho_k : U_n\to U_{n+k}$ gives rise to the embedding $\varrho_k : \mathcal{W}_n^\circ\to\mathcal{W}_{n+k}^\circ$, where $\varrho_k\psi_n(\Delta) = \psi_{n+k}\rho_k(\Delta)$ at $\Delta\in A_n$.*

Proof. The map $\varrho_k : U_n\to U_{n+k}$ gives rise to the correspondence $\varrho_k(\mathcal{W}(\Delta)) = \frac{1}{k!}\mathcal{W}(\Delta^k)$.

In accordance with our definitions, $\psi_n(\Delta) = \kappa(\Delta)\sum_{a_1,\ldots,a_n\in\mathbb{N}} : D_{a_1a_{g(1)}}\cdots D_{a_na_{g(n)}}$: and $\varrho_k\psi_n(\Delta) = \frac{1}{k!}\kappa(\Delta)\sum_{c_1,\ldots,c_k,a_1,\ldots,a_n\in\mathbb{N}} : D_{c_1c_1}\cdots D_{c_kc_k}D_{a_1a_{g(1)}}\cdots D_{a_na_{g(n)}}$:

On the other hand, $\rho_k(\Delta) = \frac{(m_1+k)!}{m_1!k!}\Delta^k$ and $\psi_{n+k}\rho_k(\Delta) = \frac{(m_1+k)!}{m_1!k!}\kappa(\Delta)\left(\frac{(m_1+k)!}{m_1!}\right)^{-1}\sum_{c_1,\ldots,c_k,a_1,\ldots,a_n\in\mathbb{N}} : D_{c_1c_1}\cdots D_{c_kc_k}D_{a_1a_{g(1)}}\cdots D_{a_na_{g(n)}} := \frac{1}{k!}\left(\prod_k m_k!k^{m_k}\right)^{-1}\sum_{c_1,\ldots,c_k,a_1,\ldots,a_n\in\mathbb{N}} : D_{c_1c_1}\cdots D_{c_kc_k}D_{a_1a_{g(1)}}\cdots D_{a_na_{g(n)}}$: □

Example 4.2. Put $\Delta = [1]$. Then $\rho_k(\Delta) = (k+1)[1,1,\ldots,1]$ and $\psi_{k+1}(\rho_k(\Delta)) = \psi_{k+1}((k+1)[1,1,\ldots,1]) = \frac{(k+1)}{(k+1)!}\sum_{c_1,\ldots,c_{k+1}\in\mathbb{N}} : D_{c_1c_1}\cdots D_{c_nc_n}$:

On the other hand, $\psi_1([1]) = \sum_{e\in\mathbb{N}} : D_{cc} :$ and $\varrho_k\left(\sum_{e\in\mathbb{N}} : D_{cc} :\right) = \frac{1}{k!}\sum_{e,e_1,\dots,e_k\in\mathbb{N}}\sum_{e\in\mathbb{N}} : D_{cc}D_{c_1c_1}\cdots D_{c_nc_n} :$

Example 4.3. Put $\Delta = [2]$. $\psi_2([2]) = \frac{1}{2}\sum_{a,b\in\mathbb{N}} : D_{ab}D_{ba} :$ and $\varrho_k\left(\frac{1}{2}\sum_{a,b\in\mathbb{N}} : D_{ab}D_{ba} :\right) = \frac{1}{2k!}\sum_{a,b,e_1,\dots,e_k\in\mathbb{N}} : D_{ab}D_{ba}D_{c_1c_1}\cdots D_{c_nc_n} :$

On the other hand, $\rho_k(\Delta) = [2, 1, \dots, 1]$ and $\psi_{k+2}(\rho_k(\Delta)) = \psi_{k+2}([2, 1, \dots, 1]) = \frac{1}{2k!}\sum_{a,b,e_1,\dots,e_k\in\mathbb{N}} : D_{ab}D_{ba}D_{c_1c_1}\cdots D_{c_nc_n} :$

Denote through $\mathcal{W}_\infty \subset U_\infty$ the subalgebra generated by the differential operators $\mathcal{W}(\Delta)$. Confronting Theorems 2.1, 2.2 and 3.1 with Lemmas 4.1 and 4.2, one obtains

Theorem 4.1. *The isomorphisms ψ_n give rise to the isomorphism of the algebras $\psi : A_\infty \to W_\infty$.*

5. Schur functions

The Schur function of n variables corresponds to the Young diagram $R = \{R_1 \geq R_2 \geq \cdots \geq R_m > 0\}$, where $n \geq m$. It is defined by the formula

$$s_R(x_1, \dots, x_n) = \frac{\det[x_i^{R_j+n-j}]_{1\leq i,j\leq n}}{\det[x_i^{n-j}]_{1\leq i,j\leq n}},$$

where $R_i = 0$ at $m < i \leq n$. The property of stability $s_R(x_1, \dots, x_n, 0) = s_R(x_1, \dots, x_n)$ allows one to define s_R on an arbitrary finite set of variables.

Define the Schur functions on finite matrices $X \in gl(n) \subset gl(\infty)$ by the formula $s_R(X) = s_R(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the eigenvalues of the matrix X. The functions $s_R(X)$ form a basis in the vector space P of all symmetric polynomials of the Miwa variables.

Polynomials of the variables $p_i = \mathrm{tr}X^i$ form another natural basis of the space P,

$$\tilde{s}_R(p) = \det[P_{R_i+j-i}(p)]_{1\leq i,j\leq n}, \qquad \exp\left(\sum_k p_k x^k\right) \equiv \sum_i P_i(p)x^i$$

where $p = (p_1, p_2\cdots)$. Then $s_R(X) = \tilde{s}_R(p)$ [12]. Associate with the Young diagram Δ the monomial $p(\Delta) = \kappa(\Delta)p_1^{m_1(\Delta)}p_2^{m_2(\Delta)}\cdots p_n^{m_n(\Delta)}$.

Put $\Delta^k = [\Delta, \underbrace{1, \dots, 1}_{k}]$. Let $\dim R$ be the dimension of representation of the symmetric group $S_{|R|}$ corresponding to the diagram R, and $\chi_R(X_\Delta)$ be the value of character of this representation on the element of the cyclic type $\Delta^{|R|-|\Delta|}$. Put $d_R = \frac{\dim R}{|R|!} = \frac{\prod_{i<j=1}^{|R|}(\mu_i-\mu_j-i+j)}{\prod_{i=1}^{|R|}(\mu_i+|R|-i)!}$ and $m_1 = m_1(\Delta)$.

Denote through $\varphi_R(\Delta)$ the function which is equal to $\frac{\kappa(\Delta)}{d_R m_1!(|R|-|\Delta|-m_1)!}\chi_R(X_\Delta)$ at $|R| - |\Delta| \geq m_1$ and 0 otherwise. Then $\varphi_R(\Delta^k) = \frac{m_1!k!}{(m_1+k)!}\varphi_R(\Delta)$ at $k = |R| - |\Delta|$.

Theorem 5.1. *The functions $s_R(X)$ are the eigenfunctions of the operators $\mathcal{W}(\Delta)$. They form a complete system of eigenfunctions for the restrictions $\mathsf{W}(\Delta) = \mathcal{W}(\Delta)|_{\mathsf{P}}$ and $\mathsf{W}(\Delta)(s_R) = \varphi_R(\Delta)s_R$.*

Proof. Consider the regular representation of the algebra $U(gl(N))$ in the algebra of polynomial functions of matrix elements of $gl(N)$. The center $Z(U(gl(N)))$ preserves the vector subspace P. The algebra $Z(U(gl(N)))$ is additively generated by the operators $T(\Delta)$ associated with the Young diagrams Δ, and $T(\Delta)(f) = \mathcal{W}(\Delta)(f)$ for $f \in \mathsf{P}$. Besides, in accordance with the Weyl theorem [13], the Schur functions $s_R(X)$ form a complete system of the eigenfunctions of the operators $T(\Delta)$. Taking the limit $N \to \infty$, one finds that the Schur functions form a complete system of the eigenfunctions of the operators $\mathsf{W}(\Delta)$.

Now we find the eigenvalues of the operators. In accordance with [12, s.1.7], $p(\Delta) = \sum_{R:|R|=|\Delta|} d_R\varphi_R(\Delta)\tilde{s}_R$. Hence,

$$p(\Delta)e^{p_1} = \sum_{k=0}^{\infty} p(\Delta)\frac{p_1^k}{k!}\sum_{k=0}^{\infty}\frac{(m_1+k)!}{m_1!k!}p(\Delta^k) = \sum_{k=0}^{\infty}\sum_{|R|=|\Delta|+k}\frac{(m_1+k)!}{m_1!k!}d_R\varphi(\Delta^k)\tilde{s}_R$$
$$= \sum_{k=0}^{\infty}\sum_{|R|=|\Delta|+k} d_R\varphi(\Delta)\tilde{s}_R = \sum_R d_R\varphi(\Delta)\tilde{s}_R.$$

On the other hand, in accordance with [12, s.1.4, example 3], $e^{p_1} = \sum_R d_R\tilde{s}_R(p)$, hence, $p(\Delta)e^{p_1} = \mathsf{W}(\Delta)(e^{p_1}) = \sum_R d_R\mathsf{W}(\tilde{s}_R)$. Thus, $\sum_R d_R\mathsf{W}(s_R) = \sum_R d_R\varphi(\Delta)s_R$.

We have already proved that s_R form a complete system of the eigenfunctions of the operator $\mathcal{W}$. Therefore, the last equality implies $\mathrm{W}(\Delta)(s_R) = \varphi_R(\Delta)s_R$. □

Corollary 5.1. *The values of* $\varphi_R(\Delta)$ *are related by the formula*

$$\varphi_R(\Delta_1)\varphi_R(\Delta_2) = \sum_{\Delta} C^{\Delta}_{\Delta_1\Delta_2}\varphi_R(\Delta)$$

where $C^{\Delta}_{\Delta_1\Delta_2}$ *are the structure constants of the algebra* A_∞*, which are obtained in Section 2.*

6. First few W-operators

Represent now the operators $\mathrm{W}(\Delta)$ as differential operators of the variables $\{p_k\}$. Then,

$$D_{ab}F(p) = X_{ac}\frac{\partial}{\partial X_{bc}}F(p) = \sum_{k=1}^{\infty} k(X^k)_{ab}\frac{\partial F(p)}{\partial p_k}.$$

Using the relation

$$D_{a'b'}(X^k)_{ab} = X_{a'c'}\frac{\partial}{\partial X_{b'c'}}(X^k)_{ab} = \sum_{j=0}^{k-1} X_{a'c'}(X^j)_{ab'}(X^{k-j-1})_{c'b} = \sum_{j=0}^{k-1}(X^j)_{ab'}(X^{k-j})_{a'b},$$

one obtains

$$D_{a'b'}D_{ab}F(p) = \sum_{k,l=1}^{\infty} kl(X^l)_{a'b'}(X^k)_{ab}\frac{\partial^2 F(p)}{\partial p_k p_l} + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} k(X^j)_{ab'}(X^{k-j})_{a'b}\frac{\partial F(p)}{\partial p_k}.$$

Thus,

$$: D_{a'b'}D_{ab} : F(p) = \sum_k \left(k\sum_{j=1}^{k-1}(X^j)_{ab'}(X^{k-j})_{a'b}\right)\frac{\partial F(p)}{\partial p_k} + \sum_{k,l} kl(X^k)_{ab}(X^l)_{a'b'}\frac{\partial^2 F(p)}{\partial p_k \partial p_l}.$$

This relation allows one to find all the operators W. In particular,

$$\mathrm{W}([1]) = \mathrm{tr}\hat{D} = \sum_{k=1} kp_k\frac{\partial}{\partial p_k}$$

$$\mathrm{W}([2]) = \frac{1}{2} : D^2 := \frac{1}{2}\sum_{a,b=1}^{\infty}\left((a+b)p_a p_b\frac{\partial}{\partial p_{a+b}} + abp_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}\right)$$

$$\mathrm{W}([1,1]) = \frac{1}{2!} : (\mathrm{tr}D)^2 := \frac{1}{2}\left(\sum_{a=1}^{\infty} a(a-1)p_a\frac{\partial}{\partial p_a} + \sum_{a,b=1}^{\infty} abp_a p_b\frac{\partial^2}{\partial p_a \partial p_b}\right)$$

$$\begin{aligned}\mathrm{W}([3]) \;=\; &\frac{1}{3} : \mathrm{tr}D^3 := \frac{1}{3}\sum_{a,b,c\geq 1}^{\infty} abcp_{a+b+c}\frac{\partial^3}{\partial p_a \partial p_b \partial p_c}\\ &+ \frac{1}{2}\sum_{a+b=c+d} cd\,(1-\delta_{ac}\delta_{bd})\,p_a p_b\frac{\partial^2}{\partial p_c \partial p_d} + \frac{1}{3}\sum_{a,b,c\geq 1}(a+b+c)\,(p_a p_b p_c + p_{a+b+c})\,\frac{\partial}{\partial p_{a+b+c}}\end{aligned}$$

$$\begin{aligned}\mathrm{W}([2,1]) \;=\; &\frac{1}{2} : \mathrm{tr}D^2\mathrm{tr}D := \frac{1}{2}\sum_{a,b\geq 1}(a+b)(a+b-2)p_a p_b\frac{\partial}{\partial p_{a+b}} + \frac{1}{2}\sum_{a,b\geq 1} ab(a+b-2)p_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}\\ &+ \frac{1}{2}\sum_{a,b,c\geq 1}(a+b)cp_a p_b p_c\frac{\partial^2}{\partial p_{a+b}\partial p_c} + \frac{1}{2}\sum_{a,b,c\geq 1} abcp_a p_{b+c}\frac{\partial^3}{\partial p_a \partial p_b \partial p_c}\end{aligned}$$

$$\begin{aligned}\mathrm{W}([1,1,1]) \;=\; &\frac{1}{3!} : (\mathrm{tr}D)^3 := \frac{1}{6}\sum_{a\geq 1} a(a-1)(a-2)p_a\frac{\partial}{\partial p_a} + \frac{1}{4}\sum_{a,b} ab(a+b-2)p_a p_b\frac{\partial^2}{\partial p_a \partial p_b}\\ &+ \frac{1}{6}\sum_{a,b,c\geq 1} abcp_a p_b p_c\frac{\partial^3}{\partial p_a \partial p_b \partial p_c}.\end{aligned}$$

7. Hurwitz numbers

Each holomorphic morphism of degree n of Riemann surfaces $f : \widetilde{\Omega} \to \Omega$ associates with the point $s \in \Omega$ a local invariant: the Young diagram $\Delta(f, s)$ of degree n with the row lengths being equal to degrees of the map f at the points of complete pre-image $f^{-1}(s) = \{s^1, \ldots, s^k\}$. More than 100 years ago Hurwitz [14] formulated a problem of calculating *the Hurwitz numbers*

$$H((s_1, \Delta_1), \ldots, (s_k, \Delta_k)|\Omega) = \sum_{f \in \mathrm{Cov}_n(\Omega, \{\alpha_1, \ldots, \alpha_s\})} \frac{1}{|\mathrm{Aut}(f)|}$$

for an arbitrary set $\{\Delta_1, \ldots, \Delta_k\}$ of Young diagrams of degree n. Here $|\mathrm{Aut}(f)|$ is the order of automorphism group of the map f, and $\mathrm{Cov}_n(\Omega, \{\alpha_1, \ldots, \alpha_s\})$ is a set of classes of the biholomorphic equivalence of the holomorphic morphisms $f : \widetilde{\Omega} \to \Omega$ with the set of critical values $s_1, \ldots, s_k \in \Omega$ and the local invariants $\alpha(f, s_i) = \alpha_i$.

This number depends only on the genus $g(\Omega)$ of the surface Ω and the diagrams $\Delta_1, \ldots, \Delta_k$. We define $\langle \Delta_1, \ldots, \Delta_k \rangle_{g(\Omega)} = H((s_1, \Delta_1), \ldots, (s_k, \Delta_k)|\Omega)$. The Hurwitz numbers of any genus are easily expressed through those at genus zero, $\langle \Delta_1, \ldots, \Delta_k \rangle = \langle \Delta_1, \ldots, \Delta_k \rangle_0$, [15].

A defining property of the Hurwitz numbers is *the associativity relation*

$$\langle \Delta_1, \ldots, \Delta_k \rangle = \sum_{\Upsilon \in \mathcal{A}_n} \langle \Delta_1, \ldots, \Delta_r, \Upsilon \rangle |\mathrm{Aut}(\Upsilon)| \langle \Upsilon, \Delta_{r+1}, \ldots, \Delta_k \rangle.$$

The Hurwitz numbers of coverings with three critical values are related to the structure constants of the algebra A_n by the formula $\langle \Delta_1, \Delta_2, \Delta_3 \rangle = C^{\Delta_3}_{\Delta_1, \Delta_2} |\mathrm{Aut}(\Delta_3)|^{-1}$. Arbitrary Hurwitz numbers are expressed through these simplest Hurwitz numbers by the formula

$$\langle \Delta_1, \ldots, \Delta_k \rangle = \sum_{\Upsilon_1, \ldots, \Upsilon_{k-1} \in \mathcal{A}_n} \langle \Delta_1, \Delta_2, \Upsilon_1 \rangle |\mathrm{Aut}(\Upsilon_1)| \langle \Upsilon_1, \Delta_3, \Upsilon_2 \rangle \times |\mathrm{Aut}(\Upsilon_2)| \cdots |\mathrm{Aut}(\Upsilon_{k-1})| \langle \Upsilon_{k-1}, \Delta_{k-1}, \Delta_k \rangle,$$

(see, e.g., [15]).

The Hurwitz numbers appear in different frameworks: strings and QCD [16], mirror symmetry [17], theory of singularities [18], matrix models [19,5], integrable systems [20–22], Yang–Mills theory [23,21] and the theory of moduli of curves [24,25,5] and other branches of string theory.

Associate with Young diagrams $\Delta_1, \ldots, \Delta_k$ and Δ, where $|\Delta_i| \le |\Delta|$ for all i, the numbers $\langle (\Delta_1, n_1), \ldots, (\Delta_k, n_k) | \Delta \rangle$ equal to the Hurwitz numbers $\langle \tilde{\Delta}_1, \ldots, \tilde{\Delta}_1, \tilde{\Delta}_2, \ldots, \tilde{\Delta}_2, \ldots, \tilde{\Delta}_k, \ldots, \tilde{\Delta}_k, \Delta \rangle$, where the Young diagram $\tilde{\Delta}_i = \rho_{|\Delta| - |\Delta_i|}(\Delta_i)$ is met exactly n_i times. We also put $\langle (\Delta_1, n_1), \ldots, (\Delta_k, n_k) | \Delta \rangle = 0$, if $|\Delta_i| > |\Delta|$ at least for one i.

Associate a variable β_Δ with each Young diagram Δ and consider the generating function for the Hurwitz numbers

$$\mathcal{Z} = \sum_{k=1}^{\infty} \sum_{\Delta, \Delta_1, \ldots, \Delta_k \in \mathcal{A}_\infty} \sum_{n_1, \ldots, n_k \in \mathbb{N}} \frac{\beta^{n_1}_{\Delta_1} \cdots \beta^{n_k}_{\Delta_n}}{n_1! \cdots n_k!} \langle \Delta_1^{n_1}, \ldots, \Delta_k^{n_k} | \Delta \rangle p(\Delta).$$

Theorem 7.1. *For any Young diagram Υ there is an equality*

$$\frac{\partial \mathcal{Z}}{\partial \beta_\Upsilon} = \mathrm{W}(\Upsilon) \mathcal{Z}.$$

Proof. The claim of the theorem implies a system of relations between the numbers $\langle \Delta_1^{n_1}, \ldots, \Delta_k^{n_k} | \Delta \rangle$. In accordance with Theorems 2.2 and 4.1, these relations are of the form $\langle \Delta_1^{n_1}, \ldots, \Delta_i^{n_i}, \ldots, \Delta_k^{n_k} | \Delta \rangle = \langle \Delta_1^{n_1}, \ldots, \Delta_i^{n_i - 1}, \ldots, \Delta_k^{n_k} | \Delta \circ \tilde{\Delta}_i \rangle$ and follow from the associativity relation. □

For the Young diagram $\Upsilon = [2]$ and $\beta_\Upsilon = 0$ at $\Upsilon \ne [2]$ Theorem 7.1 is equivalent to the "cut-and-join" relation [10]. Using the equations with the initial data $\mathcal{Z}_0 = e^{p_1}$ at all $\beta_\Upsilon = 0$ allows one to represent $\mathcal{Z}$ as the exponential of the operators $\mathrm{W}(\Upsilon)$ acting on $\mathcal{Z}_0$ and calculate this way any Hurwitz number.

The simplest equations of this kind for the Hurwitz numbers for the surfaces with boundaries [26,27] are found in [28].

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