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# A formally Kähler structure on a knot space of a $G_2$-manifold

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**Abstract** A knot space in a manifold $M$ is a space of oriented immersions $S^1 \hookrightarrow M$ up to $\mathsf{Diff}(S^1)$. J.-L. Brylinski has shown that a knot space of a Riemannian threefold is formally Kähler. We prove that a space of knots in a holonomy $G_2$ manifold is formally Kähler.

**Keywords** $G_2$-manifold · Knot space · Infinite-dimensional manifold · Kähler manifold · Symplectic manifold · Fréchet manifold

**Mathematics Subject Classification (2010)** 53B35 · 53C38 · 53C29 · 58D10

## 1 Introduction

Let $M$ be an oriented Riemannian 3-fold and $\mathrm{Knot}(M)$ be its knot space, that is, a space of non-parametrized, immersed, oriented loops, represented by a map that is injective outside a finite set. J.-L. Brylinski has proved that $\mathrm{Knot}(M)$ is an infinite-dimensional formally Kähler Fréchet manifold (see Sect. 2 for an explanation of these terms). This formal Kähler structure is easy to construct, though the proof of its formal integrability is non-trivial. Given a knot $S \subset M$, its tangent space $T_S \mathrm{Knot}(M)$ is a space of sections of its normal bundle NS, which is 2-dimensional, oriented and orthogonal. A 2-dimensional oriented Euclidean vector space has a natural complex structure, which

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is defined through counter-clockwise turns. Therefore, the bundle NS is a 1-dimensional complex Hermitian bundle. Therefore, the space of sections of NS is a complex Hermitian Fréchet vector space. The corresponding Hermitian form is easy to obtain from the volume 3-form on $M$ by integration along the knots (Definition 4.14).

$G_2$-manifolds appear naturally as a main object of "octonionic algebraic geometry", playing the same role for octonions as hyperkähler and hypercomplex manifolds play for quaternions. The main engine for the study of quaternionic geometry is the twistor construction, which makes a complex manifold from a manifold with a quaternionic structure. It is well known that the twistor data can be used to reconstruct the quaternionic structure. Singularities in hyperkähler and hypercomplex geometry and many natural geometric objects can also be studied in terms of twistors [22,23].

One would expect the hypothetical octonion twistor space (if it exists) to bring similar benefits. However, none of the usual approaches to constructing complex structures on twistor manifolds works for $G_2$-geometry, and it seems that something must be sacrificed. In the present paper, we sacrifice finite-dimensionality of a twistor space.

We propose a twistor-like construction resulting in a formally Kähler structure on the knot space of a $G_2$-manifold (Theorem 4.17). This construction is similar in flavor to one of J.-L. Brylinski; in fact, our approach to the proof of formal integrability is similar to the argument of L. Lempert [16], who used a CR-twistor space constructed for $G_2$-manifolds by C. LeBrun. A $G_2$-analog of LeBrun's twistor space was constructed in [24], and now we use it to study the complex structure on the knot space. We also interpret several objects of $G_2$-geometry (instanton bundles, associative subvarieties) as holomorphic objects on the knot space (Sect. 5).

The symplectic structure that appears in this construction was previously obtained by M. Movshev [18].

## 2 Fréchet manifolds and formally Kähler geometry

In this section, we briefly introduce Fréchet manifolds and basic geometric structures on such manifolds. For a detailed exposition, please see [17].

### 2.1 Fréchet manifolds and knot spaces

Recall that a **Fréchet space** is an infinite-dimensional topological vector space $V$ admitting a translation-invariant complete metric. It is equivalent to say that $V$ has a countable family of seminorms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_3 \ldots$, and the topology on $V$ is induced by a complete, translation-invariant metric

$$\mathrm{d}(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \min(\|x - y\|_i, 1).$$

[3]. **A differentiable map** of Fréchet spaces is a map which can be approximated at each point by a continuous linear map, up to a term that decays faster than linear, in the sense of this metric. In a similar way one defines smooth (infinitely differentiable) maps of Fréchet spaces.

A **Fréchet manifold** is a ringed space, locally modeled on a space of differentiable functions on a Fréchet space, with transition functions smooth.

When $M$ is a compact finite-dimensional manifold, the space $C^\infty M$ of smooth functions on $M$ has a sequence of norms $C_k$, with

$$\|f\|_{C^k} = \sum_{i=0}^{k} \sup_M |f^{(i)}|,$$

where $f^{(i)}$ denotes the $i$-th derivative. It is well known that this system of seminorms is complete on $C^\infty(M)$, giving a structure of Fréchet space on $C^\infty(M)$. Similar constructions allow one to define the Fréchet structure on the space of smooth sections of a vector bundle.

This is used to define a structure of a Fréchet manifold on various infinite-dimensional spaces arising in geometry, in particular on a space $\mathrm{Imm}(X, M)$ of smooth immersions $X \hookrightarrow M$, and on a group of diffeomorphisms, which becomes a Fréchet Lie group.

The quotient $\mathrm{Imm}(X, M)/\mathsf{Diff}(X)$ is a Fréchet orbifold, locally modeled on the total space $NX$ of its normal bundle. To see this, one needs to construct a slice of the $\mathsf{Diff}(X)$-action, which can be done easily using a Riemannian metric. The orbifold points correspond to those maps which are wrapped several times on themselves. Denote by $\mathrm{Imm}_0(X, M)$ the space of immersions that are injective outside of a positive codimension set of self-intersections. Clearly, then $\mathrm{Imm}_0(X, M)/\mathsf{Diff}(X)$ is a Fréchet manifold.

For the present paper, the most important Fréchet manifold is a space $\mathrm{Knot}(M) := \mathrm{Imm}_0(S^1, M)/\mathsf{Diff}_+(S^1)$ of oriented knots (non-parametrized immersed loops, injective outside of a finite set) in $M$. We could work with the orbifold $\mathrm{Imm}(S^1, M)/\mathsf{Diff}_+(S^1)$ instead, and all the results would remain valid in the orbifold context. To simplify terminology, we work with manifolds and restrict ourselves to $\mathrm{Imm}_0(S^1, M)/\mathsf{Diff}_+(S^1)$.

## 2.2 Formally complex Fréchet manifolds

Let $F$ be a Fréchet manifold. One can define **the sheaf of vector fields** $TF$ on $F$ as a shief of continuous derivations of its structure sheaf. A commutator of two derivations is again a derivation. This gives a Lie algebra structure on the sheaf of vector fields.

The formally integrable almost complex structures are defined as usual, in the following way.

**Definition 2.1** Let $F$ be a Fréchet manifold, and $I : TF \longrightarrow TF$ a smooth $C^\infty F$-linear endomorphism of the tangent bundle satisfying $I^2 = -1$. Then $I$ is called **an almost complex structure on** $F$.

*Remark 2.2* Clearly, $I$ defines a decomposition $TF \otimes \mathbb{C} = T^{1,0}F \oplus T^{0,1}F$, where $T^{1,0}F$ is the $\sqrt{-1}$ -eigenspace of $I$, and $T^{0,1}F$ the $-\sqrt{-1}$ -eigenspace. Indeed, $x = \frac{1}{2}(x + \sqrt{-1}\, Ix) + \frac{1}{2}(x - \sqrt{-1}\, Ix)$.

**Definition 2.3** An almost complex structure on a Fréchet manifold $(F, I)$ is called **formally integrable**, if $[T^{1,0}F, T^{1,0}F] \subset T^{1,0}F$, where $[\cdot, \cdot]$ denotes the commutator of vector fields. In this case, $(F, I)$ is called **a formally complex manifold**.

*Remark 2.4* Just as it happens in the finite-dimensional case, the projection of $[T^{1,0}F, T^{1,0}F]$ to $T^{0,1}F$ is always $C^\infty F$-linear. This gives an operation

$$\Lambda^2 T^{1,0}F \xrightarrow{N} T^{0,1}F,$$

called **the Nijenhuis tensor**. The Nijenhuis tensor of an almost complex Fréchet manifold $(F, I)$ vanishes if and only if it is formally integrable.

**Definition 2.5** A function $f$ on an almost complex Fréchet manifold is called **holomorphic** if $\langle df, X\rangle = 0$ for any vector field $X \in T^{0,1}F$. A smooth map to a complex topological vector space is called **holomorphic** if its composition with continuous complex linear functionals is always holomorphic.

**Definition 2.6** An almost complex structure on a Fréchet manifold $F$ is called **strongly integrable** if there exists an atlas of local coordinate charts which are given by holomorphic maps to complex Fréchet spaces. In this case, $F$ is called **holomorphic**.

*Remark 2.7* Every holomorphic Fréchet manifold is formally integrable, which is obvious, because one can locally generate $T^{1,0}F$ by coordinate vector fields that commute. The converse implication is known to be false. For finite-dimensional manifolds, formal integrability implies integrability of a complex structure, by a deep analytic result called the Newlander–Nirenberg theorem [19]. An infinite-dimensional version of Newlander–Nirenberg theorem is false (see [16]).

**Definition 2.8** Let $(F, I)$ be a formally integrable almost complex Fréchet manifold, $g$ a Hermitian structure on $F$, and $\omega$ be the corresponding $(1, 1)$-form. We say that $(F, I, g)$ is **formally Kähler** if $\omega$ is closed.

### 2.3 Formally Kähler structures on knot spaces

Let $M$ be a smooth manifold and $\mathrm{Knot}(M)$ be its knot space. As we have mentioned already, $\mathrm{Knot}(M)$ is a Fréchet manifold. Locally at $l \in \mathrm{Knot}(M)$, this manifold is modeled on the space of smooth sections of a normal bundle $Nl$.

Geometric structures on the space of knots on an oriented 3-manifold $M^3$ were a subject of much research (see e.g. [5,15,16], and the book [6]). In [5], a formally Kähler structure on $\mathrm{Knot}(M^3)$ was constructed. In [16], it was shown that this formally complex structure is never strongly integrable.

In his book [6], J.-L. Brylinski gives many uses for the formal Kähler structure on the space of knots. Another possible application of the formally Kähler structure on the space of knots (not explored much, so far) is to the spaces of discriminants of knots and their cohomology. V. A. Vassiliev defined the eponymous knot invariants by considering the stratification on the space of knots by successive discriminant spaces. Later, M. Kontsevich redefined some of these cohomology spaces and proved that

they carry the mixed Hodge structure. It is easy to see that the discriminant spaces are in fact complex subvarieties, in the sense of formally complex structure on the knots. One would expect that the mixed Hodge structure on Vassiliev invariants comes from this complex stratification, just as it would happen in the finite-dimensional case.

The aim of this paper is to generalize these results to $G_2$-manifolds, which are 7-dimensional Riemannian manifolds with special holonomy group that lies in $G_2$.

The formally complex structure on the space of knots of a 3-dimensional manifold $M$ can be defined in terms of the vector product on $TM$. Indeed, let $S$ be a knot in $M$, and $\gamma'$ a unit tangent vector field to $S$. A vector product with $\gamma'$ defines a complex structure on the normal bundle NS, which is used to define the formal complex structure on $\mathrm{Knot}(M)$. In [13] and [14], geometry of manifolds with vector products was explored at some length, and many results about the knot spaces and instantons were obtained from a similar vector product construction.

## 3 $G_2$-manifolds

### 3.1 $G_2$-geometry: basic notions

$G_2$-manifolds originally appeared in Berger's classification of Riemannian holonomy [1,2]. The first examples of $G_2$-manifolds were obtained by R. Bryant and S. Salamon [4]. The compact examples of $G_2$-manifolds were constructed by D. Joyce [11,12]. In this introduction, we follow the approach to $G_2$-geometry, which is due to N. Hitchin (see [10]).

**Definition 3.1** Let $\rho \in \Lambda^2\mathbb{R}^7$ be a 3-form on $\mathbb{R}^7$. We say that $\rho$ is **non-degenerate** if the dimension of its stabilizer is maximal:

$$\dim \mathrm{St}_{GL(7)}\rho = \dim GL(7) - \dim \Lambda^3(\mathbb{R}^7) = 49 - 35 = 14.$$

In this case, $\mathrm{St}(\rho)$ is one of two real forms of a 14-dimensional Lie group $G_2(\mathbb{C})$. We say that $\rho$ is **non-split** if it satisfies $\mathrm{St}(\rho|_x) \cong G_2$, where $G_2$ denotes the compact real form of $G_2(\mathbb{C})$. **A $G_2$-structure** on a 7-manifold is a 3-form $\rho \in \Lambda^3(M)$, which is non-degenerate and non-split at each point $x \in M$. We shall always consider a $G_2$-manifold as a Riemannian manifold, with the Riemannian structure induced by the $G_2$-structure as follows.

*Remark 3.2* A form $\rho$ defines a $\Lambda^7 M$-valued metric on $M$:

$$g(x, y) = (\rho \rfloor x) \wedge (\rho \rfloor y) \wedge \rho \tag{3.1}$$

(we denote by $\rho \rfloor x$ the contraction of $\rho$ with a vector field $x$). The Riemannian volume form associated with this metric gives a section of $\Lambda^7 M \otimes (\Lambda^7 M)^{7/2}$. Squaring and taking the 9-th degree root, we obtain a trivialization of the volume. Then (3.1) defines a metric $g$ on $M$, by construction $G_2$-invariant.

**Definition 3.3** An $G_2$-structure is called **an integrable $G_2$-structure**, if $\rho$ is preserved by the corresponding Levi-Civita connection. An integrable $G_2$-manifold is a

manifold equipped with an integrable $G_2$-structure. Holonomy group of such a manifold clearly lies in $G_2$; for this reason, the integrable $G_2$-manifolds are often called **holonomy $G_2$-manifolds**.

*Remark 3.4* In the literature, "the $G_2$-manifold" often means a "holonomy $G_2$-manifold" and "$G_2$-structure" "an integrable $G_2$-structure." A $G_2$-structure that is not necessarily integrable is called "an almost $G_2$-structure," taking analogy from almost complex structures. Further on in this paper, we shall follow this usage, unless specified otherwise.

*Remark 3.5* As shown in [9], integrability of a $G_2$-structure induced by a 3-form $\rho$ is equivalent to $\mathrm{d}\rho = \mathrm{d}(*\rho) = 0$. For this reason, the 4-form $*\rho$ is called **the fundamental 4-form of a $G_2$-manifold**, and $\rho$ **the fundamental 3-form**.

*Remark 3.6* Let $V = \mathbb{R}^7$ be a 7-dimensional real space equipped with a non-degenerate 3-form $\rho$ with $\mathrm{St}_{GL(7)}(\rho) = G_2$. As in Remark 3.2, one can easily see that $V$ has a natural $G_2$-invariant metric. For each vector $x \in V$, $|x| = 1$, its stabilizer $\mathrm{St}_{G_2}(x)$ in $G_2$ is isomorphic to $SU(3)$. Indeed, the orthogonal complement $x^\perp$ is equipped with a symplectic form $\rho\rfloor x$, which gives a complex structure $g^{-1} \circ (\rho\rfloor x)$ as usual. This gives an embedding $\mathrm{St}_{G_2}(x) \hookrightarrow U(3)$. Since the space of such $x$ is $S^6$, and the action of $G_2$ in $S^6$ is transitive, one has $\dim \mathrm{St}_{G_2}(x) = \dim G_2 - \dim S^6 = 8 = \dim U(3) - 1$. To see that $\mathrm{St}_{G_2}(x) = SU(3) \subset U(3)$ and not some other codimension 1 subgroup, one should notice that $\mathrm{St}_{G_2}(x)$ preserves two 3-forms $\rho\Big|_{x^\perp}$ and $\rho^*\rfloor x\Big|_{x^\perp}$, where $\rho^* = *\rho$ is the fundamental 4-form of $V$. A simple linear-algebraic calculation implies that $\rho\Big|_{x^\perp} + \sqrt{-1}\,\rho^*\rfloor x\Big|_{x^\perp}$ is a holomorphic volume form on $x^\perp$, which is clearly preserved by $\mathrm{St}_{G_2}(x)$. Therefore, the natural embedding $\mathrm{St}_{G_2}(x) \hookrightarrow U(3)$ lands $\mathrm{St}_{G_2}(x)$ to $SU(3)$. Using the dimension count $\dim \mathrm{St}_{G_2}(x) = \dim SU(3)$ (see above), we show that the embedding $\mathrm{St}_{G_2}(x) \hookrightarrow SU(3)$ is also surjective.

### 3.2 Octonion structure and a vector product

Let $V = \mathbb{R}^7$ be a 7-dimensional space equipped with a non-degenerate, non-split constant 3-form $\rho$ inducing a $G_2$-action on $V$. Then $V$ is equipped with the **vector product**, defined as follows: $x \star y = \rho(x, y, \cdot)^\sharp$. Here $\rho(x, y, \cdot)$ is a 1-form obtained by contraction, and $\rho(x, y, \cdot)^\sharp$ is its dual vector field.

*Remark 3.7* The complex structure on an orthogonal complement $v^\perp$ is given by a vector product: $x \longrightarrow v \star x$, if $|v| = 1$.

It is not hard to see that $(V, \star)$ becomes isomorphic to the imaginary part of the octonion algebra, with $\star$ corresponding to half of the commutant. In fact, this is one of the many ways used to define an octonion algebra. The whole octonion algebra is obtained as $\mathbb{O} := V \oplus \mathbb{R}$, with the product given by

$$(x, t)(y, t') = (ty + t'x + x \star y, g(x, y) + tt')$$

Here, $x$, $y$ and $ty + t'x + x \star y$ are vectors in $V$, and $t, t', g(x, y) + tt' \in \mathbb{R}$.

Given two non-collinear vectors in $V$, they generate a quaternion subalgebra in octonions. When these two vectors satisfy $|v| = |v'| = 1$, $v \perp v'$, the standard basis $I, J, K$ in imaginary quaternions can be given by a triple $v, v', v \star v' \in V$.

A 3-dimensional subspace $A \subset V$ is called **associative** if it is closed under the vector product. The set of associative subspaces is in bijective correspondence with the set of quaternionic subalgebras in octonions.

### 3.3 A CRtwistor space of a $G_2$-manifold

**Definition 3.8** Let $M$ be a smooth manifold, $B \subset TM$ be a sub-bundle in its tangent bundle, and $I \in \operatorname{End} B$ be an automorphism, satisfying $I^2 = -\operatorname{Id}_B$. Consider the (1,0) and (0,1)-bundles $B^{1,0}, B^{0,1} \subset B \otimes \mathbb{C}$, which are the eigenspaces of $I$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. The sub-bundle $B^{1,0} \subset TM \otimes \mathbb{C}$ is called **a CR-structure on** $M$ if it is *involutive*, that is, it satisfies $[B^{1,0}, B^{1,0}] \subset B^{1,0}$.

Let $M$ be an almost $G_2$-manifold. From Remark 3.6 it follows that with every vector $x \in TM$, $|x| = 1$, one can associate a complex Hermitian structure on its orthogonal complement $x^\perp$. The easiest way to define this structure is to notice that $x^\perp$ is equipped with a symplectic structure $\rho \rfloor x$ and a metric $g\Big|_{x^\perp}$, which can be considered as real and imaginary parts of a complex-valued Hermitian product. Then the complex structure is obtained as usual, as $I := (\rho \rfloor x) \circ g^{-1}$.

**Corollary 3.9** *Let $(M, \rho)$ be an almost $G_2$ manifold, $m \in M$ a point, and $x \in T_m M$ a non-zero vector. Then the symplectic form $\rho \rfloor x$ is a Hermitian form of a natural complex structure on $x^\perp \subset T_m M$.*

**Definition 3.10** Consider the unit sphere bundle $S^6 M$ over $M$, with the fiber $S^6$, and let $T_{\mathsf{hor}} S^6 M \subset T S^6 M$ be the horizontal sub-bundle corresponding to the Levi-Civita connection. This sub-bundle has a natural section $\theta$; at each point $(x, m) \in S^6 M$, $m \in M$, $x \in T_m M$, $|x| = 1$, we take $\theta\Big|_{(x,m)} = x$, using the standard identification $T_{\mathsf{hor}} S^6 M\Big|_{(x,m)} = T_m M$. Denote by $B \subset T_{\mathsf{hor}} S^6 M$ the orthogonal complement to $\theta$ in $T_{\mathsf{hor}} S^6 M$. Since at each point $(x, m) \in S^6 M$, the restriction $B\Big|_{(x,m)}$ is identified with $x^\perp \subset T_m M$, this bundle is equipped with a natural complex structure, that is, an operator $I \in \operatorname{End} B$, $I^2 = -\operatorname{Id}_B$.

**Theorem 3.11** [24] *Let $M$ be an almost $G_2$-manifold, $S^6 M \subset TM$ its unit sphere bundle, and $B \subset TS^6 M$ a sub-bundle of its tangent bundle constucted above and equipped with the complex structure $I$. Then $B^{0,1} \subset B \otimes \mathbb{C} \subset T S^6 M \otimes \mathbb{C}$ is involutive if and only if $M$ is a holonomy $G_2$-manifold.*

**Definition 3.12** Let $M$ be a holonomy $G_2$-manifold, and

$$\operatorname{Tw}(M) := (S^6 M, B, I)$$

the CR-manifold constructed in Theorem 3.11. Then $\operatorname{Tw}(M)$ is called **a CR-twistor space of** $M$.

The key argument in the proof of Theorem 3.11 is achieved by constructing a certain 3-form $\Omega \in \Lambda^3(S^6M, \mathbb{C})$. This form is of type $(3,0)$ on $B$ with respect to the complex structure constructed in Definition 3.10, and satisfies $\mathrm{d}\Omega\big|_B = 0$. The restriction $\Omega\big|_B$ is determined uniquely from the $SU(3)$-structure on $B$: we construct $\Omega$ in such a way that $\Omega\big|_B$ is equal to the holomorphic volume form associated with the $SU(3)$-structure. Since we are going to use this form and the expression for $\mathrm{d}\Omega$ obtained in [24], we describe it explicitly below.

Consider the fundamental 3-form and 4-form $\rho$ and $\rho^* := *(\rho)$ on a holonomy $G_2$-manifold $M$, and let $\theta \in T_{\mathsf{hor}}S^6M$ be the tautological vector field constructed in Definition 3.10. Denote by $\pi : S^6M \longrightarrow M$ the standard projection. Then

$$\Omega := \pi^*\rho + \sqrt{-1}\,(\pi^*\rho^* \rfloor \theta), \tag{3.2}$$

where $\pi^*\rho^* \rfloor \theta$ denotes the contraction of $\pi^*\rho^*$ and $\theta \in T_{\mathsf{hor}}S^6M \subset TS^6M$.

The 4-form $\mathrm{d}\Omega := \sqrt{-1}\,\mathrm{d}(\pi^*\rho^* \rfloor \theta)$ was computed in [24] explicitly, as follows.

Consider a natural embedding

$$S^6M \stackrel{\varphi}{\hookrightarrow} \mathrm{Tot}(\Lambda^3 M) \tag{3.3}$$

into the total space of $\Lambda^3 M$, mapping $(v, m)$ to $(\pi^*\rho^* \rfloor v, m)$. Let $\Xi$ be a 4-form on $\mathrm{Tot}(\Lambda^3 M)$ written in local coordinates $p_1, \ldots, p_7$ as

$$\Xi = \sum_{i_1<i_2<i_3} \mathrm{d}q_{i_1,i_2,i_3} \wedge \mathrm{d}p_{i_1} \wedge \mathrm{d}p_{i_2} \wedge \mathrm{d}p_{i_3} \tag{3.4}$$

where $q_{i_1,i_2,i_3}$ is a function on $\mathrm{Tot}(\Lambda^3 M)$, linear on fibers and expressed as

$$q_{i_1,i_2,i_3} := \frac{\mathrm{d}}{\mathrm{d}p_{i_1}} \wedge \frac{\mathrm{d}}{\mathrm{d}p_{i_3}} \wedge \frac{\mathrm{d}}{\mathrm{d}p_{i_3}}$$

(here we identify the 3-vector fields on $M$ with linear functions on $\mathrm{Tot}(\Lambda^3 M)$). The form $\Xi$ is a 4-dimensional analog of the usual Hamiltonian 2-form on $\mathrm{Tot}(\Lambda^1 M)$ and satisfies similar standard properties. It is called "the Hamiltonian 4-form" in [24].

In [24], the following result was proven.

**Proposition 3.13** *Let $M$ be a holonomy $G_2$-manifold, and $\Omega$ be the 3-form on $S^6M$ constructed above. Then $-\sqrt{-1}\,\mathrm{d}\Omega = \mathrm{d}(\pi^*\rho^* \rfloor \theta)$ is equal to $\varphi^*\Xi$, where $S^6M \stackrel{\varphi}{\hookrightarrow} \mathrm{Tot}(\Lambda^3 M)$ is the embedding defined in* (3.3).

## 4 A Kähler structure on a knot space of a $G_2$-manifold

### 4.1 Knot spaces on CR-manifolds

Let $\mathbb{S}$ and $M$ be smooth manifolds, $\mathrm{Imm}(\mathbb{S}, M)$ the set of immersions from $\mathbb{S}$ to $M$, and $\mathrm{Knot}(\mathbb{S}, M) := \mathrm{Imm}(\mathbb{S}, M)/\mathsf{Diff}(\mathbb{S})$ the corresponding knot orbifold. Clearly,

$\mathrm{Knot}(\mathbb{S}, M)$ is a Fréchet orbifold, modeled on the space of sections of a normal bundle NS in a neighborhood of a point $S \in \mathrm{Knot}(\mathbb{S}, M)$.

**Definition 4.1** Suppose that $(M, B, I)$ is a CR-manifold. A knot $S \in \mathrm{Knot}(\mathbb{S}, M)$ is called **transversal**, if for all $s \in S$, the intersection of $T_s S \cap B\big|_s = 0$, that is, $B$ is transversal to $TS$ everywhere. Denote the space of transversal knots by $\mathrm{Knot}_B(\mathbb{S}, M)$.

*Remark 4.2* Let $(M, B, I)$ be a CR-manifold, and $\mathrm{Knot}_B(\mathbb{S}, M)$ be the space of transversal knots. Suppose that $\dim \mathbb{S} = \operatorname{codim} B$. For each knot $S \in \mathrm{Knot}_B(\mathbb{S}, M)$, one has $\mathrm{NS} \cong B\big|_S$, hence $\mathrm{Knot}_B(\mathbb{S}, M)$ has a natural almost complex structure.

**Theorem 4.3** *Let $(M, B, I)$ be a CR-manifold, $\mathbb{S}$ be a smooth manifold for which $\dim \mathbb{S} = \operatorname{codim} B$, and $\mathrm{Knot}^0_B(\mathbb{S}, M)$ the space of transversal knots which are embedded to $M$ (that is, have no self-intersection), equipped with a complex structure as above. Then $\mathrm{Knot}^0_B(\mathbb{S}, M)$ is a formally complex Fréchet manifold.*

*Proof* Consider an embedded knot $S \in \mathrm{Knot}^0_B(\mathbb{S}, M)$. The space $T^{1,0}_S \mathrm{Knot}^0_B(\mathbb{S}, M)$ of $(1,0)$-tangent vectors is by definition equal to the space of sections of $B^{1,0}\big|_S$. Let $X, Y \in B^{1,0}\big|_S$ be some sections of the bundle $B^{1,0}\big|_S$. We extend $X, Y$ to sections $X_1, Y_1$ of $B^{1,0}$ in some neighborhood of $S$ (this is possible, because $S$ has no self-intersections). The vector fields $X_1, Y_1 \in TM$ are used in a usual way to define vector fields $\widetilde{X}, \widetilde{Y} \in T^{1,0}\mathrm{Knot}^0_B(\mathbb{S}, M)$ satisfying $\widetilde{X}\big|_S = X$, $\widetilde{Y}\big|_S = Y$. Denote by $Z_1 = [X_1, Y_1]$ the commutator of $X_1, Y_1$. Since $(M, B, I)$ is a CR-manifold, $Z_1 \in B^{1,0}$. Denote by $\widetilde{Z} \in T^{1,0}\mathrm{Knot}^0_B(\mathbb{S}, M)$ the corresponding vector field on $\mathrm{Knot}^0_B(\mathbb{S}, M)$, defined in a neighborhood of $S$. Clearly, $[\widetilde{X}, \widetilde{Y}] = \widetilde{Z}$, and its $(0,1)$-component vanishes. Therefore, the Nijenhuis tensor

$$N : T^{1,0}_S \mathrm{Knot}^0_B(\mathbb{S}, M) \times T^{1,0}_S \mathrm{Knot}^0_B(\mathbb{S}, M) \longrightarrow T^{0,1}_S \mathrm{Knot}^0_B(\mathbb{S}, M)$$

vanishes on arbitrarily chosen vectors $X, Y$. We proved that $N = 0$, hence $\mathrm{Knot}^0_B(\mathbb{S}, M)$ is integrable. □

The same argument also proves the following theorem.

**Theorem 4.4** *Let $(M, B, I)$ be a CR-manifold, and $F \subset TM$ be a sub-bundle of $TM$ containing $B$. Consider the space $\mathrm{Knot}^0_F(\mathbb{S}, M)$, defined as above, and let $\mathcal{B} \subset T\,\mathrm{Knot}^0_F(\mathbb{S}, M)$ be a sub-bundle consisting of all $\xi \in T_S \mathrm{Knot}^0_F(\mathbb{S}, M)$ which belong to $\Gamma(B\big|_S) \subset \Gamma(F\big|_S) = T_S \mathrm{Knot}^0_F(\mathbb{S}, M)$ (here $\Gamma$ denotes the space of global sections of a vector bundle). Consider a complex structure operator on $\mathcal{B}$ induced by $I : B\big|_S \longrightarrow B\big|_S$. Then $(B, I)$ is an integrable CR-structure on $\mathrm{Knot}^0_F(\mathbb{S}, M)$.*

## 4.2 Two failed proofs of formal integrability and a definition of $L$-knots

It looks natural to obtain the integrability of the almost complex structure on $\mathrm{Knot}(M)$ for a $G_2$-manifold $M$ directly from Theorem 4.4. The first naive approach is to use the standard projection $\mathrm{Knot}^0_F(S^6 M) \xrightarrow{\pi} \mathrm{Knot}(M)$, where $\mathrm{Knot}^0_F(S^6 M) =$

$\operatorname{Knot}^0_F(S^1, S^6M)$ is the space of knots transversal to the bundle $F = T_{\mathsf{ver}}S^6M \oplus B$. It is easy to see that the projection of a knot in $S^6M$ transversal to $T_{\mathsf{ver}}S^6M$ is a knot in $M$. By Theorems 4.4 and 3.11, $\operatorname{Knot}^0_F(S^6M)$ is equipped with an integrable CR-structure $\mathcal{B}^{1,0}$ induced from $B^{1,0} \subset TS^6M \otimes \mathbb{C}$. The projection $\operatorname{Knot}^0_F(S^6M) \stackrel{\pi}{\longrightarrow} \operatorname{Knot}(M)$ induces a map

$$\mathrm{d}\pi : \mathcal{B} \longrightarrow T\operatorname{Knot}(M) \tag{4.1}$$

which is an isomorphism at each point of $S^6M$ (it follows directly from an isomorphism $T_S \operatorname{Knot}(M) = \Gamma(B\big|_S)$, where $\Gamma(B\big|_S)$ is the space of sections of the restriction $B\big|_S$). Were $\mathrm{d}\pi$ also compatible with the complex structures, integrability of the complex structure on $\operatorname{Knot}(M)$ would follow immediately from the integrability of $\mathcal{B}^{1,0}$. Unfortunately, this is not so. In fact, it is easy to see that the projection (4.1) is compatible with the complex structures only at the one point in each fiber of $\pi$ (Remark 4.7). This one point corresponds to a special class of knots in $S^6M$ called *Legendrian* in [16]. To avoid confusion with the usual Legendrian knots on the contact manifold $S^6M$, we call the Lempert's Legendrian knots **the $L$-knots**.

**Definition 4.5** Let $M$ be a Riemannian 7-manifold, $S \in \operatorname{Knot}_{T_{\mathsf{ver}}S^6M}(S^6M)$ a transversal knot on $S^6M$, and $S_1 \in \operatorname{Knot}(M)$ its projection to $M$. Consider a unit speed parametrization $\gamma_1 : S^1 \longrightarrow S_1$, and let $\gamma : S^1 \longrightarrow S$ be the corresponding parametrization in $S$. Each point $t \in S^1$ gives a unit vector $\dot\gamma_1(t) \in T_{\gamma_1(t)}M$. Assume that $\gamma(t) = (\dot\gamma_1(t), \gamma_1(t))$. Then $S$ is called **an $L$-knot**.

We return now to the case when $M$ is a $G_2$-manifold.

*Remark 4.6* Clearly, there is precisely one $L$-knot in every fiber of the projection $\operatorname{Knot}^0_{T_{\mathsf{ver}}S^6M}(S^6M) \stackrel{\pi}{\longrightarrow} \operatorname{Knot}(M)$. This gives a smooth section $\Psi : \operatorname{Knot}(M) \longrightarrow \mathsf{LKnot}(M)$, where $\mathsf{LKnot}(M) \subset \operatorname{Knot}^0_F(S^6M)$ denotes the space of $L$-knots.

*Remark 4.7* For each $L$-knot $S$, the map

$$\mathrm{d}\pi : \mathcal{B}\big|_S \longrightarrow T_{\pi(S)}\operatorname{Knot}(M) \tag{4.2}$$

is complex linear. Indeed, for each $(v, m) \in S$, the complex structure in $B\big|_S$ is given by a vector product with a vector $v$, and the complex structure in the normal bundle $N\pi(S)$ is given by a vector product with a unit tangent vector to $\pi(S)$. These vectors are equal precisely when $S$ is an $L$-knot.

From this argument, it is clear that (4.2) is complex linear only if $S$ is an $L$-knot.

This leads us to the second failed argument for the proof of integrability of $\operatorname{Knot}(M)$. It is natural to expect the embedding

$$\Psi : \operatorname{Knot}(M) \longrightarrow \mathsf{LKnot}(M) \subset \operatorname{Knot}_F(S^6M)$$

together with Theorem 3.11 to bring us integrability of the complex structure on $\operatorname{Knot}(M)$. Unfortunately, this argument does not work, because the bundle $\mathcal{B} \subset T\operatorname{Knot}_F(S^6M)$ is not tangent to the submanifold $\mathsf{LKnot}(M) \subset \operatorname{Knot}_F(S^6M)$.

This is why a direct application of Theorem 3.11 to the integrability of complex structure on $\mathrm{Knot}(M)$ does not work. Instead, we use a different approach, which employs Theorem 3.11 only as a framework.

### 4.3 The tangent space to the space of $L$-knots

To prove the integrability of the knot space, we use an explicit description of the tangent space $T\,\mathsf{LKnot}(M)$. The following claim is trivial.

**Claim 4.8** *Consider the natural action of the group* $\mathsf{Diff}(M)$ *on the space* $S^6M$*, interpreted as a double cover of the projectivization* $\mathbb{P}TM$*. Then* $\mathsf{LKnot}(M) \subset \mathrm{Knot}(S^6M)$ *is a* $\mathsf{Diff}(M)$*-invariant subset of the corresponding knot space.*

*Proof* The definition of $\mathsf{LKnot}(M)$ is functorial with respect to diffeomorphisms, hence $\mathsf{LKnot}(M)$ is clearly $\mathsf{Diff}(M)$-invariant. □

Since the natural projection $\pi : \mathsf{LKnot}(M) \longrightarrow \mathrm{Knot}(M)$ is a diffeomorphism of Fréchet manifolds, to describe the tangent space $T\,\mathsf{LKnot}(M) \subset T\,\mathrm{Knot}(S^6M)$, we need to describe the image of vector fields $X \in T\,\mathrm{Knot}(M)$ in $T\,\mathsf{LKnot}(M) \subset T\,\mathrm{Knot}(S^6M)\Big|_{\mathsf{LKnot}(M)}$.

**Proposition 4.9** *We consider knot spaces on a Riemannian manifold $M$ of dimension 7.*[1] *Let $S_1 \in \mathrm{Knot}(M)$, and let $S \in \mathsf{LKnot}(M)$ be the corresponding $L$-knot. Consider the decomposition*

$$T_S\,\mathrm{Knot}(S^6M) = \Gamma\left(T_{\mathsf{ver}}(S^6M)\big|_S\right) \oplus \Gamma\left(B\big|_S\right) \tag{4.3}$$

*obtained from the Levi-Civita-induced orthogonal decomposition $TS^6M = T_{\mathsf{ver}}(S^6M) \oplus B \oplus \mathbb{R}\cdot\theta$ (Definition 3.10). Let $\gamma_1$ be a unit speed parametrization of $S_1$. Consider a vector $X \in T_S\,\mathsf{LKnot}(M) \subset T_S\,\mathrm{Knot}(S^6M)$, and let $X_1 \in T_{S_1}\,\mathrm{Knot}(M) = \Gamma(B\big|_S)$ be the corresponding tangent vector to $S_1$. Then the decomposition of $X$ into components of (4.3) is written as*

$$X = -\nabla_{\dot\gamma_1} X_1 + X_1, \tag{4.4}$$

*where $\nabla_{\dot\gamma_1} X_1$ is a $\Gamma(T_{\mathsf{ver}}(S^6M)\big|_S)$-component of $X$, and $X_1$ the $\Gamma(B\big|_S)$-component.*

*Proof* Clearly, it would suffice to prove (4.4) for a dense open subset $\mathrm{Knot}^0(M) \subset \mathrm{Knot}(M)$ of embedded knots. For such a knot, we may extend $X_1$ to a vector field $\widetilde{X}_1 \in TM$. Let $\widetilde{X}$ be the correspoding vector field tangent to $\mathsf{LKnot}(M)$ (Claim 4.8). Clearly, $\widetilde{X}\big|_S = X$. The action of $\widetilde{X}$ on $\mathsf{LKnot}(M)$ is a restriction of a vector field $\widetilde{X}_{S^6M} \in T\,\mathrm{Knot}(S^6M)$ acting on $S^6M$ by functoriality (here we again identify $S^6M$ with a double cover of $\mathbb{P}TM$). To finish the proof of Proposition 4.9 it would remain to compute the vector field $\widetilde{X}_{S^6M}$.

[1] In fact, the proof of Proposition 4.9 would hold for any dimension $> 2$.

At a point $(\dot\gamma_1, m) \in S \subset S^6M$, the vector field $\widetilde{X}_{S^6M}$ is equal to $(-\nabla_{\dot\gamma_1} X_1, X_1)$, which gives

$$\widetilde{X}_{S^6M}\big|_S = (-\nabla_{\dot\gamma_1} X_1, X_1). \tag{4.5}$$

Indeed, for any section $V \in TM$,

$$\operatorname{Lie}_{\widetilde{X}} V = [\widetilde{X}, V] = \nabla_{\widetilde{X}} V - \nabla_V \widetilde{X},$$

and choosing $V$ in such a way that $\nabla_{\widetilde{X}} V = 0$, and $V = v$, we obtain that

$$e^{t\widetilde{X}}(v, m) = (v - t\nabla_v \widetilde{X} + o(t^2), e^{t\widetilde{X}} m).$$

Then (4.5) clearly follows. From (4.5), (4.4) is apparent. □

For the proof of integrability of the complex structure on $\operatorname{Knot}(M)$, the following proposition is used.

**Proposition 4.10** *Let $M$ be a holonomy $G_2$-manifold, and $\Xi \in \Lambda^4(S^6M)$ the 4-form obtained as in (3.4). Consider the corresponding 4-form $\widetilde{\Xi}$ on $\operatorname{Knot}(S^6M)$ mapping vector fields $X_1, \dots, X_4 \in T_S \operatorname{Knot}(S^6M)$ to the integral $\int_\gamma \Xi(X_1, \dots, X_4)\mathrm{d}t$, where $\gamma(t)$ is a unit speed parametrization of $S$. Then $\widetilde{\Xi}\Big|_{\mathsf{LKnot}(M)} = 0$.*

*Proof* Let $X_i^{\mathsf{ver}}$, $X_i^{\mathsf{hor}}$ be the vertical and horizontal components of $X_1 \in T_S \operatorname{Knot}(S^6M)$, under the natural decomposition

$$T_S \operatorname{Knot}(S^6M) = \Gamma\left(T_{\mathsf{ver}}(S^6M)\big|_S\right) \oplus \Gamma\left(B\big|_S\right).$$

In [24], it was shown that $\Xi\Big|_{T_{\mathsf{hor}}S^6M}$ vanishes. Also, from the local formula (3.4) it is apparent that $\Xi(v_1, v_2, \cdot, \cdot) = 0$, where $v_1, v_2 \in T_{\mathsf{ver}}(S^6M)$. Therefore,

$$\Xi(X_1, \dots, X_4) = \operatorname{Alt} \rho^*(X_1^{\mathsf{ver}}, X_2^{\mathsf{hor}}, X_3^{\mathsf{hor}}, X_4^{\mathsf{hor}}),$$

where Alt denotes the skew-symmetrization over the indexes 1, 2, 3, 4. However, by Proposition 4.9, $X_i^{\mathsf{ver}} = -\nabla_{\dot\gamma} X_i^{\mathsf{hor}}$, hence

$$\widetilde{\Xi}(X_1, \dots, X_4) = \int_\gamma \operatorname{Alt} \rho^*(\nabla_{\dot\gamma} X_1^{\mathsf{hor}}, X_2^{\mathsf{hor}}, X_3^{\mathsf{hor}}, X_4^{\mathsf{hor}}). \tag{4.6}$$

Since $M$ is a holonomy $G_2$-manifold, $\nabla_{\dot\gamma}\rho^* = 0$, and (4.6) gives

$$\widetilde{\Xi}(X_1, \dots, X_4) = \int_\gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^*(X_1^{\mathsf{hor}}\Big|_{\gamma(t)}, \dots, X_4^{\mathsf{hor}}\Big|_{\gamma(t)}\right)\mathrm{d}t,$$

which is equal 0 by Stokes' theorem. □

### 4.4 Non-degenerate (3, 0)-forms

**Definition 4.11** Let $M$ be a smooth or Fréchet manifold, equipped with an almost complex structure, and $\Omega \in \Lambda^{3,0}(M)$ a (3, 0)-form. We say that $\Omega$ is **non-degenerate** if for any $X \in T^{1,0}(M)$ there exist $Y, Z \in T^{1,0}(M)$ such that $\Omega(X, Y, Z) \neq 0$.

The utility of non-degenerate (3,0)-forms is due to the following simple theorem.

**Theorem 4.12** *Let $M$ be a smooth or a Fréchet manifold equipped with an almost complex structure, and $\Omega \in \Lambda^{3,0}(M)$ a non-degenerate $(3, 0)$-form. Assume that $\mathrm{d}\Omega = 0$. Then $M$ is formally integrable.*

*Proof* Let $X, Y \in T^{1,0}(M)$ and $Z, T \in T^{0,1}(M)$. Since $\Omega$ is a (3,0)-form, it vanishes on (0, 1)-vectors. Then Cartan's formula together with $\mathrm{d}\Omega = 0$ implies that

$$0 = \mathrm{d}\Omega(X, Y, Z, T) = \Omega(X, Y, [Z, T]). \tag{4.7}$$

From the non-degeneracy of $\Omega$, we obtain that unless $[Z, T] \in T^{0,1}(M)$, for some $X, Y \in T^{1,0}M$ one would have $\Omega(X, Y, [Z, T]) \neq 0$. Therefore, (4.7) implies $[Z, T] \in T^{0,1}(M)$, for all $Z, T \in T^{0,1}(M)$, which means that $M$ is integrable. □

For the proof of integrability of the almost complex structure on the knot space, we also need the following trivial lemma.

**Lemma 4.13** *Let $M$ be a $G_2$-manifold, and $\Omega := \pi^*\rho + \sqrt{-1}(\pi^*\rho^* \rfloor \theta) \in \Lambda^3(S^6M)$ be the 3-form constructed in 3.2. Denote by $\widetilde{\Omega}$ the corresponding form on $\mathsf{LKnot}(M)$, mapping $X_1, X_2, X_3 \in T_S\,\mathsf{LKnot}(S^6M) \subset T_S\,\mathrm{Knot}(S^6M)$ to $\int_{\gamma(t)} \Omega(X_1, X_2, X_3)\mathrm{d}t$, where $\gamma(t)$ is a unit speed parametrization of $S$. Then $\Omega$ a non-degenerate $(3, 0)$-form on $\mathsf{LKnot}(M) \cong \mathrm{Knot}(M)$.*

*Proof* The space $T_S\,\mathrm{Knot}(M)$ is identified with the space of sections $\Gamma(B\big|_S)$, where $B$ is an $SU(3)$-bundle defined in Definition 3.10. From its construction, it is clear that $\Omega\Big|_{\tau_{\mathsf{ver}}S^6M}$ vanishes, and on $B$ the form $\Omega$ is equal to the standard complex volume form. Therefore, $\Omega$ is of type $(3, 0)$, and for each $X \in B^{1,0}\big|_S$, there exist $Y, Z \in B^{1,0}\big|_S$ such that $\Omega(X, Y, Z)$ is everywhere real, non-negative, and positive at any point where $X \neq 0$. Then, $\widetilde{\Omega}(X, Y, Z)$ is real and positive, unless $X = 0$. □

### 4.5 The proof of integrability of the knot space

Let $M$ be a holonomy $G_2$-manifold, and $S \subset M$ *a knot*, that is, a class of an immersion $S^1 \overset{\gamma}{\hookrightarrow} M$ up to oriented reparametrizations, and injective outside of a finite set. We can assume that $S$ is parametrized, with $|\gamma'| = \mathsf{const}$ (such a parametrization is obviously unique).

The tangent space $T_S\,\mathrm{Knot}(M)$ is identified with the space of sections of the normal bundle NS. At each point $s \in S$, $N_sS = (T_sS)^{\perp}$ is the orthogonal complement to an oriented line $T_sS \subset T_sM$. Then Remark 3.6 gives a complex structure on $N_sS$. This defines an almost complex structure on $\mathrm{Knot}(M)$. To define a Hermitian form, the following construction is used.

**Definition 4.14** Let $\operatorname{Knot}^m(M) \subset \operatorname{Knot}(M) \times M$ be the space of *marked knots*, that is, pairs $(S^1 \stackrel{\gamma}{\hookrightarrow} M, s \in S^1)$, where $\gamma$ is injective somewhere, and $|\gamma'| = 1$. Clearly, the forgetful map $\operatorname{Knot}^m(M) \stackrel{\pi}{\longrightarrow} \operatorname{Knot}(M)$ is an $S^1$-fibration. The fiberwise integration map

$$\Lambda^i(\operatorname{Knot}^m(M)) \stackrel{\pi_*}{\longrightarrow} \Lambda^{i-1}(\operatorname{Knot}(M))$$

is defined as usual,

$$\pi_*(\alpha)\big|_S := \int\limits_{\pi^{-1}(S)} \left(\alpha \lrcorner \frac{\mathrm{d}}{\mathrm{d}t}\right) \mathrm{d}t$$

where $t$ is a parameter on $S$. It is easy to check that $\pi_*$ commutes with the de Rham differential. Define $\sigma : \operatorname{Knot}^m(M) \longrightarrow M$ as follows,

$$\sigma\left(S^1 \stackrel{\gamma}{\hookrightarrow} \operatorname{Knot}(M), s \in S^1\right) := \gamma(s).$$

This gives an interesting map

$$\pi_*\sigma^* : \Lambda^i(M) \longrightarrow \Lambda^{i-1}(\operatorname{Knot}(M))$$

commuting with the de Rham differential.

For a $G_2$-manifold $(M, \rho)$, the 2-form $\pi_*\sigma^*(\rho)$ was computed by M. Movshev in [18], who proved that it is symplectic.

**Claim 4.15** *Let $(M, \rho)$ be an almost $G_2$-manifold, $S \in \operatorname{Knot}(M)$ a knot, and $\alpha, \beta \in NS$ two sections of a normal bundle, considered as tangent vectors $a, b \in T_S \operatorname{Knot}(M)$. Consider the integral $S(a, b) := \int_S \rho(a, b, \cdot)\big|_S$. Then $\pi_*\sigma^*(\rho)(a, b) = S(a, b)$.*

*Proof* This claim is essentially a restatement of a definition (see [18] for more detail). □

Comparing Claim 4.15 and Corollary 3.9, we obtain the following result.

**Proposition 4.16** *Let $(M, \rho)$ be an almost $G_2$-manifold $\omega := \pi_*\sigma^*(\rho)$ the Movshev's 2-form on $\operatorname{Knot}(M)$, and $I$ the almost complex structure on $\operatorname{Knot}(M)$ constructed above. Then $(\operatorname{Knot}(M), I, \omega)$ is an almost complex Hermitian Fréchet manifold.* □

The main result of this paper is the following theorem.

**Theorem 4.17** *Let $M$ be a holonomy $G_2$-manifold, and $(\operatorname{Knot}(M), I, \omega)$ an almost complex Hermitian Fréchet manifold constructed above. Then $(\operatorname{Knot}(M), I, \omega)$ is formally Kähler.*

*Remark 4.18* The manifold $(\operatorname{Knot}(M), \omega)$ is symplectic [18]. This is clear from the construction of $\omega = \pi_*\sigma^*(\rho)$, because $\pi_*\sigma^*$ commutes with the de Rham differential.

*Proof of Theorem 4.17* To prove integrability of the almost complex structure on $\operatorname{Knot}(M)$, we identify $\operatorname{Knot}(M)$ with the space of $L$-knots $\mathsf{LKnot}(M) \subset \operatorname{Knot}(S^6 M)$

as in Remark 4.6. As follows from Lemma 4.13, the space $\mathsf{LKnot}(M)$ is equipped with a non-degenerate $(3,0)$-form $\widetilde{\Omega}$. From Proposition 3.13 it follows that $\mathrm{d}\widetilde{\Omega} = \sqrt{-1}\,\widetilde{\Xi}$, and from Proposition 4.10 that $\widetilde{\Xi}\Big|_{\mathsf{LKnot}(M)} = 0$, hence the form $\widetilde{\Omega}$ is closed on $\mathsf{LKnot}(M)$. Now, integrability of the almost complex structure on $\mathrm{Knot}(M)$ follows from Theorem 4.12. □

## 5 The complex structure on the knot space and $G_2$-geometry

### 5.1 Associative subvarieties of a $G_2$-manifold and complex subvarieties in its knot space

The complex geometry of a knot space can be used to study the geometry of a $G_2$-manifold. Many notions of a $G_2$-geometry can be directly translated to the language of complex geometry, as follows.

**Definition 5.1** Let $X \subset M$ be a 3-dimensional subvariety of a $G_2$-manifold. We say that $X$ is **associative** if $T_x X \subset T_x M$ is an associative subspace for each smooth point $x \in X$ (see Sect. 3.2 for a definition of an associative subspace).

**Proposition 5.2** *Let $M$ be a holonomy $G_2$-manifold, and $A \subset \mathrm{Knot}(M)$ a 1-dimensional complex subvariety. Denote by $\widetilde{A} \subset M$ the union of all knots in $A$. Then $\widetilde{A}$ is an associative subvariety of $M$.*

*Proof* Let $\gamma \in A$ be a knot, and $x, y \in T_\gamma A$ two tangent vectors, considered as sections of a normal bundle $N\gamma$, with $I(x) = y$. A complex structure on $T_\gamma A$ is given by a vector product with the unit vector field $\frac{\gamma'}{|\gamma'|}$ (Remark 3.7). Therefore, the 3-dimensional space $\langle x, y, \gamma' \rangle$ is closed under the vector product. □

**Proposition 5.3** *Let $M$ be a holonomy $G_2$-manifold, and $X \subset M$ a subvariety, $1 < \dim X < 7$. Then $\mathrm{Knot}(X) \subset \mathrm{Knot}(M)$ is a formally complex subvariety if and only if $X$ is an associative subvariety.*

*Proof* The same argument as in Proposition 5.2 proves that $T_x X \subset T_x M$ is closed with respect to the vector product, for any smooth point $x \in X$. However, any proper subalgebra of octonions is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, as an easy algebraic argument implies. Since $1 < \dim X < 7$, this is a quaternion subalgebra, and the subspace $T_x X$ 3-dimensional and associative. □

### 5.2 Holomorphic bundles on a knot space

$G_2$ instanton bundles were introduced in [8] and much studied since then. This notion is a special case of a more general notion of an instanton on a calibrated manifold, which is already well developed. Many estimates known for 4-dimensional manifolds (such as Uhlenbeck's compactness theorem) can be generalized to the calibrated case [20,21].

Recently, $G_2$-instantons became a focus of much activity because of attempts to construct a higher-dimensional topological quantum field theory, associated with $G_2$ and 3-dimensional Calabi–Yau manifolds [7].

**Definition 5.4** Let $M$ be a $G_2$-manifold, and $\Lambda^2 M = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M)$ the irreducible decomposition of the bundle of 2-forms $\Lambda^2(M)$ associated with the $G_2$-action. A vector bundle $(B, \nabla)$ with connection is called **a $G_2$-instanton** if its curvature lies in $\Lambda^2_{14}(M) \otimes \operatorname{End}(B)$.

*Remark 5.5* Since the curvature of a holonomy $G_2$-manifold lies in $\Lambda^2 M \otimes \mathfrak{g}_2$, and $g_2 \subset \mathfrak{so}(TM)$ is identified with $\Lambda^2_{14}$ under the identification $\mathfrak{so}(TM) = \Lambda^2(M)$, the curvature of $TM$ lies in $\Lambda^2_{14} M \otimes \operatorname{End}(TM)$. Therefore, the tangent bundle and all its tensor powers are $G_2$-instantons.

*Remark 5.6* Let $M$ be a finite-dimensional complex manifold, and $E$ a Hermitian bundle on $M$. Recall that a holomorphic structure on $E$ induces a unique Hermitian connection $\nabla$ on $E$ which its curvature $\Theta$ satisfying $\Theta \in \Lambda^{1,1}(M) \otimes \operatorname{End} E$ and $\nabla^{0,1} = \overline{\partial}$, where $\overline{\partial}$ is the holomorphic structure operator (this connection is called **the Chern connection**). This motivates the following definition.

**Definition 5.7** Let $(F, I)$ be a formally complex Fréchet manifold, and $(E, \nabla)$ a Hermitian bundle with connection. We say that $(E, \nabla)$ is **formally holomorphic** if the curvature $\Theta$ of $\nabla$ satisfies $\Theta \in \Lambda^{1,1}(F) \otimes \operatorname{End} E$.

*Remark 5.8* Let $E$ be a vector bundle with connection on a Riemannian manifold $M$, and $\operatorname{Knot}(M)$ be the knot space of $M$. For a given $S \in \operatorname{Knot}(M)$, consider the space $E(S)$ of sections of $E\big|_S$. Consider an infinite-dimensional bundle $\widetilde{E}$ on $\operatorname{Knot}(M)$ with fiber $E(S)$ at $S \in \operatorname{Knot}(M)$. This bundle can be obtained as $\pi_\bullet \sigma^* E$, where $\sigma : \operatorname{Knot}^m(M) \longrightarrow M$, $\pi : \operatorname{Knot}^m(M) \longrightarrow \operatorname{Knot}(M)$ are the maps defined in Sect. 4.5, and $\pi_\bullet$ is a pushforward, considered in the sense of sheaf theory. Also, every connection on $E$ induces a connection $\widetilde{\nabla} := \pi_\bullet \sigma^* \nabla$ on $\widetilde{E}$.

**Theorem 5.9** *Let $M$ be a $G_2$-manifold,* $\operatorname{Knot}(M)$ *its knot space equipped with a natural formally Kähler structure, $(E, \nabla)$ a Hermitian vector bundle with connection, and $(\widetilde{E}, \widetilde{\nabla})$ the corresponding bundle on* $\operatorname{Knot}(M)$. *Then $(\widetilde{E}, \widetilde{\nabla})$ is formally holomorphic if and only if $\nabla$ is a $G_2$-instanton.*

*Proof* Clearly, the curvature $\widetilde{\Theta}$ of $\widetilde{E}$ is obtained by lifting the curvature $\Theta$ of $E$ to $\operatorname{Knot}(M)$ in a natural way. From [24, Proposition 3.2], it follows that a form belongs to $\Lambda^2_{14}(M)$ if and only if its restriction to each 6-dimensional subspace $x^\perp \subset T_m M$ is of type (1, 1). This is equivalent to $\widetilde{\Theta}$ being of type (1,1) on $\operatorname{Knot}(M)$.

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