The Maximum Independent Set Problem in Planar Graphs

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Abstract. We study the computational complexity of finding a maximum independent set of vertices in a planar graph. In general, this problem is known to be NP-hard. However, under certain restrictions it becomes polynomial-time solvable. We identify a graph parameter to which the complexity of the problem is sensible and produce a number of both negative (intractable) and positive (solvable in polynomial time) results, generalizing several known facts.

1 Introduction

Planar graphs form an important class both from a theoretical and practical point of view. The theoretical importance of this class is partly due to the fact that many algorithmic graph problems that are NP-hard in general remain intractable when restricted to the class of planar graphs. In particular, this is the case for the MAXIMUM INDEPENDENT SET (MIS) problem, i.e., the problem of finding in a graph a subset of pairwise non-adjacent vertices (an *independent set*) of maximum cardinality. Moreover, the problem is known to be NP-hard even for planar graphs of maximum vertex degree at most 3 [9] or planar graphs of large girth [15]. On the other hand, the problem can be solved in polynomial-time in some subclasses of planar graphs, such as outerplanar graphs [5] or planar graphs of bounded chordality [10].

Which other graph properties are crucial for the complexity of the problem in the class of planar graphs? Trying to answer this question, we focus on graph properties that are *hereditary* in the sense that whenever a graph possesses a

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certain property the property is inherited by all induced subgraphs of the graph. In other words, a class of graphs is hereditary if deletion of a vertex from a graph in the class results in a graph in the same class. Many important graph classes, such as bipartite graphs, perfect graphs, graphs of bounded vertex degree, graphs of bounded chordality, etc., are hereditary, including the class of planar graphs itself.

Any hereditary property can be described by a unique set of minimal graphs that do not possess the property – the so-called *forbidden induced subgraphs*. We shall denote the class of graphs containing no induced subgraphs from a set M by $Free(M)$. Any graph in $Free(M)$ will be called M-free. All our results are expressed in terms of some restrictions on the set of forbidden induced subgraphs M. In particular, in Section 2 we will impose a condition on the set M that will imply NP-hardness of the maximum independent set problem in the class of planar M-free planar graphs. In Section 3, by violating this condition we will reveal new polynomially solvable cases of the problem that generalize some of the previously studied classes.

All graphs in this paper will be finite, undirected, without loops or multiple edges. For a vertex $x \in V(G)$, we denote by $N(x)$ the neighborhood of x, that is, the set of vertices adjacent to x. The *degree* of x, $deg(x)$, is the size of its neighborhood. The *independence number* of a graph G is the maximum cardinality of an independent set in G. The *girth* of a graph is the length of its smallest cycle, while the *chordality* of a graph is the length of its largest chordless cycle. A *subdivision* of an edge uv consists in replacing the edge with a new vertex adjacent to u and v. For two graphs G and H, we denote by $G+H$ the disjoint union of G and H. In particular, nG is the disjoint union of n copies of G. As usual, P_n, C_n and K_n denote the chordless path, the chordless cycle and the complete graph on n vertices, respectively. $K_{n,m}$ is the complete bipartite graph with parts of size n and m. By T_s we denote the graph obtained by subdividing each edge of the complete bipartite graph $K_{1,s}$ exactly once. Also, A_k is the graph obtained by adding to a chordless cycle C_k a new vertex adjacent to exactly one vertex of the cycle. Following [6] we call this graph an *apple* of size k. $S_{i,j,k}$ and H_i are the two graphs shown in Figure 1.

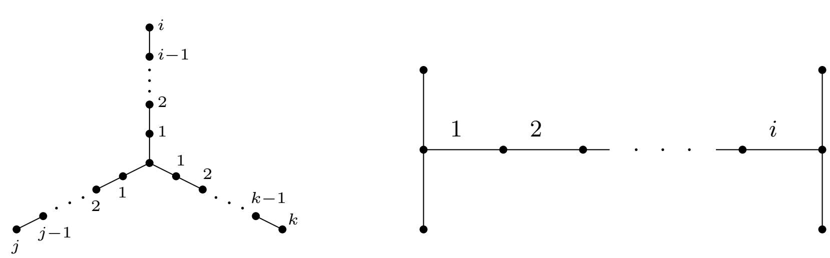

Fig. 1. Graphs $S_{i,j,k}$ (left) and H_i (right)

2 A Hardness Result

From [9] we know that the MIS problem is NP-hard for planar graphs of vertex degree at most 3. Murphy strengthened this result by showing that the problem is NP-hard for planar graphs of degree at most 3 and large girth [15]. This immediately follows from the fact that double subdivision of an edge increases the independence number of the graph by exactly one. The same argument can be used to show the following lemma, which, for the case of general (not necessarily) planar graphs, was first shown in [1].

Lemma 1. *For any k, the* MAXIMUM INDEPENDENT SET *problem is NP-hard in the class of planar $(C_3, \ldots, C_k, H_1, \ldots, H_k)$-free graphs of vertex degree at most 3.*

We now generalize this lemma in the following way. Let $\mathcal{S}_k$ be the class of $(C_3, \ldots, C_k, H_1, \ldots, H_k)$-free planar graphs of vertex degree at most 3. To every graph G we associate the parameter $\kappa(G)$, which is the maximum k such that $G \in \mathcal{S}_k$. If G belongs to no class $\mathcal{S}_k$, we define $\kappa(G)$ to be 0, and if G belongs to all classes $\mathcal{S}_k$, then $\kappa(G)$ is defined to be ∞. Finally, for a set of graphs M, we define $\kappa(M) = \sup\{\kappa(G) \ : \ G \in M\}$.

Theorem 1. *Let M be a set of graphs and X the class of M-free planar graphs of degree at most 3. If $\kappa(M) < \infty$, then the* MAXIMUM INDEPENDENT SET *problem is NP-hard in the class X.*

Proof. To prove the theorem, we will show that there is a k such that $\mathcal{S}_k \subseteq X$. Denote $k := \kappa(M) + 1$ and let G belong to $\mathcal{S}_k$. Assume that G does not belong to X. Then G contains a graph $A \in M$ as an induced subgraph. From the choice of G we know that A belongs to $\mathcal{S}_k$, but then $k \leq \kappa(A) \leq \kappa(M) < k$, a contradiction. Therefore, $G \in X$ and hence, $\mathcal{S}_k \subseteq X$. By Lemma 1, this implies NP-hardness of the problem in the class X. □

This negative result significantly reduces the area for polynomial-time algorithms. But still this area contains a variety of unexplored classes. In the next section, we analyze some of them.

3 Polynomial Results

Unless $P = NP$, the result of the previous section suggests that the MIS problem is solvable in polynomial time for graphs in a class of M-free planar graphs only if $\kappa(M) = \infty$. Let us distinguish a few major ways to push $\kappa(M)$ to infinity.

One of the possible ways to unbind $\kappa(M)$ is to include in M a graph G with $\kappa(G) = \infty$. According to the definition, in order for $\kappa(G)$ to be infinite, G must belong to every class $\mathcal{S}_k$. It is not difficult to see that this is possible only if every connected component of G is of the form $S_{i,j,k}$ represented on the left of Figure 1. Let us denote the class of all such graphs by $\mathcal{S}$. More formally, $\mathcal{S} := \bigcap_{k \geq 3} \mathcal{S}_k$. Any other way to push $\kappa(M)$ to infinity requires the inclusion in M of infinitely many graphs. In particular, we will be interested in classes where the set of forbidden subgraphs M contains graphs with arbitrarily large chordless cycles.

The literature does not contain many results when M includes a graph from the class $\mathcal{S}$, that is, a graph G with $\kappa(G) = \infty$, and only a few classes of this type are defined by a single forbidden induced subgraph. Minty [14] and Sbihi [18] independently of each other found a solution for the problem in the class of claw-free (i.e., $S_{1,1,1}$-free) graphs. This result was then generalized to $S_{1,1,2}$-free graphs (see [3] for unweighted and [11] for weighted version of the problem) and to $S_{1,1,1} + K_2$-free graphs [13]. Another important example of this type is the class of mP_2-free graphs (where m is a constant). A solution to the problem in this class is obtained by combining an algorithm to generate all maximal independent sets in a graph [20] and a polynomial upper bound on the number of maximal independent sets in mP_2-free graphs [2, 8].

Observe that all these results hold for general (not necessarily planar) graphs. In the case of planar graphs, the result for mP_2-free graphs can be further extended to the class of P_k-free graphs (for an arbitrary k) via the notion of tree-width. In fact, the diameter of P_k-free planar graphs is bounded by a constant. Therefore, the tree-width of P_k-free planar graphs is bounded by a constant, since the tree-width of planar graphs is bounded by a function of its diameter [7]. An extension of the result for P_k-free planar graphs was recently proposed in [10] where the authors show that the tree-width is bounded in the class of $(C_k, C_{k+1}, \ldots)$-free planar graphs (for any fixed k).

Below we report further progress in this direction. In particular, we show that the MIS problem can be solved in polynomial time in the class of $(A_k, A_{k+1}, \ldots)$-free planar graphs. We thus generalize the result not only for $(C_k, C_{k+1}, \ldots)$-free planar graphs, but also $S_{1,1,1}$-free, $S_{1,1,2}$-free and $S_{1,1,1} + K_2$-free planar graphs. Observe that, in contrast to planar graphs of bounded chordality, the tree-width of $(A_k, A_{k+1}, \ldots)$-free planar graphs is not bounded, which makes it necessary to employ more techniques for the design of a polynomial-time algorithm. One of the techniques we use in our solution is known as *decomposition by clique separators* [19, 21]. It reduces the problem to connected graphs without separating cliques, i.e., without cliques whose deletion disconnects the graph. We also use the notion of graph compression defined in the next section. In addition, in Section 3.2 we prove some auxiliary results related to the notion tree-width. Finally, in Section 3.3 we describe the solution.

3.1 Graph Compressions and Planar T_s-free Graphs

A *compression* of a graph $G = (V, E)$ is a mapping $\phi : V \to V$ which maps any two distinct non-adjacent vertices into non-adjacent vertices and which is not an automorphism. Thus, a compression maps a graph into its induced subgraph with the same independence number. Two particular compressions of interest will be denoted $\binom{a}{b}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

By $\binom{a}{b}$ we mean the compression which maps a to b and leaves all other vertices fixed. This map is a compression if and only if $ab \in E(G)$ and $N(b) - \{a\} \subseteq N(a) - \{b\}$.

The compression $\binom{a\ b}{c\ d}$ is defined as follows: $\phi(a) = c$, $\phi(b) = d$ and the remaining vertices of the graph are fixed. This map is a compression if

- $c \neq d$,
- $ac, bd \in E$ and $ab, cd \notin E$,
- every vertex adjacent to c different from a and b is also adjacent to a,
- every vertex adjacent to d different from a and b is also adjacent to b.

A graph which admits neither $\binom{a}{b}$ nor $\binom{a\ b}{c\ d}$ will be called *incompressible*.

Lemma 2. *Let G be an incompressible T_s-free planar graph and a, b two vertices of distance 2 in G. Then $|N(a) \cap N(b)| \leq 4s + 1$.*

Proof. Let us call a vertex $x \in N(a) \cap N(b)$

- *specific* if every neighbor of x, other then a and b, belongs to $N(a) \cap N(b)$,
- *a-clear* (*b-clear*) if x has a neighbor non-adjacent to a (to b).

Notice that every vertex in $N(a) \cap N(b)$ is either specific or a-clear or b-clear. Let us estimate the number of vertices of each type in $N(a) \cap N(b)$.

First, suppose that $N(a) \cap N(b)$ contains 4 specific vertices. Then, due to planarity of G, two of these vertices are non-adjacent, say x and y. But then $\binom{a\ b}{x\ y}$ is a compression. Therefore, $N(a) \cap N(b)$ contains at most 3 specific vertices.

Now suppose $N(a) \cap N(b)$ contains $2s$ a-clear vertices. Consider a plane embedding of G. This embedding defines a cyclic order of the neighbors of each vertex. Let $x_1, x_2, \ldots, x_{2s}$ be the a-clear vertices listed in the cyclic order with respect to a. Also, for each $i = 1, 2, \ldots, 2s$, denote by y_i a vertex adjacent to x_i and non-adjacent to a. Some of the vertices in the set $\{y_1, y_2, \ldots, y_{2s}\}$ may coincide but the vertices $\{y_1, y_3, y_5 \ldots, y_{2s-1}\}$ must be pairwise distinct and non-adjacent. But then the set $\{a, x_1, x_3, x_5 \ldots, x_{2s-1}, y_1, y_3, y_5 \ldots, y_{2s-1}\}$ induces a T_s. This contradiction shows that there are at most $2s - 1$ a-clear vertices. Similarly, there are at most $2s - 1$ b-clear vertices. □

Lemma 3. *Let G be an incompressible T_s-free planar graph. Then the degree of each vertex in G is at most $(4s + 1)(4s - 1)$.*

Proof. Let a be a vertex in G, A the set of neighbors of a and B the set of vertices of distance 2 from a. Consider the bipartite subgraph H of G formed by the sets A and B and all the edges connecting vertices of A to the vertices of B. Let the size of a maximum matching in H be π and the size of a minimum vertex cover in H be β. According to the theorem of König, $\pi = \beta$.

Observe that every vertex $x \in A$ has a neighbor in B, since otherwise $\binom{a}{x}$ is a compression. Thus, H contains a set D of $deg(a)$ edges no two of which share a vertex in A. By Lemma 2, the degree of each vertex of B in the graph H is at most $4s + 1$. Therefore, to cover the edges of D we need at least $deg(a)/(4s + 1)$ vertices, and hence $\pi \geq deg(a)/(4s + 1)$.

In the graph H, consider an arbitrary matching M with π edges. In the graph G, contract[5] each edge of M into a single vertex obtaining in this way a planar graph G', and denote the subgraph of G' induced by the set of "contracted" vertices (i.e., those corresponding to the edges of M) by H'. If $deg(a) > (4s+1)(4s-1)$, then H' contains at least $4s$ vertices. By the Four Color Theorem [17], it follows that H' contains an independent set of size s. The vertices of this set correspond to s edges in the graph G that induce an sK_2. Together with vertex a these edges induce a T_s, a contradiction. □

3.2 Tree-width and Planar Graphs

In this section, we derive several auxiliary results on the tree-width of planar graphs. More generally, our results are valid for any class of graphs excluding an apex graph as a minor. An *apex graph* is a graph that contains a vertex whose deletion leaves a planar graph. A graph H is said to be a *minor* of a graph G if H can be obtained from G by means of vertex deletions, edge deletions and edge contractions. We say that a class of graphs is *minor closed* if with every graph G it contains all minors of G. Both graphs of bounded tree-width and planar graphs are minor closed.

If H is not a minor of a graph G, we say that G is *H-minor-free* and call H a *forbidden minor* for X. It is well known that any minor-closed graph class can be described by a unique *finite* set of minimal forbidden minors. For instance, the class of planar graphs is exactly the class of $(K_5, K_{3,3})$-minor-free graphs.

For brevity, let us call a family of graphs *apex-free*, if it is defined by a single forbidden minor H, which is an apex graph.

An $n \times n$ grid G_n is the graph with the vertex set $\{1,\ldots,n\} \times \{1,\ldots,n\}$ such that (i,j) and (k,l) are adjacent if and only if $|i-k|+|j-l| = 1$. By a result of Robertson and Seymour [16], graphs of large tree-width must contain a large grid as a minor. For apex-free graph families, even more is true. In the following lemma, an *augmented grid* is a grid G_n augmented with additional edges (and no additional vertices). Vertices (i,j) with $\{i,j\} \cap \{1,n\} \neq \emptyset$ are *boundary vertices* of the grid; the other ones are *nonboundary.*

Lemma 4. *[7] Let H be an apex graph. Let $r = 14|V(H)|-22$. For every integer k there is an integer $g_H(k)$ such that every H-minor-free graph of tree-width at least $g_H(k)$ can be contracted into an $k' \times k'$ augmented grid R such that $k' \geq k$, and each vertex $v \in V(R)$ is adjacent to less than $(r+1)^6$ nonboundary vertices of the grid.*

With extensive help of this lemma we shall derive the main result of this section, which we state now.

Lemma 5. *For any apex graph H and integers k, s and d, there is an integer $N = N(H,k,s,d)$ such that for every H-minor-free graph G of tree-width at*

[5] The *contraction of an edge uv* consists of replacing the two vertices u and v with a single vertex x adjacent to every vertex in $(N(u) \cup N(v)) \setminus \{u,v\}$.

least N and every nonempty subset $S \subseteq V(G)$ of at most s vertices, the graph G contains a chordless cycle C such that:

- *every vertex $v \in V(G)$ is non-adjacent to at least k consecutive vertices of C.*
- *the distance between C and S is at least d.*

To prove Lemma 5, we will need a few auxiliary results. First, we recall that in apex-free graphs, large tree-width forces the presence of arbitrarily long chordless cycles [10]. More formally:

Lemma 6. *For every apex graph H and every integer k there is an integer $f_H(k)$ such that every H-minor-free graph of tree-width at least $f_H(k)$ contains a chordless cycle of order at least k.*

Next, we prove two additional lemmas that will be needed in the proof of Lemma 5.

Lemma 7. *For every apex graph H and every integer k there is an integer $f(H,k)$ such that every H-minor-free graph G of tree-width at least $f(H,k)$ contains a chordless cycle C such that every vertex $v \in V(G)$ is non-adjacent to at least k consecutive vertices of C.*

Proof. Let $r = 14|V(H)| - 22$, let f_H be the function given by Lemma 6, and let g_H be the function given by Lemma 4. Furthermore, let $f(H,k) = g_H\left(f_H\left((k+1)(r+1)^6\right)+2\right)$. We will show that the function $f(H,k)$ satisfies the claimed property.

Let G be an H-minor-free graph of tree-width at least $f(H,k)$. By Lemma 4, G can be contracted into an $k' \times k'$ augmented grid R where $k' \geq f_H((k+1)(r+1)^6)+2$ and such that each vertex $v \in V(R)$ is adjacent to less than $(r+1)^6$ non-boundary vertices of the grid. For $i,j \in \{1,\ldots,k'\}$, let $V(i,j)$ denote the subset of $V(G)$ that gets contracted to the vertex (i,j) of the grid. Furthermore, let R_0 denote the $(k'-2)\times(k'-2)$ augmented sub-grid, induced by the nonboundary vertices of R. Since the tree-width of an $n \times n$ grid is n, and the tree-width cannot decrease by adding edges, we conclude that the tree-width of R_0 is at least $k'-2 \geq f_H((k+1)(r+1)^6)$. Moreover, as R_0 is H-minor-free, Lemma 6 implies that R_0 contains a chordless cycle C_0 of length at least $(k+1)(r+1)^6$. By the above, every vertex $v \in V(R)$ is adjacent to less than $(r+1)^6$ vertices of R_0. Therefore, the neighbors of v on C_0 (if any) divide the cycle into less than $(r+1)^6$ disjoint paths whose total length is at least $|V(C_0)|-(r+1)^6$. In particular, this implies every vertex of $V(R)$ is non-adjacent to at least $\frac{|V(C_0)|-(r+1)^6}{(r+1)^6} \geq k$ consecutive vertices of C_0.

Let the cyclic order of vertices of R_0 on C_0 be given by $((i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s))$. To complete the proof, we have to lift the cycle C_0 to a chordless cycle C in G. Informally, we will replace each pair of incident edges $(i_{p-1},j_{p-1})(i_p,j_p)$ and $(i_p,j_p)(i_{p+1},j_{p+1})$ in C_0 with a shortest path connecting vertex (i_{p-1},j_{p-1}) to vertex (i_{p+1},j_{p+1}) in the graph G whose internal vertices all belong to V_{i_p,j_p}. Implementation details of this "lifting" procedure are omitted due to the lack of space. □

Our second preliminary lemma states that the tree-width of apex-free graphs cannot be substantially decreased by contracting the set of vertices at constant distance from some set of constantly many vertices. We remark that this fails for minor-closed families that exclude no apex graph (in the statement of the lemma, take G to be the graph obtained from an $n \times n$ grid by adding to it a dominating vertex).

Lemma 8. *Let H be an apex graph, and let s, d and m be integers. Then, there is an integer $t = t(H, s, d, m)$ such that the following holds:*

Let G be an H-minor-free graph of tree-width at least t, and let $S \subseteq V(G)$ be a set of at most s vertices of G. Furthermore, let U be the set of vertices in G that are at distance less than d from S, and let G' be the graph obtained from G by contracting the set U into a single vertex. Then, the tree-width of G' is at least m.

Proof. By an easy inductive argument on the number of connected components of $G[S]$, we may assume that S induces a connected subgraph of G. If $d = 0$, then $G' = G$, and we have $t = m$.

Let now $d \geq 1$. For $i = 1, \dots, d$, let $G^{(i)}$ denote the graph obtained from G by contracting the set $V^{(i)}$ of vertices at distance less than i from S into a single vertex $v^{(i)}$. Furthermore, let $r = 14|V(H)| - 22$. Also, let g_H be the function given by Lemma 4.

Consider the following recursively defined function $h : \{1, \dots, d\} \to \mathbb{N}$: $h(1) = m$, and $h(i+1) = g_H(2(r+1)^3 h(i))$, for all $i = 1, \dots, d-1$. Let $t := t(H, s, d, m) := h(d) + s$.

With the above notation, we have $G' = G^{(d)}$. So, it suffices to show the following:

Claim. For all $i = 1, \dots, d$, the tree-width of $G^{(i)}$ is at least $h(d + 1 - i)$.

We now prove the claim by induction on i. For $i = 1$, note that $G^{(1)}$ contains $G - S$ as an induced subgraph, and therefore $\mathrm{tw}(G^{(1)}) \geq \mathrm{tw}(G-S) \geq \mathrm{tw}(G) - s \geq t - s = h(d) = h(d - i + 1)$ (where $\mathrm{tw}(K)$ denotes the tree-width of a graph K).

For the induction hypothesis, assume that the statement holds for some $i \geq 1$: the tree-width of $G^{(i)}$ is at least $h(d + 1 - i) = g_H(2(r+1)^3 h(d-i))$. By Lemma 4, $G^{(i)}$ can be contracted into an $k \times k$ augmented grid R such that $k \geq 2(r+1)^3 h(d-i)$, and each vertex $v \in V(R)$ is adjacent to less than $(r+1)^6$ nonboundary vertices of the grid.

Therefore, R must contain a large subgrid R' such that $v^{(i)} \in V(G^{(i)})$ does not belong to R', and has no neighbors in R'. More precisely, R' can be chosen to be of size $k' \times k'$, where $k' \geq \lfloor \frac{k-2}{\sqrt{(r+1)^6}} \rfloor \geq \frac{k}{2(r+1)^3} \geq h(d-i)$. By definition of $V^{(i+1)}$ and since $v^{(i)}$ has no neighbors in R', we conclude that the graph $G^{(i+1)}$ contains the grid R' as a minor. Thus, the tree-width of $G^{(i+1)}$ is at least the tree-width of R', which is at least $h(d-i) = h(d + 1 - (i+1))$. The proof is complete. $\square$

We conclude this section with a short proof of Lemma 5, based on Lemmas 7 and 8.

Proof. (Lemma 5) Let $f(H, k)$ be given by Lemma 7. We let $N := N(H, k, s, d) := t(H, s, d+1, f(H, k))$, where t is given by Lemma 8.

Let G' be the graph obtained from G by contracting the set of vertices at distance less than $d+1$ form S into a single vertex. Then, by Lemma 5, the tree-width of G' is at least $f(H, k)$. By Lemma 7, G' contains a chordless cycle C such that every vertex $v \in V(G')$ is non-adjacent to at least k consecutive vertices of C.

Using the same argument as in the proof of Lemma 5, C' can be lifted to a chordless cycle C of G such that every vertex $v \in V(G)$ is non-adjacent to at least k consecutive vertices of C. □

3.3 Solution to the Problem for Planar $(A_k, A_{k+1}, \ldots)$-free Graphs

In this section, we prove polynomial-time solvability of the MIS problem in the class of planar $(A_k, A_{k+1}, \ldots)$-free graphs, for an arbitrary integer k.

Theorem 2. *For any k, the* MAXIMUM INDEPENDENT SET *problem can be solved in the class of planar $(A_k, A_{k+1}, \ldots)$-free graphs in polynomial time.*

Proof. Let k be an integer and G be a planar $(A_k, A_{k+1}, \ldots)$-free graph. Without loss of generality we can assume that G is incompressible and has no clique separators. If G is T_{11}-free, then by Lemma 3 the degree of vertices in G is bounded by a constant. It was recently shown in [12] that the MIS problem in the class of $(A_k, A_{k+1}, \ldots)$-free graphs of bounded vertex degree is polynomial-time solvable. This enables us to assume that G contains a T_{11} as an induced subgraph. In this subgraph, we will denote the vertex of degree 11 by a, the vertices of degree 2 by $b_1, \ldots, b_{11}$ and the respective vertices of degree 1 by $c_1, \ldots, c_{11}$.

Let $N = N(K_5, 6k+8, 23, k+2)$ be the constant defined in Lemma 5. We shall show that the tree-width of G is less than N. Assume by contradiction that the tree-width of G is at least N. Then by Lemma 5, with $S = V(T_{11})$, the graph G contains a chordless cycle C such that

- every vertex of G is non-adjacent to at least $6k+8$ consecutive vertices of C.
- the distance between C and T_{11} is at least $k+2$.

Fact 1. No vertex of G can have more than 4 neighbors on C. Moreover, if a vertex v has 3 neighbors on C, then these neighbors appear in C consecutively. If v has 4 neighbors, they can be split into two pairs of consecutive vertices. If v has 2 neighbors, they are either adjacent or of distance 2 in C.

Indeed, if v has more then 4 neighbors on C, then a large portion of C containing at least $6k+8$ consecutive vertices together with v and one of its neighbors create a forbidden induced apple. The rest of Fact 1 also follows from $(A_k, A_{k+1}, \ldots)$-freeness of G, which can be verified by direct inspection.

Claim. G has a chordless cycle containing vertex a and some vertices of C.

Proof. Since G has no clique separators, it is 2-connected. Therefore, there exist two vertex-disjoint paths connecting a to C. Let $P = (x_1, \ldots, x_p)$ and $Q = (y_1, \ldots, y_q)$ be two such paths, where x_1 and y_1 are adjacent to a, while x_p and y_q have neighbors on C. Without loss of generality, we shall assume that the total length of P and Q is as small as possible. In particular, this assumption implies that x_1 and y_1 are the only neighbors of a on P, Q, and no vertex of P or Q different from x_p and y_q has a neighbor on C. Any edge connecting a vertex of P to a vertex of Q will be called a (P, Q)-*chord*.

Fact 3. The neighborhood of x_p on C consists of two adjacent vertices and the neighborhood of y_q on C consists of two adjacent vertices.

Obviously, to avoid a big induced apple, x_p must have at least two neighbors on C. Consider a longest sub-path P' of C such that x_p has no neighbors on P'. We know that P' has at least $6k+8$ vertices. Moreover, by maximality of P', x_p is adjacent to the two (distinct!) vertices u, v on C outside P' each of which is adjacent to an endpoint of P'. Then, u and v must be adjacent, for otherwise G would contain a forbidden apple induced by the vertex set $P' \cup \{u, v, x_p, x_{p-1}\}$. The same reasoning shows that the neighborhood of y_q on C consists of two adjacent vertices.

Fact 3. The neighborhood of x_p on C does not coincide with the neighborhood of y_q on C, and there are no (P, Q)-chords different from x_1y_1.

For the sake of contradiction, suppose that $N(x_p) \cap C = N(y_q) \cap C = \{x_{p+1}, y_{q+1}\}$. Denote by T^1 the triangle x_p, x_{p+1}, y_{q+1} and by T^2 the triangle y_q, x_{p+1}, y_{q+1}. To avoid a separating clique (one triangle inside the other), we must conclude that, without loss of generality, x_p is inside C while y_q is outside C in the planar embedding of G. If additionally a is inside C, then Q meets the cycle before it meets y_q. This contradiction completes the proof of the first part of Fact 2.

To prove the second part, suppose that G contains a (P, Q)-chord different from x_1y_1. Let x_iy_j be such a chord with maximum value of $i + j$. In order to prevent a large induced apple, x_i must be adjacent to y_{j-1}. By symmetry, y_j must be adjacent to x_{i-1}. This implies, in particular, that both $i > 1$ and $j > 1$. Denote by T^1 the triangle x_{i-1}, x_i, y_j, by T^2 the triangle y_{j-1}, x_i, y_j and by C' the cycle formed by vertices $x_p, x_{p-1}, \ldots, x_i, y_j, \ldots, y_q$ and a portion of C. The rest of the proof of Fact 3 is identical to the above arguments.

From Fact 3 we conclude that if x_1y_1 is not a chord, then G has a desired cycle, i.e., a chordless cycle containing a and some vertices of C. From now on, assume x_1 is adjacent to y_1. Denote by C^* a big chordless cycle formed of P, Q and a portion of C containing at least half of its vertices, i.e., a portion containing at least $3k+4$ consecutive vertices of C. We will denote this portion by P^*.

Observe that among vertices $b_1, \ldots, b_{11}$ there is a vertex, name it z_1, which is adjacent neither to x_1 nor to y_1, since otherwise G has a separating clique (a triangle with a vertex inside it). Let us show that z_1 has no neighbors on C^*.

Indeed, z_1 cannot have neighbors on P^*, since the distance between z_1 and P^* is at least $k+2$. If z_1 has both a neighbor on P and a neighbor on Q, then G contains a big induced apple. If z_1 is adjacent to a vertex $x_i \in P$ and have no neighbors on Q, then either the pair of paths P, Q is not of minimum total length (if $i > 2$) or G has a big induced apple (if $i = 2$).

Since G has no clique separators, vertex z_1 must be connected to the cycle C^* by a path avoiding the clique $\{a, x_1, y_1\}$. Let $R = (z_1, \ldots, z_r)$ be a shortest path of this type. Since z_1 has no neighbors on C^*, r must be strictly greater than 1. According to Fact 1, z_r cannot have more then 4 neighbors on P^*. Moreover, these neighbors partition P^* into at most 3 portions (of consecutive non-neighbors of z_r) the largest of which has at least k vertices. Therefore, z_r has at least k consecutive non-neighbors on the cycle C^*. By analogy with Fact 2, we conclude that the neighborhood of z_r on C^* consists of two adjacent vertices. Also, by analogy with Fact 3, we conclude that the only possible chord between R and the other path connecting z_1 to C^* (i.e. (z_1, a)) is the edge az_2. Therefore, G has a chordless cycle containing vertex a and some vertices of C, and the proof of the claim is completed. □

Denote by $C^a = (a, v_1, v_2, \ldots, v_s)$ a chordless cycle containing the vertex a and a part of C. The vertices of C^a belonging to C will be denoted $v_i, v_{i+1}, \ldots, v_j$. Since the distance between T_{11} and C is at least $k+2$, none of the vertices $b_1, b_2, \ldots, b_{11}$ is adjacent to any of the vertices $v_{i-k}, v_{i-k+1}, \ldots, v_{k+j}$. Clearly among vertices $b_1, b_2, \ldots, b_{11}$ at least 9 do not belong to C^a. Among these 9, at least 5 vertices are adjacent neither to v_1 nor to v_s (since otherwise G contains a separating clique, i.e., a triangle with a vertex inside it). Without loss of generality, let the vertices $b_1, b_2, \ldots, b_5$ be not in C^a and non-adjacent to v_1, v_s. It is not difficult to see that none of these 5 vertices has a neighbor in the set $\{v_3, v_4, \ldots, v_{s-3}, v_{s-2}\}$, since otherwise a big induced apple arises (remember that none of these 5 vertices is adjacent to any of $v_{i-k}, v_{i-k+1}, \ldots, v_{k+j}$). For the same reason, none of $b_1, b_2, \ldots, b_5$ can be adjacent simultaneously to v_2 and v_{s-1} and none of them can be non-adjacent simultaneously to v_2 and v_{s-1}. Therefore, we may assume without loss of generality that in a fixed plane embedding of G, among these 5 vertices there are 2, say b_i, b_j, such that b_i is inside the 4-cycle a, b_j, v_2, v_1. Due to planarity, vertex c_i has no neighbors on the cycle C^a except possibly v_1 and v_2. However, regardless of the adjacency c_i to v_1 or v_2, the reader can easily find a big induced apple in G. This contradiction shows that if G contains a T_{11}, then the tree-width of G is bounded by a constant, which completes the proof of the theorem. □

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