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Inserted Perturbations Generating Asymptotical Integrability*

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Abstract—We discuss the general opportunity to create (asymptotically) a completely integrable system from the original perturbed system by inserting additional perturbing terms. After such an artificial insertion, there appears an opportunity to make the secondary averaging and secondary reduction of the original system. Thus, in this way, the $3D$-system becomes 1-dimensional. We demonstrate this approach by the example of a resonance Penning trap.

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1. INTRODUCTION

Physical systems are often presented as ε-perturbations of some integrable systems. The leading integrable Hamiltonians possess algebras of symmetries, and one can average and reduce the whole system by using these symmetries. For example, if the original system is 3-dimensional, then the reduced system has only 2 degrees of freedom. But this is still not enough, since we do not attain complete integrability after such a primary reduction.

Now assume that we artificially insert an additional perturbing term (perturbing field, structure, ...) into the original physical system in such a way that, after the primary reduction, this term, averaged by the primary symmetries, becomes the new leading Hamiltonian. Since we control the artificial additional term, we can try to make this secondary leading Hamiltonian integrable. Then one obtains an opportunity to make the secondary reduction by its symmetry algebra.

Thus, the original $3D$ physical perturbed system is subjected to a double averaging and double reduction procedure. The number of degrees of freedom is reduced twice and becomes 1. The resulting reduced system now turns out to be completely integrable (of course, only asymptotically, with accuracy $O(\varepsilon^\infty)$).

In this note, we demonstrate how such a general idea works by using the resonance Penning trap as an example. The term “Penning trap” is applied to devices based on the use of an electric field created by a cylinder-like or ring-like condenser or electrodes and of an axially directed magnetic field intended to hold an electric charge in a compact domain near the center of the trap [1]–[9].

The mathematical model of an ideal Penning trap is equivalent to a harmonic $3D$-oscillator of hyperbolic type (with a saddle point). Three normal frequencies of this oscillator depend on the external parameters: the magnetic field magnitude, the electric voltage, and the geometry of the electrodes. We assume that these frequencies are in resonance as $2 : (-1) : 2$. This assumption makes the leading Hamiltonian of the trap superintegrable with a noncommutative symmetry algebra.

The whole Penning-trap Hamiltonian also contains some perturbations: inhomogeneous magnetic corrections and anharmonic corrections in the electric potential; see, for instance, [1]–[9]. The direct averaging and reduction of these perturbations by the oscillator symmetry algebra generate a non-integrable system with 2 degrees of freedom.

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In order to make this system (asymptotically) integrable, we apply the general idea of "artificially inserted perturbation" described above. Namely, we assume that there is a small deviation of the homogeneous magnetic field from the axial direction. The corresponding artificial perturbing term in the Hamiltonian is very simple: just a quadratic form in canonical phase variables. After the primary averaging, this term generates an integrable system. Under a specific choice of the deviation angle, this system has a resonance regime with its own noncommutative symmetry algebra. Thus, there arises a nonobvious secondary resonance and a secondary symmetry algebra. This algebra is of non-Lie type. Below we demonstrate its generators and permutation relations.

The inhomogeneous and anharmonic perturbations in the original physical trap can now be subjected to double averaging and double reduction. This procedure generates an integrable system with one degree of freedom. Thus, we essentially simplify the study of a given physical system (the Penning trap) just by inserting a deviation of the magnetic field from its original axial direction.

2. HAMILTONIAN WITH ARTIFICIALLY INSERTED PERTURBING TERM

The Hamiltonian of the trap with all types of perturbations is of the form

$$\widehat{H} = \widehat{H}_0 + \varepsilon \widehat{H}_1 + \varepsilon^2 \widehat{H}_2 + O(\varepsilon^3). \tag{2.1}$$

Here $\widehat{H}_0$ is the Hamiltonian of the ideal trap,

$$\widehat{H}_0 = \frac{1}{2}\big[\widehat{p}_1^2 + \widehat{p}_2^2 + \widehat{p}_3^2 + 2\omega(\widehat{p}_1 q_2 - \widehat{p}_2 q_1) + (\omega^2 - \omega_0^2)(q_1^2 + q_2^2) + 2\omega_0^2 q_3^2\big], \tag{2.2}$$

where by $\widehat{p}_j = -i\hbar\partial/\partial q_j$ we denote the momentum operators corresponding to the Cartesian coordinates q_j $(j = 1, 2, 3)$ and by ω, ω_0 positive parameters obeying condition $\omega > \omega_0$. The homogeneous magnetic field of the ideal trap is directed along the third axis. The electric potential of the ideal trap has the form $U_0 = (\omega_0^2/2)(2q_3^2 - q_1^2 - q_2^2)$. The real physical electric potential U obeys the Laplace equation $\Delta U = 0$ and is approximated by the function U_0 near the center $q = 0$ of the trap:

$$U = U_0 + \text{third- and fourth-degree terms} + \dots .$$

After rescaling the coordinates in a small domain near the center and after averaging, the terms of the third and fourth degrees together generate the summand $\widehat{H}_2$ in (2.1), and the higher-degree terms are put in the remainder $O(\varepsilon^3)$.

The term $\widehat{H}_1$ in (2.1) is due to the artificially inserted deviation of the homogeneous magnetic field from its "ideal" direction along the third axis. The direction of the deviation is given by the vector $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$. Thus,

$$\widehat{H}_1 = \frac{1}{2}\widehat{k}\cdot[q\times\mathcal{B}], \qquad k \stackrel{\text{def}}{=} (p_1 + \omega q_2, p_2 - \omega q_1, p_3). \tag{2.3}$$

By making the change of coordinates

$$\begin{aligned} q_1 &= \frac{1}{\sqrt{2}\sqrt[4]{\omega^2-\omega_0^2}}(x_+ + x_-), \qquad & p_1 &= \frac{\sqrt[4]{\omega^2-\omega_0^2}}{\sqrt{2}}(p_+ + p_-),\\ q_2 &= \frac{1}{\sqrt{2}\sqrt[4]{\omega^2-\omega_0^2}}(p_+ - p_-), \qquad & p_2 &= \frac{\sqrt[4]{\omega^2-\omega_0^2}}{\sqrt{2}}(x_- - x_+),\\ q_3 &= \frac{1}{\sqrt[4]{2}\sqrt{\omega_0}}x_0, \qquad & p_3 &= \sqrt[4]{2}\sqrt{\omega_0}p_0, \end{aligned} \tag{2.4}$$

we transform the Hamiltonian (2.2) to the normal form

$$\widehat{H}_0 = \frac{1}{\sqrt{2}}\big[\omega_+(\widehat{p}_+^2 + x_+^2) - \omega_-(\widehat{p}_-^2 + x_-^2) + \omega_0(\widehat{p}_0^2 + x_0^2)\big], \qquad \text{where} \quad \omega_\pm = \frac{\omega^2 \pm (\omega^2-\omega_0^2)^{1/2}}{\sqrt{2}}. \tag{2.5}$$

Now let us assume that

$$\omega^2 = \frac{9}{8}\,\omega_0^2. \tag{2.6}$$

Then $\omega_+ = \omega_0$, $\omega_- = \omega_0/2$ and the Hamiltonian (2.5) becomes

$$\widehat{H}_0 = \frac{\omega_0}{\sqrt{2}}[2\widehat{z}_+^*\widehat{z}_+ - \widehat{z}_-^*\widehat{z}_- + 2\widehat{z}_0^*\widehat{z}_0] + \frac{3\hbar\omega_0}{2\sqrt{2}}\,, \tag{2.7}$$

where

$$\widehat{z}_\pm = \frac{1}{\sqrt{2}}(x_\pm + i\widehat{p}_\pm), \qquad \widehat{z}_0 = \frac{1}{\sqrt{2}}(x_0 + i\widehat{p}_0). \tag{2.8}$$

So, under condition (2.6), the Hamiltonian of the ideal Penning trap is the linear combination of three one-dimensional oscillators whose frequencies are in resonance as $2 : (-1) : 2$.

3. PRIMARY SYMMETRY ALGEBRA

The spectrum of $\widehat{H}_0$ is discrete and infinitely degenerate under the resonance condition (2.6). The degeneracy is controlled by the symmetry algebra, i.e., by the algebra of all operators commuting with $\widehat{H}_0$. The following operators can be chosen as the generators of this algebra:

$$S_\pm = \widehat{z}_\pm^*\widehat{z}_\pm, \qquad S_0 = \widehat{z}_0^*\widehat{z}_0, \tag{3.1}$$

$$A_\rho = \widehat{z}_+^*\widehat{z}_0, \qquad A_\sigma = \widehat{z}_+^*(\widehat{z}_-^*)^2, \qquad A_\theta = (\widehat{z}_-^*)^2\widehat{z}_0^*. \tag{3.2}$$

The commutation relations between these generators (and the conjugate operators) are the following ones:

$$\begin{aligned}
&[S_+, A_\rho] = \hbar A_\rho, && [S_0, A_\rho] = -\hbar A_\rho,\\
&[S_+, A_\sigma] = \hbar A_\sigma, && [S_-, A_\sigma] = 2\hbar A_\sigma,\\
&[S_-, A_\theta] = 2\hbar A_\theta, && [S_0, A_\theta] = \hbar A_\theta,\\
&[A_\rho, A_\sigma^*] = -\hbar A_\theta^*, && [A_\rho, A_\theta] = \hbar A_\sigma,\\
&[A_\rho^*, A_\rho] = \hbar(S_0 - S_+), && [A_\sigma, A_\theta^*] = -4\hbar\big(S_- + \frac{\hbar}{2}\big)A_\rho,\\
&[A_\sigma^*, A_\sigma] = \hbar(4S_+S_- + S_-^2 + 2\hbar S_+ + 3\hbar S_- + 2\hbar^2),\\
&[A_\theta^*, A_\theta] = \hbar(S_-^2 + 4S_-S_0 + 3\hbar S_- + 2\hbar S_0 + 2\hbar^2).
\end{aligned} \tag{3.3}$$

Relations (3.3) have the following three Casimir operators:

$$\begin{aligned}
C_\rho &= A_\rho A_\rho^* - S_+(S_0 + \hbar),\\
C_\sigma &= A_\sigma A_\sigma^* - S_+S_-(S_- - \hbar),\\
C_\theta &= A_\theta A_\theta^* - S_0S_-(S_- - \hbar).
\end{aligned} \tag{3.4}$$

In the realization (3.1), (3.2), these three operators vanish.

We also have the following "quasi-Casimir" operators:

$$\begin{aligned}
C_+ &= A_\rho A_\sigma^* - S_+A_\theta^*,\\
C_- &= A_\sigma A_\theta^* - S_-(S_- - \hbar)A_\rho,\\
C_0 &= A_\rho A_\theta - (S_0 + \hbar)A_\sigma.
\end{aligned} \tag{3.5}$$

The commutators of these operators with all generators are proportional to operators (3.5). In the realization (3.1), (3.2), the operators (3.5) also vanish.

Thus, the symmetry algebra of the resonance oscillator (2.6) can be defined as an algebra with nine generators and relations (3.3) factorized by the ideal generated by elements (3.4) and (3.5).

In this quotient algebra, we still have one additional Casimir element

$$C = 2S_+ - S_- + 2S_0, \tag{3.6}$$

which is, in fact, the operator $\widehat{H}_0$ (2.7); namely,

$$\widehat{H}_0 = \frac{\omega_0}{\sqrt{2}}\Big(C + \frac{3\hbar}{2}\Big).$$

The general theory of resonance symmetry algebras can be found in [10]–[14].

4. PRIMARY AVERAGING

We follow the general scheme of algebraic averaging [11]–[13], [15]–[18]. Let us construct a unitary

$$V = \exp\Big\{-\frac{i\varepsilon R}{\hbar}\Big\} \qquad \text{such that} \quad V^{-1}\cdot \widehat{H}\cdot V = \widehat{H}_0 + \varepsilon \widehat{H}_{10} + \varepsilon^2 \widehat{H}_{20} + O(\varepsilon^3), \tag{4.1}$$

where

$$[\widehat{H}_0, \widehat{H}_{10}] = [\widehat{H}_0, \widehat{H}_{20}] = 0. \tag{4.2}$$

The operator $R = R_0 + \varepsilon R_1$ is obtained from the homological equations

$$\frac{i}{\hbar}\big[\widehat{H}_0, R_0\big] = \widehat{H}_1 - \widehat{H}_{10}, \qquad \frac{i}{\hbar}\big[\widehat{H}_0, R_1\big] = \widehat{H}_2 + \frac{i}{2\hbar}\big[R_0, \widehat{H}_1 + \widehat{H}_{10}\big] - \widehat{H}_{20}. \tag{4.3}$$

These equations can be solved as follows:

$$\widehat{H}_{10} = \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\widehat{H}_1 e^{\frac{it}{\hbar}C}\,dt, \qquad \widehat{H}_{20} = \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\Big(\widehat{H}_2 + \frac{i}{2\hbar}\big[R_0, \widehat{H}_1 + \widehat{H}_{10}\big]\Big)e^{\frac{it}{\hbar}C}\,dt, \tag{4.4}$$

$$R_0 = \frac{\sqrt{2}}{2\pi\omega_0}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\widehat{H}_1 e^{\frac{it}{\hbar}C}t\,dt, \qquad R_1 = \frac{\sqrt{2}}{2\pi\omega_0}\int_0^{2\pi} e^{-\frac{it}{\hbar}C}\Big(\widehat{H}_2 + \frac{i}{2\hbar}\big[R_0, \widehat{H}_1 + \widehat{H}_{10}\big]\Big)e^{\frac{it}{\hbar}C}t\,dt, \tag{4.5}$$

The operators on the right-hand sides in (4.4), (4.5) are computed by using the fact that the operator C is the linear combination (3.6) of the "action" operators $S_\pm$, S_0 and by taking into account that the evolution of the coordinates $\widehat{z}_\pm$, $\widehat{z}_0$ with respect to the actions can be derived explicitly.

The final formulas for the perturbing terms $\widehat{H}_{10}$, $\widehat{H}_{20}$ in the Hamiltonian (4.1) are

$$\widehat{H}_{10} = \frac{\xi}{\sqrt{2}}\Big(\eta\Big(2S_+ + S_- + \frac{3\hbar}{2}\Big) - \sqrt{1-\eta^2}(A_\rho + A_\rho^*)\Big), \tag{4.6}$$

$$\widehat{H}_{20} = \text{linear and quadratic combinations of generators (3.1), (3.2)}, \tag{4.7}$$

where we use the notation

$$\xi^2 = \mathcal{B}_1^2 + 2\mathcal{B}_3^2, \qquad \eta = \sqrt{2}\mathcal{B}_3/\xi, \tag{4.8}$$

and we choose $\mathcal{B}_2 = 0$ without loss of generality.

The explicit expression for $\widehat{H}_{20}$ in (4.7) can be obtain from the concrete information about the inhomogeneous part of the magnetic field and the anharmonic part of the electric potential of the trap.

5. SECONDARY INTEGRABLE HAMILTONIAN AND SECONDARY RESONANCE

The perturbing terms $\widehat{H}_{10}, \widehat{H}_{20}, \dots$ in (4.1) commute with the leading term $\widehat{H}_0$. Therefore, we now have to study the Hamiltonian

$$\widehat{H}_{10} + \varepsilon\widehat{H}_{20} + O(\varepsilon^2) \tag{5.1}$$

on the eigenspaces of the resonance oscillator $\widehat{H}_0$.

The symmetry algebra of the operator $\widehat{H}_{10}$ (4.6) is trivial in general position (commutative and generated by $\widehat{H}_0$, S_- and $\widehat{H}_{10}$ itself), and its spectrum is nondegenerate. However, under a special resonance condition imposed on the components of the perturbing magnetic field $\mathcal{B}$, this algebra becomes noncommutative, and a spectral degeneracy of $\widehat{H}_{10}$ appears.

First, let us note that it follows from the commutation relations (3.3) that, on the eigensubspace where $\widehat{H}_0$ takes the value $(\hbar\omega_0/\sqrt{2})(n+3/2)$, the spectrum of $\widehat{H}_{10}$ consists of the following numbers:

$$\frac{\xi\hbar}{\sqrt{2}}\biggl((1+3\eta)n_+ - (1-3\eta)n_- - \eta\biggl(n-\frac{3}{2}\biggr)\biggr)$$

with integers n_+, n_-, and n. Thus, the degeneracy of the spectrum is equivalent to the resonance condition

$$(1+3\eta):(1-3\eta) = k:l \qquad \text{or} \qquad (k-l)\mathcal{B}_1 = 4\sqrt{(k+l)^2 + kl/2}\,\mathcal{B}_3, \tag{5.2}$$

where k, l are positive coprime integers.

Below we choose the lowest resonance $k=1$, $l=0$, i.e., $\eta = \frac{1}{3}$ or $4\mathcal{B}_3 = \mathcal{B}_1$.

Theorem 5.1. *Under condition* (5.2)*, the symmetry algebra of the secondary integrable Hamiltonian* $\widehat{H}_{10}$ (4.6) *is noncommutative. The generators of this secondary resonance algebra are*

$$A_0 \stackrel{\text{def}}{=} S_-,$$

$$A_\pm \stackrel{\text{def}}{=} \biggl(1 \pm \frac{1}{3}\biggr)S_+ + \biggl(1 \mp \frac{1}{3}\biggr)S_0 \mp \frac{2\sqrt{2}}{3}(A_\rho + A_\rho^*), \tag{5.3}$$

$$B \stackrel{\text{def}}{=} \frac{\sqrt{2}}{\sqrt{3}}A_\sigma + \frac{2}{\sqrt{3}}A_\theta. \tag{5.4}$$

The commutation relations between generators (5.3), (5.4) *read*

$$[A_0, B] = 2\hbar B, \qquad [A_-, B] = 2\hbar B, \qquad [A_+, B] = 0,$$

$$[B^*, B] = 2\hbar(A_0^2 + 2A_0A_- + 3\hbar A_0 + \hbar A_- + 2\hbar^2). \tag{5.5}$$

The Casimir elements of this algebra are

$$M = A_+ - A_- + A_0, \qquad C = A_+ + A_- - A_0, \qquad K = BB^* - A_0(A_0 - \hbar)A_-. \tag{5.6}$$

In the realization (3.1), (3.2)*, the Casimir elements* (5.6) *read*

$$M = \frac{2}{3\mathcal{B}_3}\widehat{H}_{10} - \frac{1}{3}(C + 3\hbar), \qquad C = \frac{\sqrt{2}}{\omega_0}\widehat{H}_0 - \frac{3\hbar}{2}, \qquad K = 0, \tag{5.7}$$

Based on this theorem, the next step is the secondary averaging of the perturbation $\widehat{H}_{20}$ from (4.7). Finally, this procedure results in the unitary transformation of the original Hamiltonian (2.1) to the new Hamiltonian of the form

$$\widehat{H}_0 + \varepsilon\widehat{H}_{10} + \varepsilon^2\widehat{H}_{200} + O(\varepsilon^3) \tag{5.8}$$

with the secondary perturbing term $\widehat{H}_{200}$ commuting with the leading parts:

$$\bigl[\widehat{H}_0, \widehat{H}_{200}\bigr] = 0, \qquad \bigl[\widehat{H}_{10}, \widehat{H}_{200}\bigr] = 0.$$

The explicit formula for $\widehat{H}_{200}$ is obtained by the same integral operation as in (4.4), but, instead of the Casimir operator C in the exponent, one now must use the Casimir operator M. The resulting reduced system (5.8) is completely integrable, but of course, only asymptotically, with accuracy $O(\varepsilon^3)$. The accuracy can be made $O(\varepsilon^\infty)$ by applying the averaging procedure (4.1)-(4.5) in all higher orders in ε.

Note that the effective averaged Hamiltonian $\widehat{H}_{200}$ of the Penning trap is a quadratic function in the generators of the symmetry algebra (5.5). This algebra is of non-Lie type (since the right-hand side of the commutation relation between B^* and B in (5.5) is nonlinear in the generators). Thus, the Hamiltonian $\widehat{H}_{200}$ represents a kind of "Euler top" over this nonlinear version of $su(1,1)$. General methods of analysis of such a non-Lie type systems via quantum complex structures and coherent transformations were developed in [16]–[20]; for applications to the case of Penning traps, see [21]–[23].

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