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On nilpotent and solvable Lie algebras of derivations

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A B S T R A C T

Let K be a field and A be a commutative associative K-algebra which is an integral domain. The Lie algebra $\mathrm{Der}_K A$ of all K-derivations of A is an A-module in a natural way, and if R is the quotient field of A then $R\,\mathrm{Der}_K A$ is a vector space over R. It is proved that if L is a nilpotent subalgebra of $R\,\mathrm{Der}_K A$ of rank k over R (i.e. such that $\dim_R RL = k$), then the derived length of L is at most k and L is finite dimensional over its field of constants. In case of solvable Lie algebras over a field of characteristic zero their derived length does not exceed $2k$. Nilpotent and solvable Lie algebras of rank 1 and 2 (over R) from the Lie algebra $R\,\mathrm{Der}_K A$ are characterized. As a consequence we obtain the same estimations for nilpotent and solvable Lie algebras of vector fields with polynomial, rational, or formal coefficients. Analogously, if X is an irreducible affine variety of dimension n over an algebraically closed field K of characteristic zero and A_X is its coordinate ring, then all nilpotent (solvable) subalgebras of $\mathrm{Der}_K A_X$ have derived length at most n ($2n$ respectively).

Introduction

Let $\mathbb{K}$ be a field and A be an associative commutative $\mathbb{K}$-algebra with unity, without zero divisors, i.e. an integral domain. The set $\mathrm{Der}_{\mathbb{K}} A$ of all $\mathbb{K}$-derivations of A, i.e.

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$\mathbb{K}$-linear operators D on A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$ is a Lie algebra over $\mathbb{K}$ and an A-module in a natural way: given $a \in A$, $D \in \operatorname{Der}_{\mathbb{K}} A$, the derivation aD sends any element $x \in A$ to $a \cdot D(x)$. The structure of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ is of great interest because, in geometric terms, derivations can be considered as vector fields on geometric objects. For example, in case $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[x_1, \dots, x_n]$, the polynomial ring, any $D \in \operatorname{Der}_{\mathbb{K}} A$ is of the form

$$D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, \quad f_i \in \mathbb{C}[x_1, \dots, x_n],$$

i.e. D is a vector field on $\mathbb{C}^n$ with polynomial coefficients. Lie algebras of vector fields with polynomial, formal power series, or analytical coefficients were studied intensively by many authors (see, for example, [7,1–4,11,12]).

In general case, when A is an integral domain, subalgebras L of $\operatorname{Der}_{\mathbb{K}} A$ such that L are submodules of the A-module $\operatorname{Der}_{\mathbb{K}} A$ were studied in [6] (see also [10,13]), and sufficient conditions were given for L to be simple. In this paper, we study subalgebras of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ at the other extreme: nilpotent and solvable, under the condition that they are of finite rank over A. Recall that if R is the quotient field of A, then the rank $\operatorname{rk}_R L$ is defined as $\operatorname{rk}_R L = \dim_R RL$. Any subalgebra L of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ determines uniquely the field $F = F(L)$ of constants consisting of all $r \in R$ such that $D(r) = 0$ for all $D \in L$. The vector space FL over the field F is actually a Lie algebra over F (note that RL being a Lie algebra over $\mathbb{K}$ is not in general a Lie algebra over R). The main results of the paper: if L is a nilpotent subalgebra of the Lie algebra $R \operatorname{Der}_{\mathbb{K}} A$ with $\operatorname{rk}_R L = k$, then the derived length of L is at most k and the Lie algebra FL is finite dimensional over F (Theorem 1). In case when L is solvable, $\operatorname{rk}_R L = k$ and $\operatorname{char} \mathbb{K} = 0$, the derived length does not exceed $2k$ (Theorem 2). If $\dim_{\mathbb{K}} L < \infty$, then the last estimation can be improved to $k + 1$.

If we consider the important case $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[[x_1, \dots, x_n]]$, the ring of formal power series, we get that nilpotent (solvable) subalgebras of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A$ of rank k over R have derived length $\leqslant k$ ($\leqslant 2k$, respectively). Note that in this particular case it was proved in [9] that all nilpotent subalgebras have derived length at most n and solvable at most $2n$ (see Corollary 3).

One can apply obtained results for vector fields on an affine variety X and obtain analogous bounds for the derived length of nilpotent and solvable subalgebras of the Lie algebra $\operatorname{Der}_{\mathbb{K}} A_X$ where A_X is the coordinate ring of X (see Corollary 4).

We also give a rough characterization of nilpotent and solvable subalgebras of rank 1 and 2 over R from the Lie algebra $R \operatorname{Der}_{\mathbb{K}} A$ (over their fields of constants). Such a characterization can be applied to study finite dimensional Lie algebras of smooth vector fields in three variables (the case of one and two variables was studied in [7,3,4]). Using the same approach we gave in [8] a description of finite dimensional subalgebras of $W(A)$ in case $A = \mathbb{K}(x, y)$, the field of rational functions.

We use standard notations, the ground field $\mathbb{K}$ is arbitrary unless otherwise stated. The quotient field of the integral domain A under consideration will be denoted by R.

Any derivation D of A can be uniquely extended to a derivation of R by the rule: $D(a/b) = (D(a)b - aD(b))/b^2$. It is obvious that $R\operatorname{Der}_{\mathbb{K}} A$ is a subalgebra of the Lie algebra $\operatorname{Der}_{\mathbb{K}} R$ and $\operatorname{Der}_{\mathbb{K}} A$ is embedded in a natural way into $R\operatorname{Der}_{\mathbb{K}} A$. We will denote $R\operatorname{Der}_{\mathbb{K}} A$ by $W(A)$, it is a vector space over R of dimension $\operatorname{rk}_R \operatorname{Der}_{\mathbb{K}} A$, and a Lie algebra over $\mathbb{K}$ but not in general case over R. All subspaces and subalgebras of $W(A)$ will be considered over the field $\mathbb{K}$ unless otherwise stated. If L is a subalgebra of the Lie algebra $W(A)$, then the field $F = \{r \in R \mid D(r) = 0 \text{ for all } D \in L\}$ will be called the *field of constants* of L. We denote by $s(L)$ the derived length of a (solvable) Lie algebra L. If a Lie algebra L contains an ideal N and a subalgebra B such that $L = N + B$, $N \cap B = 0$, then we write $L = B \leftthreetimes N$ for the semidirect sum of B and N. Let V be a vector space of dimension n over $\mathbb{K}$ and $gl(V)$ the general linear Lie algebra of V. The external semidirect sum $gl(V) \leftthreetimes V$ (with the natural action of $gl(V)$ on V) will be called the general affine Lie algebra and denoted by $ga_n(\mathbb{K})$ (in case $\mathbb{K} = \mathbb{R}$ it is the Lie algebra of the general affine group $GA_n(\mathbb{R})$).

1. Nilpotent subalgebras of finite rank of the Lie algebra $W(A)$

We will use the next statement which can be immediately checked.

Lemma 1. *Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then*:

1. $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$;
2. *If $a, b \in \ker D_1 \cap \ker D_2$, then $[aD_1, bD_2] = ab[D_1, D_2]$.*

Let L be a nonzero subalgebra of rank k over R of the Lie algebra $W(A)$ and let $\{D_1, \dots, D_k\}$ be a basis of L over R. Recall that the set RL of $W(A)$ consists of all linear combinations of elements aD, where $a \in R$, $D \in L$; analogously one can define the set FL ($F = F(L)$ is the field of constants of L).

Lemma 2. *Let L be a nonzero subalgebra of $W(A)$ and FL, RL be $\mathbb{K}$-spaces defined as above. Then*:

1. *FL and RL are $\mathbb{K}$-subalgebras of the Lie algebra $W(A)$. Moreover, FL is a Lie algebra over the field F.*
2. *If the algebra L is abelian, nilpotent, or solvable then the Lie algebra FL has the same property, respectively.*

Proof. Immediate check. □

Lemma 3. *Let L be a subalgebra of finite rank over R of the Lie algebra $W(A)$, $Z = Z(L)$ be the center of L and F be the field of constants of L. Then $\operatorname{rk}_R Z = \dim_F FZ$ and FZ*

is a subalgebra of the center $Z(FL)$. In particular, if L is abelian, then FL is an abelian subalgebra of $W(A)$ and $\operatorname{rk}_R L = \dim_F FL$.

Proof. Let $\{D_1, \dots, D_k\}$ be a basis of Z over R. Take any element $D \in Z$ and write $D = a_1 D_1 + \dots + a_k D_k$, where $a_i \in R$. Then for any element $S \in L$ we have:

$$0 = [S, D] = [S, a_1 D_1 + \dots + a_k D_k] = S(a_1) D_1 + \dots + S(a_k) D_k.$$

Since the elements $D_1, \dots, D_k$ are linearly independent over R it follows from the last relation that $S(a_i) = 0$, $i = 1, \dots, k$. Hence $a_i \in F$, $i = 1, \dots, k$ and $\{D_1, \dots, D_k\}$ is a basis of FZ over F. The latter means that $\operatorname{rk}_R Z = \dim_F FZ$. $\square$

Lemma 4. *Let L be a subalgebra of the Lie algebra $W(A)$ and I be an ideal of L. Then the vector space $RI \cap L$ (over $\mathbb{K}$) is also an ideal of L.*

Proof. Take any element $\sum_{k=1}^{m} r_k i_k \in RI \cap L$ with $r_k \in R$, $i_k \in I$, $k = 1, \dots, m$. Then for an arbitrary element $D \in L$ we obtain:

$$\left[D, \sum_{k=1}^{m} r_k i_k\right] = \sum_{k=1}^{m} \bigl(D(r_k) i_k + r_k [D, i_k]\bigr) \in RI \cap L.$$

This completes the proof of the lemma. $\square$

Lemma 5. *Let L be a nilpotent subalgebra of rank $k > 0$ over R of the Lie algebra $W(A)$. Then*:

1. *L contains a series of ideals*

$$0 = I_0 \subset I_1 \subset I_2 \subset \dots \subset I_k = L \tag{1}$$

   such that $\operatorname{rk}_R I_s = s$, $s = 0, \dots, k$.
2. *L possesses an R-basis $\{D_1, \dots, D_k\}$ such that $I_s = L \cap (RD_1 + \dots + RD_s)$, $s = 1, \dots, k$ and $[L, D_s] \subset I_{s-1}$.*
3. *$\dim_F FL/FI_{k-1} = 1$.*
4. *$[I_j, I_j] \subset I_{j-1}$, $j = 1, \dots, k$.*

Proof. 1–2. Take a nonzero element $D_1 \in Z(L)$ and put $I_1 = RD_1 \cap L$. Then I_1 is an ideal of L by Lemma 4. Assume that we have built the set of elements $D_1, \dots, D_j$ such that the $\mathbb{K}$-spaces $I_s = L \cap (RD_1 + \dots + RD_s)$, $s = 1, \dots, j$ are ideals of the Lie algebra L and $[L, D_s] \subset I_{s-1}$ for $s = 1, \dots, j$ with $\operatorname{rk}_R I_s = s$. Take a one-dimensional ideal $\langle D_{j+1} \rangle + I_j$ of the (nilpotent) quotient algebra L/I_j. Then $[L, D_{j+1}] \subset I_j$ and the elements $D_1, \dots, D_{j+1}$ are linearly independent over R. Put $I_{j+1} = L \cap (RD_1 + \dots + RD_{j+1})$. Then I_{j+1} is an ideal of L by Lemma 4 and $\operatorname{rk}_R I_{j+1} = j + 1$. Therefore we obtain by induction the chain (1) of ideals and a basis $\{D_1, \dots, D_k\}$ of L. This basis satisfies obviously condition 2 of the lemma.

3. Take an arbitrary element $D = a_1 D_1 + \dots + a_k D_k \in L$ and any element D_i from the basis $\{D_1, \dots, D_k\}$. Then using Lemma 1 we get:

$$\left[D_i, \sum_{j=1}^{k} a_j D_j\right] = \sum_{j=1}^{k} D_i(a_j) D_j + \sum_{j=1}^{k} a_j [D_i, D_j].$$

Since $[D_i, I_s] \subseteq I_{s-1}$ we see from the last relation that $D_i(a_k) = 0$, $i = 1, \dots, k$. The latter means $a_k \in F$ and therefore $\dim_F FL/FI_{k-1} = 1$. Part 3 of the lemma is proved.

4. This part of the lemma is a consequence of its parts 2 and 3. □

Corollary 1. *Let L be a nilpotent subalgebra of rank k over R of the Lie algebra $W(A)$. Then the derived length of L is at most k.*

Proof. See part 4 of Lemma 5. □

Remark 1. We will use the next almost obvious statement: If V is a vector space over the field $\mathbb{K}$ and U, W are subspaces of V of finite codimension, then the subspace $U \cap W$ is also of finite codimension in V.

Lemma 6. *Let V be a vector space over the field $\mathbb{K}$ and L be a finite dimensional $\mathbb{K}$-subspace of $\operatorname{End}(V)$. Suppose that L acts nilpotently on V (i.e. $L^n(V) = 0$ for some $n \geqslant 1$). If the vector space $V_0 = \{v \in V \mid Lv = 0\}$ is finite dimensional over $\mathbb{K}$, then $\dim V < \infty$.*

Proof. Induction on the smallest number n such that $L^n(V) = 0$. If $n = 1$ then $LV = 0$, $V = V_0$, hence $\dim(V) < \infty$ by the conditions of the lemma. Consider the $\mathbb{K}$-subspace $U = L(V)$ of V. The vector space $U_0 = \{u \in U \mid Lu = 0\}$ has obviously finite dimension over $\mathbb{K}$ and $L^{n-1}(U) = 0$. By the inductive assumption $\dim U < \infty$. Choose a basis $\{g_1, g_2, \dots, g_k\}$ of L over $\mathbb{K}$. It follows from the proven above that $\dim V/\ker g_i < \infty$ because the linear operator g_i maps V into U and $\dim U < \infty$. But then $\dim V/\bigcap_{i=1}^{k} \ker g_i < \infty$ by Remark 1 and therefore V_0 is of finite codimension in V. Since $\dim V_0 < \infty$ by the conditions of the lemma, we obtain $\dim V < \infty$. □

Theorem 1. *Let L be a nilpotent subalgebra of finite rank over R from the Lie algebra $W(A)$ and $F = F(L)$ be the field of constants of L. Then the Lie algebra FL is finite dimensional over F.*

Proof. Let $k = \operatorname{rk}_R L$ and $0 = I_0 \subseteq I_1 \subseteq \dots \subseteq I_k = L$ be the series of ideals of L, constructed in Lemma 5. Take a basis $\{D_1, \dots, D_k\}$ of L over R obtained in such a way as in Lemma 5. We prove by induction on i that $\dim_F FL/FI_{k-i} < \infty$. It is true for $i = 1$ by Lemma 5, part 3. Assume that $\dim_F FL/FI_j < \infty$ for $j = k - i$ and consider the natural action (by multiplication) of FL on the F-space $V = FI_j/FI_{j-1}$. It holds $[FI_j, FI_j] \subset FI_{j-1}$ by Lemma 5 and therefore $FI_jV = 0$. Hence V is a module over the finite dimensional (over F) Lie algebra FL/FI_j. The Lie algebra FL/FI_j acts nilpotently on V because the algebra FL is nilpotent. Let $V_0 = \{v \in V \mid (FL/FI_j)v = 0\}$ and $D = a_1D_1 + \cdots + a_jD_j$ a representative of an arbitrary element from $V_0 \subseteq V = FI_j/FI_{j-1}$. Then for any $i = 1, \ldots, k$ we have

$$\begin{aligned}[D_i, D] &= [D_i, a_1D_1 + \cdots + a_jD_j] \\ &= [D_i, a_1D_1 + \cdots + a_{j-1}D_{j-1}] + a_j[D_i, D_j] + D_i(a_j)D_j \in I_{j-1}.\end{aligned}$$

The first and second summands in the right side of the last equality lie in I_{j-1}, so $D_i(a_j)D_j \in I_{j-1}$. The latter means that $D_i(a_j) = 0$, $i = 1, \ldots, k$ and therefore $a_j \in F$ by definition of the field F. Thus $\dim_F V_0 = 1$ and Lemma 6 yields $\dim_F FI_j/FI_{j-1} < \infty$. But then $\dim_F FL/FI_{j-1} = \dim_F FL/FI_{k-(i+1)} < \infty$. When $i = k$ we obtain the inequality $\dim_F FL < \infty$. $\square$

Proposition 1. *Let L be a nilpotent subalgebra of $W(A)$ and $F = F(L)$ be its field of constants. Then*:

1. *If $\mathrm{rk}_R L = 1$, then L is abelian and $\dim_F FL = 1$.*
2. *If $\mathrm{rk}_R L = 2$, then there exist elements $D_1, D_2 \in FL$ and $a \in R$ such that*

$$FL = F\left\langle D_1, aD_1, \ldots, \frac{a^k}{k!}D_1, D_2\right\rangle, \quad k \geqslant 0 \ \big(\textit{if } k = 0, \ \textit{then put } FL = F\langle D_1, D_2\rangle\big),$$

where $[D_1, D_2] = 0$, $D_1(a) = 0$, $D_2(a) = 1$.

Proof. 1. It follows from Lemma 5, part 3.

2. Let $\mathrm{rk}_R L = 2$. Suppose that $\mathrm{rk}_R Z(L) = 2$ and let $\{D_1, D_2\}$ be a basis of $Z(L)$ over R. Put $I_k = RD_k \cap L$, $k = 1, 2$. Since $I_1 \cap I_2 = 0$ and $\dim_F FL/FI_k = 1, k = 1, 2$ by Lemma 5 we see that $\dim_F FL = 2$ and $FL = F\langle D_1, D_2 \mid [D_1, D_2] = 0\rangle$ is of type 2 of the lemma. Let now $\mathrm{rk}_R Z(L) = 1$, $D_1 \in Z(L)$ be a nonzero element and $I_1 = RD_1 \cap L$. Then I_1 is an ideal of L and $\dim_F FL/FI_1 = 1$ by Lemma 5. Choose any nonzero element $D_2 \in L \setminus I_1$. The elements D_1, D_2 form a basis of L over R and $[D_1, D_2] = 0$. Since the Lie algebra L is nilpotent the operator $\operatorname{ad} D_2$ acts nilpotently on the abelian ideal FI_1 of the algebra FL over the field F.

Let us show that $\operatorname{ad} D_2$ has in some basis of FI_1 (over F) the matrix which is a single Jordan block. Really, any Jordan chain for $\operatorname{ad} D_2$ on FI_1 contains an element of the form aD_1 such that $[D_2, aD_1] = 0$. But then $D_2(a) = 0$ and taking into account the equality $D_1(a) = 0$ we get $a \in F$. The latter means that $\operatorname{ad} D_2$ has the only Jordan chain $\{D_1, a_1D_1, \ldots, a_kD_1\}$ on I_1 with $a_i \in R$ and its matrix in this basis is a single Jordan block. Since $[D_2, D_1] = 0, [D_2, a_1 D_1] = D_1, \dots, [D_2, a_k D_1] = a_{k-1} D_1$ we have

$$D_2(a_1) = 1, \qquad D_2(a_2) = a_1, \quad \dots, \quad D_2(a_k) = a_{k-1}.$$

Denoting $a = a_1$ we obtain $D_2(a_2 - a^2/2!) = 0$. Since also $D_1(a_2 - a^2/2!) = 0$ we get $a_2 - a^2/2! \in F$. But then without loss of generality we can take $a_2 = a^2/2!$. Repeating these considerations we obtain a basis $\{D_1, aD_1, \dots, (a^n/n!)D_1\}$ of the ideal FI_1. $\square$

Remark 2. Let $A = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial algebra in n variables over $\mathbb{K}$. Then $\operatorname{Der}_{\mathbb{K}} A = W_n(\mathbb{K})$ is the Lie algebra of all vector fields on $\mathbb{K}^n$ with polynomial coefficients. Take the elements $D_1 = \frac{\partial}{\partial x_1}$, $D_2 = \frac{\partial}{\partial x_2}$ from the Lie algebra $W_n(\mathbb{K})$ and put $a = x_2 \in A$. It is obvious that the Lie algebra

$$L_n = F\left\langle \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_1}, \dots, (x_2^n/n!) \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle,$$

where $F = \mathbb{K}(x_3, \dots, x_n)$ is the field of constants for L, is nilpotent of nilpotency class $n-2$. If we consider the union $L = \bigcup_{i=1}^{\infty} L_i$ of the ascending chain of Lie algebras $L_1 \subset L_2 \subset L_3 \subset \dots \subset L_n \subset \cdots$, then the algebra L is solvable of derived length 2 and infinite dimensional over F (one can show that L is locally nilpotent but not nilpotent).

2. Solvable subalgebras of $W(A)$

Lemma 7. *Let L be a solvable subalgebra of rank 1 over R from the Lie algebra $W(A)$ and $F = F(L)$ be its field of constants. Then:*

1. *If L is abelian, then FL is one dimensional over F.*
2. *If L is nonabelian, then $\dim_F FL = 2$. In particular, $s(L) = 2$.*

Proof. 1. Let L be abelian. Since L is nilpotent of rank 1 over R, it follows from Proposition 1 that $\dim_F FL = 1$.

2. Suppose that L is nonabelian and take a maximal (by inclusion) abelian ideal $I \subset L$ and a nonzero element $D_1 \in I$. Then $FI = FD_1$ is of dimension 1 over F by the proven above. Choose any two elements $b_1 D_1, b_2 D_1 \in L \setminus I$ (recall that all elements of L are of the form aD_1 for some $a \in R$). Since I is a maximal abelian ideal we have $C_L(I) = I$ and therefore $[D_1, b_i D_1] = D_1(b_i) D_1 \neq 0$, $i = 1, 2$. Denoting $D_1(b_i) = a_i$, $i = 1, 2$ we obtain from the last relations that a_1, a_2 are nonzero elements of the field $F = \ker D_1$. But then $D_1(a_1^{-1} b_1 - a_2^{-1} b_2) = 0$ and therefore $a_1^{-1} b_1 - a_2^{-1} b_2 \in F$. The latter means that the elements $b_1 D_1$, $b_2 D_1$ are linearly dependent over F and FL is nonabelian of dimension 2 over F. $\square$

Remark 3. How to construct solvable subalgebras of rank 1 from $W(A)$? The answer is as following: to build an abelian Lie algebra one should take any $\mathbb{K}$-subspace V from the subfield $\ker D_1$ and set $L = VD_1$. Then L is abelian and every abelian Lie algebra of rank 1 over R can be obtained in such a way. To construct a nonabelian Lie algebra one should take a derivation D_1 possessing an element $b \in R$ such that $D_1(b) = 1$. Then L is a subalgebra of the Lie algebra $(\ker D_1)D_1 + (b \ker D_1)D_1$. The latter Lie algebra is isomorphic to the general affine Lie algebra $ga_1(\ker D_1)$.

Lemma 8. *Let L be a solvable subalgebra of rank k over R from the Lie algebra $W(A)$, $F = F(L)$ be its field of constants and I an ideal of L such that $I = RI \cap L$. If $\operatorname{rk}_R I = k-1$, then $\dim_F FL/FI \leqslant 2$, in particular, $s(L/I) \leqslant 2$. Besides, if $\dim_F FL/FI = 2$, then $s(L/I) = 2$.*

Proof. Take an R-basis $\{D_1, \ldots, D_k\}$ of L such that the elements $D_1, \ldots, D_{k-1}$ form an R-basis of I. Consider the following $\mathbb{K}$-subspace $M \subset RL$:

$$M = \{a_k D_k \mid \exists a_1, \ldots, a_{k-1} \text{ with } a_1 D_1 + \cdots + a_{k-1} D_{k-1} + a_k D_k \in L\}.$$

It is easy to see that M is a subalgebra of rank 1 over R from the Lie algebra RL. Since the subalgebra M has derived length $\leqslant 2$ by Lemma 7 and $L/I \simeq M$ we get that $s(L/I) \leqslant 2$.

Take any nonzero abelian ideal J/I of L/I. Any element $D \in J \setminus I$ can be written in the form $D = a_1 D_1 + \cdots + a_k D_k$ with $a_i \in R$ and $a_k \neq 0$. Then

$$\begin{aligned}[D_i, D] &= \left[D_i, \left(\sum_{i=1}^{k-1} a_i D_i\right) + a_k D_k\right] = \left[D_i, \left(\sum_{i=1}^{k-1} a_i D_i\right)\right] + [D_i, a_k D_k] \\ &= i_1 + D_i(a_k) D_k + a_k [D_i, D_k], \quad i = 1, \ldots, k-1,\end{aligned}$$

where $i_1 = [D_i, \sum_{i=1}^{k-1} D_i] \in I$ by the choice of the basis $\{D_1, \ldots, D_k\}$. Since $[D_i, D] \in I$, $i = 1, \ldots, k-1$ we get $D_i(a_k) = 0$, $i = 1, \ldots, k-1$. Further, we can assume without loss of generality that $D_k \in J$. As J/I is abelian we see that $[D_k, a_k D_k] \in I$ and therefore $D_k(a_k) = 0$. Then $a_k \in F$ and $\dim_F FJ/FI = 1$. In particular, if $\dim_F FL/FI = 2$, then $s(L/I) = 2$. If the quotient algebra L/I is abelian, then it holds $\dim_F FL/FI = 1$. Let FL/FI be nonabelian. Then its derived subalgebra is abelian and therefore is one-dimensional over F by Lemma 7. We may assume that $FD_k + FI/FI$ is the derived subalgebra of FL/FI. If $\dim_F FL/FI > 2$, then there exists an element $i_1 + aD_k \in FL \setminus (FD_k + FI)$ with $i_1 \in RI$ and $a \in R$ such that $[D_k, i_1 + a_k D_k] \in I$. The latter means that $D_k(a) = 0$ and taking into account the equalities $D_1(a_k) = 0, i = 1, \ldots, k-1$ obtained analogously, we see that $a_k \in F$. But then $i_1 + a_k D_k \in FD_k + FI$. The latter is impossible by the choice of this element and the obtained contradiction shows that $\dim_F FL/FI = 2$. □

The next statement can be easily deduced from the Lie Theorem for solvable Lie algebras over fields of characteristic zero and its modification over fields of positive characteristic (see, for example, [5], Theorem 4.1 and Exercise 2 on p. 20). We do not require the ground field to be algebraically closed because one can always consider all the objects over the algebraic closure $\overline{\mathbb{K}}$.

Lemma 9. *Let $\mathbb{K}$ be a field, V be a vector space of dimension n over $\mathbb{K}$ and L a solvable subalgebra of $gl_n(\mathbb{K})$. If $\operatorname{char}\mathbb{K} = 0$ or $\operatorname{char}\mathbb{K} > n$, then $s(L) \leqslant n$.*

Theorem 2. *Let L be a solvable subalgebra of rank k over R of the Lie algebra $W(A)$. If the ground field $\mathbb{K}$ is of characteristic zero, then the derived length $s(L)$ of L does not exceed $2k$. Moreover, if L is finite dimensional over its field of constants, then $s(L) \leqslant k+1$.*

Proof. Since $s(L) = s(FL)$ we can assume $L = FL$. Let J_1 be an abelian ideal of L of maximal rank over R, let $\operatorname{rk}_R J_1 = k_1$. Take a basis $D_1, \dots, D_{k_1}$ of J_1 over R and denote $I_1 = RJ_1 \cap L$. Then I_1 is also an ideal of L by Lemma 4 and $\operatorname{rk}_R I_1 = k_1$. Let J_2/I_1 be an abelian ideal of L/I_1 such that J_2 has maximal rank over R. Denote $I_2 = RJ_2 \cap L$. Then I_2 is an ideal of L of rank k_2 over R. As above take a basis $D_{k_1+1}, \dots, D_{k_2}$ of J_2/I_1. Continuing this consideration we can construct the series of ideals:

$$0 \subset J_1 \subseteq I_1 \subset \cdots \subset J_s \subseteq I_s = L,$$

with $\operatorname{rk}_R I_j = \operatorname{rk}_R J_j = k_j$, J_j/I_{j-1} is abelian, $I_j = RJ_j \cap L$, $j = 1, \dots, s$. Simultaneously we obtain an R-basis $\{D_1, \dots, D_{k_s}\}$ of L such that $D_{k_{j-1}+1}, \dots, D_{k_j}$ is a basis of J_j/I_{j-1}, $j = 1, \dots, s$.

Let us prove the statement of the theorem by induction on s. If $s = 0$ then $L = \{0\}$ and the proof is completed. Let $s \geqslant 1$. By the inductive assumption $s(I_{s-1}) \leqslant 2k_{s-1}$. Let us show that the abelian ideal J_s/I_{s-1} is of dimension $k_s - k_{s-1}$ over F. Really, for any element

$$D = c_1 D_1 + \cdots + c_{k_{s-1}} D_{k_{s-1}} + c_{k_{s-1}+1} D_{k_{s-1}+1} + \cdots + c_{k_s} D_{k_s} \in J_s$$

we have $[D_j, D] \in I_{s-1}$, $j = 1, \dots, k_s$. One can write:

$$[D_j, D] = \sum_{i=k_{s-1}+1}^{k_s} \bigl(D_j(c_i) D_i + c_i [D_j, D_i]\bigr) + \left[D_j, \sum_{i=1}^{k_{s-1}} c_i D_i\right].$$

Since $[D_j, D_i] \in I_{s-1}$, $i = k_{s-1}+1, \dots, k_s$ and the second sum in the right side lies in I_{s-1} we obtain that $D_j(c_i) = 0$, $j = 1, \dots, k_s$, $i = k_{s-1}+1, \dots, k_s$. Hence $c_i \in F$, $i = k_{s-1}+1, \dots, k_s$ by definition of F. Thus $\dim_F J_s/I_{s-1} = k_s - k_{s-1}$.

Note that we have also proved that the centralizer of J_s/I_{s-1} in the Lie algebra L/I_{s-1} coincides with J_s/I_{s-1}. Therefore L/J_s acts exactly on the F-vector space J_s/I_{s-1} of dimension $k_s - k_{s-1}$ over F. Since $C_{L/J_s}(J_s/I_{s-1}) = J_s/I_{s-1}$, then the solvable Lie algebra

L/J_s can be embedded isomorphically into the general linear Lie algebra $gl_{k_s-k_{s-1}}(F)$. As solvable subalgebras of this Lie algebra have derived length $\leqslant k_s - k_{s-1}$ (by Lemma 9), we see that $s(L/J_s) \leqslant k_s - k_{s-1}$. But then $s(L) \leqslant 2k_{s-1} + k_s - k_{s-1} \leqslant 2k_s = 2k$.

If L is finite dimensional over F, then $[L, L]$ is nilpotent of derived length $\leqslant k$ by Corollary 1. Therefore $s(L) \leqslant k + 1$. This completes the proof of the theorem. □

Remark 4. The first part of Theorem 2 remains valid also in the case of positive characteristic of the ground field $\mathbb{K}$ provided that $\operatorname{char} \mathbb{K} > k$ (because its proof uses only Lemma 9 with this restriction on the rank k).

Corollary 2. *Let $\mathbb{K}$ be a field and A be one of the following algebras over $\mathbb{K}$:*

(1) $\mathbb{K}[x_1, \dots, x_n]$ *the polynomial algebra*;
(2) $\mathbb{K}[[x_1, \dots, x_n]]$ *the algebra of formal power series*;
(3) $\mathbb{K}(x_1, \dots, x_n)$ *the field of rational functions*;
(4) $\mathbb{K}((x_1, \dots, x_n))$ *the fraction field of the algebra* $\mathbb{K}[[x_1, \dots, x_n]]$.

Let $\mathfrak{D}(A)$ be the Lie algebra of all $\mathbb{K}$-derivations D of A of the form $D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ with $f_i \in A$ (in cases (1) and (2) $\mathfrak{D}(A)$ obviously coincides with $\operatorname{Der}_{\mathbb{K}} A$). If L is a nilpotent subalgebra of $\mathfrak{D}(A)$, then L is finite dimensional over its field of constants and $s(L) \leqslant n$. If L is solvable and the ground field $\mathbb{K}$ is of characteristic zero, then $s(L) \leqslant 2n$.

Let $\mathbb{K} = \mathbb{C}$, $A = \mathbb{C}[[x_1, \dots, x_n]]$, and $\overline{W}_n(\mathbb{K}) = \operatorname{Der}_{\mathbb{K}} A$ be the Lie algebra of all vector fields with formal power series coefficients.

Corollary 3. *Let L be a nilpotent (solvable) subalgebra of $\overline{W}_n(\mathbb{K})$. Then the derived length of L does not exceed n ($2n$ respectively).*

The last statement was proved recently in [9], where it was used to study groups of automorphisms of formal power series rings. As the next example shows, the bound in Theorem 2 cannot be improved (see also [9]).

Example 1. Let $L = \{\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \in \overline{W}_n(\mathbb{K}) \mid a_j \in \mathbb{K}[[x_1, \dots, x_{j-1}]] + x_j \mathbb{K}[[x_1, \dots, x_{j-1}]]\}$. Then the derived length of L equals $2n$.

Corollary 4. *Let X be an irreducible affine variety of dimension n over an algebraically closed field $\mathbb{K}$ of characteristic zero and A_X be its coordinate ring. If L is a nilpotent (solvable) subalgebra of $\operatorname{Der}_{\mathbb{K}} A_X$, then the derived length of L is at most n ($2n$ respectively).*

If L is a solvable subalgebra of rank 2 over R of the Lie algebra $W(A)$, then L is contained in a maximal (by inclusion) solvable subalgebra of rank 2 over R. Really, let

S_2 be the set of all solvable subalgebras of rank 2 over R from $W(A)$. Using Theorem 2 one can easily show that the set S_2 is inductively ordered (by inclusion), so there exists by Zorn's Lemma at least one maximal element of S_2. The next statement shows the possible types of such maximal solvable subalgebras of rank 2 over R. Since any solvable subalgebra L of rank 2 over R from $W(A)$ is contained in a maximal subalgebra of the same type we get a rough characterization of such Lie algebras.

Proposition 2. *Let L be a solvable subalgebra of $W(A)$ which is maximal (by inclusion) among all solvable subalgebras of rank 2 over R from $W(A)$ and let $F = F(L)$ be its field of constants. If the ground field $\mathbb{K}$ is of characteristic zero, then L is a Lie algebra over F, the algebra L contains elements D_1, D_2 with $[D_2, D_1] = aD_1$ for some $a \in F_1 = \ker D_1$ and L is one of the following algebras over the field F:*

1. *$L = \langle D_2\rangle \leftthreetimes F_1D_1$.*
2. *$L = \langle D_2\rangle \leftthreetimes (F_1D_1 + bF_1D_1)$, where $b \in R$, $D_1(b) = 1$, $D_2(b) = ab + a_1$ for some $a_1 \in F_1$.*
3. *$L = (\langle D_2\rangle \leftthreetimes \langle cD_1 + dD_2\rangle) \leftthreetimes F_1D_1$, where $c \in R$, $d \in F_1$ such that $D_1(c) \in F_1$, $D_2(d) = 1$, $D_2(c) = -ac + r$ for some $r \in F_1$.*
4. *$L = (\langle D_2\rangle \leftthreetimes \langle cD_1 + dD_2\rangle) \leftthreetimes (F_1D_1 + F_1bD_1)$, where $D_1(b) = 1$, $D_2(d) = 1$, $d \in F_1$, $D_1(c) \in F_1$, $D_2(c) = ac + r$, $D_2(b) = ab + a_1$ for some $r, a_1 \in F_1$.*
5. *L is isomorphic to a solvable subalgebra of the affine Lie algebra $ga_2(F)$ containing F^2, in particular $2 \leqslant \dim_F L \leqslant 5$.*

Proof. Let L be a subalgebra of the Lie algebra $W(A)$ satisfying all the conditions of this proposition. Then FL as a Lie algebra over the field $\mathbb{K}$ also satisfies these conditions and $L \subseteq FL$. Therefore $FL = L$ because of maximality of L and L is a Lie algebra over the field F. We consider two cases dependent on properties of maximal abelian ideals of L:

Case 1. Every maximal abelian ideal of L is of rank 1 over R. Take any two such ideals I and J of L and let $D_1 \in I$, $D_2 \in J$ be nonzero elements. If D_1 and D_2 are linearly independent over R, then $I \cap J = 0$ and $I + J$ is an abelian ideal of rank 2 over R from L. But then $I + J$ is contained in a maximal abelian ideal of rank 2 over R from L which contradicts to our assumption. Therefore D_1 and D_2 are linearly dependent over R and $I + J$ is of rank 1 over R. Since $I + J$ is a nilpotent ideal of L it follows from Proposition 1 that $I + J$ is abelian. But then $I = J$ and I is the only maximal abelian ideal of rank 1 from L. Denote $I_1 = RI \cap L$. The ideal I_1 has rank 1 over R and $\dim_F L/I_1 \leqslant 2$ by Lemma 8. Take any nonzero element D_1 from I_1 provided that I_1 is abelian, or from the abelian ideal $[I_1, I_1]$ in other case (recall that I_1 has derived length at most 2). It can be easily shown that $[D_2, D_1] = aD_1$ for some element $a \in F_1 = \ker D_1$ and F_1I_1 is a subalgebra of $W(A)$ of rank 1 over R. It is easy to prove that $[D_2, F_1I_1] \subseteq F_1I_1$ and therefore $L + F_1I_1$ is a solvable subalgebra of rank 2 from $W(A)$. But then $L = L + F_1I_1$ because of maximality of L and hence $F_1I_1 \subseteq L$. The latter means that $F_1I_1 = I_1$ and I_1 is a Lie algebra over the field F_1.

Subcase 1. The ideal I_1 is abelian. If $\dim_F L/I_1 = 1$, then choosing any element $D_2 \in L \setminus I_1$ we see that $L = \langle D_2 \rangle \leftthreetimes F_1 D_1$ is a Lie algebra of type 1. Let $\dim_F L/I_1 = 2$. Then L/I_1 is nonabelian by Lemma 8. Take the one-dimensional ideal $\langle D_2 \rangle + I_1/I_1$ from the quotient algebra L/I_1. Take also any element $cD_1 + dD_2 \in L$ such that $[D_2, cD_1 + dD_2] = D_2 + rD_1$ for some element $rD_1 \in I_1$. This gives the equality $D_2(c)D_1 + caD_1 + D_2(d)D_2 = D_2 + rD_1$ which implies $D_2(d) = 1$, and $D_2(c) = -ac + r$. Besides, from the inclusion $[D_1, cD_1 + dD_2] \in I_1$ we get that $D_1(d) = 0$, i.e. $d \in F_1$. The same relation also gives $D_1(c) \in F_1$. We see that L is a Lie algebra of type 3 of the proposition.

Subcase 2. The ideal I_1 is nonabelian. Suppose first that $\dim_F L/I_1 = 1$ and take any element $D_2 \in L \setminus I_1$. In view of Lemma 7, $I_1 = F_1 D_1 + F_1 bD_1$ for some $b \in R$ such that $D_1(b) = 1$. Since $[D_2, D_1] = aD_1$ for some $a \in F_1$, it holds $[D_1, D_2](b) = aD_1(b) = a$. On the other hand $(D_1D_2 - D_2D_1)(b) = D_1(D_2(b)) = a$. But then $D_1(ba - D_2(b)) = a - a = 0$ and hence $ba - D_2(b) \in F_1$. Then $D_2(b) = ba + a_1$ for some element $a_1 \in F_1$ and L is a Lie algebra of type 2. Let now $\dim_F L/I_1 = 2$. The quotient algebra FL/FI_1 is nonabelian by Lemma 8. Take the one-dimensional ideal $\langle D_2 + I_1 \rangle$ from the quotient algebra L/I_1 (over F) and let $cD_1 + dD_2$ be such an element that $[D_2, cD_1 + dD_2] = D_2 + rD_1$ for some element $rD_1 \in I_1$. It follows from this relation that $D_2(d) = 1$ and $D_2(d) = -ac + r$ for $r \in F_1$. Further we have from the inclusion $[D_1, cD_1 + dD_2] \in I_1$ that $D_1(d) = 0$. This means that $d \in F_1$. Using the same inclusion we get $D_1(c) \in F_1$. Further, as above one can show that $D_2(b) = ab + a_1$ for some element $a_1 \in F_1$ and L is a Lie algebra of type 4.

Case 2. L contains at least one maximal abelian ideal of rank 2 over R. Denote it by J and choose any two elements D_1 and D_2 from J linearly independent over R. If $D = u_1 D_1 + u_2 D_2 \in J$, then from the equality

$$0 = [D_i, D] = [D_i, u_1 D_1 + u_2 D_2] = D_i(u_1)D_1 + D_i(u_2)D_2, \quad i = 1, 2$$

we obtain $D_i(u_j) = 0$. The latter means that $u_i \in F$, i.e. $\dim_F J = 2$. Since J is a maximal abelian ideal of L it holds $C_L(J) = J$. Therefore $\dim_F L/J \leqslant 3$ because of solvability of L/J and equality $\dim J = 2$. Let us consider the case $\dim L/J = 1$ and take any element $D_3 \in L \setminus J$. Then $D_3 = u_1 D_1 + u_2 D_2$ for some $u_1, u_2 \in R$. As

$$[D_i, D_3] = D_i(u_1)D_1 + D_i(u_2)D_2 \in J$$

we obtain $D_i(u_j) \in F$, $i, j = 1, 2$. If the matrix

$$\begin{pmatrix} D_1(u_1) & D_2(u_1) \\ D_1(u_2) & D_2(u_2) \end{pmatrix} \tag{2}$$

is nonsingular, then applying an appropriate linear transformation we can write

$$u_1 = \alpha_{11} v_1 + \alpha_{12} v_2, \qquad u_2 = \alpha_{21} v_1 + \alpha_{22} v_2$$

for some $\alpha_{ij} \in F$ and $D_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. It is obvious that $L_1 = F\langle D_1, D_2, v_i D_j \mid i, j = 1, 2\rangle$ is a Lie algebra of dimension 6 over F isomorphic to the general affine Lie algebra $ga_2(F)$. But then

$$D_3 = u_1 D_1 + u_2 D_2 = (\alpha_{11} v_1 + \alpha_{12} v_2) D_1 + (\alpha_{21} v_1 + \alpha_{22} v_2) D_2$$

is an element of L_1 and L is a subalgebra of L_1.

Let now the matrix (2) be degenerated. Since $D_3 \in L \setminus J$, at least one of the rows of the matrix (2) is nonzero, let the first. Without loss of generality we can assume that $D_1(u_1) = 1$, $D_2(u_1) = \gamma$ for some $\gamma \in F$. The second row of the matrix (2) is proportional to the first one and therefore $u_2 = \alpha u_1 + \beta$ for some $\alpha, \beta \in F$. Then we have $D_3 = u_1 D_1 + (\alpha u_1 + \beta) D_2$. Replacing the element D_3 by the element $D_3 - \beta D_2$ we can assume that $D_3 = u_1 D_1 + \alpha u_1 D_2$. If $\gamma = 0$, then $D_1(u_1) = 1, D_2(u_1) = 0$ and L is isomorphic to a subalgebra of $ga_2(F)$. In case $\gamma \neq 0$ we choose the basis $D_1' = D_1$, $D_2' = D_1 - \gamma^{-1} D_2$ of the abelian ideal J. Then we obtain $D_1'(u_1) = 1$, $D_2'(u_1) = 0$ and all is done. Analogously one can consider the cases $\dim L/J = 2$ and $\dim L/J = 3$ and show that L is isomorphic to a subalgebra of $ga_2(F)$. $\square$

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