ISSN 0278-6419, Moscow University Computational Mathematics and Cybernetics, 2016, Vol. 40, No. 1, pp. 19–27. © Allerton Press, Inc., 2016.
Original Russian Text © T.S. Yakushkina, 2016, published in Vestnik Moskovskogo Universiteta. Vychislitel'naya Matematika i Kibernetika, 2016, No. 1, pp. 19–27.

A Distributed Replicator System Corresponding to a Bimatrix Game

T. S. Yakushkina

Department of Computational Mathematics and Cybernetics, Moscow State University, Moscow, 119991 Russia
Higher School of Economics, National Research University, Moscow, 101000 Russia
e-mail: tyakushkina@hse.ru

Received April 17, 2015

Abstract—Reaction–diffusion type replicator systems are investigated for the case of a bimatrix. An approach proposed earlier for formalizing and analyzing distributed replicator systems with one matrix is applied to asymmetric conflicts. A game theory interpretation of the problem is described and the relation between dynamic properties of systems and their game characteristics is determined. The stability of a spatially homogeneous solution for a distributed system is considered and a theorem on maintaining stability is proved. The results are illustrated with two-dimensional examples in the case of distribution.

Keywords: Evolutionary game theory, reaction–diffusion systems, replicator equations.

DOI: 10.3103/S0278641916010064

1. INTRODUCTION

Models of evolutionary game theory find wide application in many fields, among which two main categories are distinguished [1]: biological and economic. Replicator equations are widely used universal tools for describing evolutionary processes; such equations are used in population genetics [2] and in the theory of prebiological evolution [3, 4]. Most studies of the evolution of cooperation also rely on replicator dynamics [5, 6]. One standard way of presenting replicator equations [7] is

$$\dot{v}_i(t) = v_i(f_i(\mathbf{v}) - f^l(\mathbf{v})), \quad i = 1,\ldots,n, \tag{1}$$

where $\mathbf{v} = (v_1(t), \ldots, v_n(t))^T$ is the vector that describes the state of a system (e.g., the probability distribution of the selection of strategies for a game in normal form). The interaction between a system's elements generates suitability (fitness) functions $f_i(\mathbf{v})$ of particular elements; here, expression $f^l(\mathbf{v})$ is an averaged characteristic of this interaction. Many models use the matrix form for describing interaction: if $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ is the payoff matrix, then $f_i(\mathbf{v}) = \sum_{j=1}^n a_{ij}v_j$.

The main assumption enabling us to use lumped replicator systems of form (1) is that there is no spatial dependence in a system. However, the idea that all elements interact with one another with identical probabilities is often not typical of actual biological systems. There are a number approaches that allow us to consider spatial structure in replicator equations: the use of spatial lattices [8, 9], random graphs [10], and reaction–diffusion systems [11, 12]. Modifications of system (1) with one matrix are considered in most works devoted to studying distributed replicator systems.

In this work, a bimatrix case of replicator systems is considered; the approach proposed in [13] is used for their analysis. We thus investigate the stability of distributed replicator systems under global regulation for problems with two matrices, show the relation between game theory concepts and the concept of stability if space is taken into account, and demonstrate the existence of spatially nonhomogeneous solutions.

2. COMMON REPLICATOR SYSTEMS: BIMATRIX GAMES

One possible biological formulation of the problem [14] considers a system with interaction between two populations, each of which consists of n types. If we denote by x_i, y_j the absolute size of particular types, the states of the system are described by vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n_+$. In this case, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are differentiable with respect to the real variable $t > 0$, which has the meaning of time. It is assumed that the pairwise interaction of the i-th and j-th types associated with different populations occurs randomly and is characterized by interaction matrices $\mathbf{A} = \{a_{ij}\}^n_{i,j=1}$ and $\mathbf{B} = \{b_{ij}\}^n_{i,j=1}$ with fixed elements.

The rate of growth in the size of a particular type is proportional to the evolutionary success of this type, so the law of population reproduction is written as

$$\dot{x}_i = x_i(\mathbf{Ay})_i, \quad \dot{y}_j = y_i(\mathbf{Bx})_j, \tag{2}$$

where $(\mathbf{Ay})_i$ and $(\mathbf{Bx})_j$ are elements of the corresponding vectors. We assume that the total size of each population is quite large and introduce frequencies (relative sizes) of the types

$$u_i = x_i/\overline{x}, \quad v_j = y_i/\overline{y}, \quad 1 \leqslant i, j \leqslant n, \quad \overline{x} = \sum_{k=1}^n x_k, \quad \overline{y} = \sum_{k=1}^n y_k. \tag{3}$$

Vector functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ then belong to simplexes of the form

$$S_n = \left\{ s(t) : \sum_{i=1}^n s_i(t) = 1, \quad s_i(t) \geqslant 0, \quad 1 \leqslant i \leqslant n \right\}.$$

Using (2) and differentiating (3), we obtain the bimatrix replicator system

$$\begin{aligned} \dot{u}_i &= u_i((\mathbf{Av})_i - (\mathbf{u}, \mathbf{Av})), \\ \dot{v}_j &= v_j((\mathbf{Bu})_j - (\mathbf{v}, \mathbf{Bu})), \quad 1 \leqslant i, j \leqslant n, \end{aligned} \tag{4}$$

where $(\mathbf{Av})_i$ and $(\mathbf{Bu})_j$ represent the suitability (fitness) of a corresponding type. The average fitness is determined as

$$(\mathbf{u}, \mathbf{Av}) = \sum_{i=1}^n u_i(\mathbf{Av})_i, \quad (\mathbf{v}, \mathbf{Bu}) = \sum_{i=1}^n v_i(\mathbf{Bu})_i,$$

System (4) is consistent with one of the basic principles of Darwinism: The reproductive success of an individual (or a group) depends on the advantage of its own fitness over the average fitness in the population. Let us write the basic characteristics of replicator systems of type (4); these characteristics must be known when studying distributed systems.

- The stationary points of system (4) are determined by the equations

$$(\mathbf{Av})_1 = \ldots = (\mathbf{Av})_n = \beta_1, \quad (\mathbf{Bu})_1 = \ldots = (\mathbf{Bu})_n = \beta_2.$$

In the general case, the Jacobian matrix at a stationary point has the form [2] $\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ \mathbf{D} & \mathbf{0} \end{bmatrix}$, where $\mathbf{0}$ is the $(n-1)$-by-$(n-1)$ zero submatrix, while $\mathbf{C}$ and $\mathbf{D}$ are submatrices formed by certain constant coefficients

- The characteristic polynomial of the system has the form $p(\lambda) = \det(\lambda^2\mathbf{I} - \mathbf{DC})$. If λ is an eigenvalue, then $-\lambda$ is also an eigenvalue. We take advantage of this below in analyzing the stability of the equilibrium position (analysis shows specifically that in a two-dimensional case, the system cannot have a stationary point of the focus or knot type).

2.1. Replicator Systems in Game Theory

The game theory interpretation of system (4) is based on a normal two-player game where the players have different finite sets of strategies and different payoff matrices $\mathbf{A}$ and $\mathbf{B}$ (games of this type are referred to as bimatrix). In this formulation of the problem, n is the number of pure strategies of players and $\mathbf{u}, \mathbf{v} \in S_n$ are the mixed strategies of the players. The dominance of one type in a population corresponds to pure strategies, while the possible simultaneous coexistence of several types accords with mixed strategies. Below, we consider a system in form (4); in doing so, we consider that the Nash equilibrium in a bimatrix game with payoff matrices $\mathbf{A}$ and $\mathbf{B}$ is a stationary point of system (4) (the reverse is generally not true) [7].

3. DISTRIBUTED REPLICATOR SYSTEM

Let us consider a replicator system with diffusion:

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= u_i((\mathbf{Av})_i - f^A(t)) + d_i^A \frac{\partial^2 u_i}{\partial x^2}, \quad 1 \leqslant i \leqslant n, \\ \frac{\partial v_j}{\partial t} &= v_j((\mathbf{Bu})_j - f^B(t)) + d_j^B \frac{\partial^2 v_j}{\partial x^2}, \quad 1 \leqslant j \leqslant n, \end{aligned} \tag{5}$$

where d_i^A and d_j^B are positive diffusion coefficients. For this system, $u_i = u_i(x, t)$ and $v_i = v_i(x, t)$, where x is a spatial variable, $t > 0$, and $f^A(t)$ and $f^B(t)$ are the fitnesses of each player.

Considering the biological and game theory premises of the model, it makes sense to examine the restricted domain of the definition of the spatial variable: $D \in \gg \mathbb{R}^k$ with piecewise smooth boundary Γ, $x \in D$ ($k = 1$, 2, or 3). We assume that $u_i(x, t)$ and $v_i(x, t)$ are differentiable with respect to t for any $x \in D$ and (as functions of x with fixed time) belong to Sobolev space $W_2^1(D)$ when $D \in \mathbb{R}^1$, or W_2^2 when $D \in \mathbb{R}^2$ ($D \in \mathbb{R}^3$).

We set a condition analogous to that of the constancy of frequencies for lumped system (4):

$$\forall t : \sum_{i=1}^{n} \int_D u_i(x,t)dx = 1, \quad \sum_{i=1}^{n} \int_D v_i(x,t)dx = 1.$$

Then fitnesses of the two populations have the forms $f^A = \int_D (\mathbf{u}, \mathbf{Av})dx$ and $f^B = \int_D (\mathbf{v}, \mathbf{Bu})dx$. On boundary Γ of set D, we set homogeneous Neumann condition $\left.\frac{\partial u_i}{\partial n}\right|_{x\in\Gamma} = 0, \left.\frac{\partial v_i}{\partial n}\right|_{x\in\Gamma} = 0$, where n is an external normal to the boundary of the set. Cauchy conditions $\mathbf{u}(x, 0) = \mathbf{u}_0^A(x)$, $\mathbf{v}(x, 0) = \mathbf{v}_0^B(x)$ are set at instant $t = 0$.

We assume that $D_t = D \times [0; \infty)$ and $S_n(D_t)$ is the set of nonnegative vector functions $\mathbf{y}(x, t)$ with the norm of elements

$$\|y_i\|_S = \max_{t\geq 0} \left\{ \|y_i(x,t)\|_{W_2^k} + \left\| \frac{\partial y_i(x,t)}{\partial t} \right\|_{W_2^k} \right\}.$$

We find the solution to the above initial-boundary value problem in class $S_n(D_t)$ of functions that satisfy the equalities

$$\int_0^\infty \int_D \frac{\partial u_i}{\partial t} \eta(x,t)dxdt = \int_0^\infty \int_D u_i[(\mathbf{Av})_i - f^A(t)]\eta(x,t)dxdt - d_i^A \int_0^\infty \int_D (\nabla u_i, \nabla \eta)dxdt,$$

$$\int_0^\infty \int_D \frac{\partial v_i}{\partial t} \eta(x,t)dxdt = \int_0^\infty \int_D v_i[(\mathbf{Bu})_i - f^B(t)]\eta(x,t)dxdt - d_i^B \int_0^\infty \int_D (\nabla v_i, \nabla \eta)dxdt,$$

met for all functions $\eta(x, t)$ (differentiable with respect to t) with a compact carrier on $[0, +\infty)$ that belong to the corresponding Sobolev space in x.

As equilibrium positions of dynamic system (5), we consider solutions $\mathbf{w}^A(x)$ and $\mathbf{w}^B(x)$ of the system of equations

$$\begin{aligned} d_i^A \Delta w_i^A(x) + w_i^A((\mathbf{Aw}^B)_i - f^A) &= 0, \quad 1 \leqslant i \leqslant n, \\ d_j^b \Delta w_j^B(x) + w_j^B((\mathbf{Bw}^A)_j - f^B) &= 0, \quad 1 \leqslant j \leqslant n, \end{aligned} \tag{6}$$

with boundary and balance conditions

$$\left.\frac{\partial w_i^A}{\partial n}\right|_{x\in\Gamma} = 0, \quad \left.\frac{\partial w_i^B}{\partial n}\right|_{x\in\Gamma} = 0, \quad \sum_{i=1}^{n} \int_D w_i^A dx = 1, \quad \sum_{i=1}^{n} \int_D w_i^B dx = 1.$$

We denote as $S_n(D)$ the set of nonnegative functions $w_i^A(x)$ and $w_i^B(x)$ that belong to the corresponding Sobolev space when $1 \leqslant i \leqslant n$ and satisfy these conditions. Average fitnesses for the problem in question are fixed, since

$$f^A = \int_D (\mathbf{w}^A, \mathbf{A}\mathbf{w}^B)dx, \quad f^B = \int_D (\mathbf{w}^B, \mathbf{B}\mathbf{w}^A)dx.$$

Stationary points of the initial system with no diffusion satisfy the steady-state equations of system (6); we shall refer to these as spatially homogeneous solutions of system (5). In this case, the reverse is also true: the spatially homogeneous solutions of system (6) are also the stationary points of initial system (4). We introduce the following definition in order to analyze the stability of the stationary solutions to system (5):

Definition 1. Stationary solution $\mathbf{w}^*(x) = (\mathbf{w}^{A*}, \mathbf{w}^{B*}) \in S_n \times S_n$ to system (6) is Lyapunov stable if for any $\varepsilon > 0$ there exists a neighborhood

$$U^\delta = \left\{(\mathbf{w}^A(x), \mathbf{w}^B(x)) \in S_n \times S_n, \quad \sum_{i=1}^n \|w_i^{\tau*}(x) - w_i^\tau(x)\|_{W_2^k} < \delta^2, \quad \tau = A, B\right\}$$

of pair $(\mathbf{w}^{A*}(x), \mathbf{w}^{B*}(x))$ such that the following inequalities hold under any initial conditions for system (5) which belong to neighborhood U^δ) at any $t \geqslant 0$:

$$\sum_{i=1}^n \|u_i(x,t) - w_i^{A*}(x)\|_S \leqslant \varepsilon^2, \quad \sum_{i=1}^n \|v_i(x,t) - w_i^{B*}(x)\|_S \leqslant \varepsilon^2.$$

Here, $u_i(x, t)$ and $v_i(x, t)$ represent the corresponding solution to system (5) with boundary conditions $w_i^A(x)$ and $w_i^B(x)$ $(1 \leqslant i \leqslant n)$.

Let us consider the boundary eigenvalue problem

$$\Delta\psi(x) + \lambda\psi(x) = 0, \quad x \in D, \quad \left.\frac{\partial\psi}{\partial n}\right|_{x\in\Gamma} = 0. \tag{7}$$

If $\psi_0(x) = 1$, then $\{\psi_i(x)\}_{i=1}^\infty$ is a complete system in Sobolev space W_2^1 such that

$$\langle\psi_i(x), \psi_j(x)\rangle = \int_D \psi_i(x)\psi_j(x)dx = \delta_{ij}, \tag{8}$$

where δ_{ij} is the Kronecker symbol. The corresponding eigenvalues satisfy condition [15] $0 = \lambda_0 < \lambda_1 \leqslant \ldots \leqslant \lambda_i \leqslant \ldots \leqslant \lim_{i\to\infty} \lambda_i = +\infty$.

Theorem 1. *Assume that the pair* $(\mathbf{u}^*, \mathbf{v}^*) \in \operatorname{int}(S_n \times S_n)$ *is the Lyapunov stable equilibrium position of system* (4); *then for any positive values of diffusion coefficients* d_i^A *and* d_i^B $(1 \leqslant i \leqslant n)$ *this position yields the stable spatially homogeneous stationary solution to distributed system* (5).

We find the solution to system (5) in the form

$$u_i(x,t) = u_i^* + w_i^A(x,t), \quad v_i(x,t) = v_i^* + w_i^B(x,t), \tag{9}$$

$$w_i^A(x,t) = C_0^i(t) + \sum_{k=1}^\infty C_k^i(t)\psi_k(x), \quad w_i^B(x,t) = E_0^i(t) + \sum_{k=1}^\infty E_k^i(t)\psi_k(x),$$

where $(\mathbf{u}^*, \mathbf{v}^*)$ is the equilibrium position of an initial system with no diffusion (4), while $C_k^i(t)$ and $E_k^i(t)$ are smooth functions that tend to zero when $t \to \infty$; in this case, ψ_i satisfy (7) and (8) for all i. Using the Cauchy conditions and the constancy condition of frequencies, we obtain

$$\sum_{i=1}^n u_i^* = 1, \quad \sum_{i=1}^n v_i^* = 1 \Rightarrow \sum_{i=1}^n C_0^i(t) = 0, \quad \sum_{i=1}^n E_0^i(t) = 0. \tag{10}$$

(i) We assume that $k = 0$, insert the solution to form (9) into system (5), and consider that equalities $(\mathbf{Av}^*)_i - (\mathbf{u}^*, \mathbf{Av}^*) = 0$ and $(\mathbf{Av}^*, \mathbf{C}_0) = 0$ hold in the equilibrium position (in view of (10)). We retain only linear terms, use the analogous procedure for $\mathbf{E}_0$, and obtain system of equations

$$\frac{d}{dt}C_0^i(t) = u_i^*((\mathbf{A}\overline{\mathbf{E}}_0)_i - (\mathbf{A}^T\mathbf{u}^*, \overline{\mathbf{E}}_0)), \quad \frac{d}{dt}E_0^i(t) = v_i^*((\mathbf{B}\overline{\mathbf{C}}_0)_i - (\mathbf{B}^T\mathbf{v}^*, \overline{\mathbf{C}}_0)), \tag{11}$$

where $\overline{\mathbf{E}}_0 = (E_0^1, \ldots, E_0^n)^T$ and $\overline{\mathbf{C}}_0 = (C_0^1, \ldots, C_0^n)^T$. The Jacobian matrix of initial system with no diffusion (4) taken at point $(\mathbf{u}^*, \mathbf{v}^*)$ coincides with the matrix of this system:

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ \mathbf{D} & \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = \{c_{ij}\} : c_{ij} = a_{ij}u_i^* - u_i^*(\mathbf{Au}^*)_i, \quad \mathbf{D} = \{d_{ij}\} : d_{ij} = b_{ij}v_i^* - v_i^*(\mathbf{Bv}^*)_i.$$

The trivial equilibrium position of system (11) is therefore also stable.

(ii) We multiply the equations of system (5) by functions $\psi_k(x)$, retaining only the linear terms upon inserting (9), and obtain system (in view of expression (7))

$$\frac{d}{dt}C_k^i(t) = u_i^*((\mathbf{A}\overline{\mathbf{E}}_k)_i - \lambda_k d_j^A C_k^i(t), \quad \frac{d}{dt}E_k^i(t) = v_i^*((\mathbf{B}\overline{\mathbf{C}}_k)_i - \lambda_k d_j^B E_k^i(t),$$

where λ_k is an eigenvalue of problem (7). The Jacobian matrix of this system has the form

$$\mathbf{J} = \begin{bmatrix} -\mathbf{\Lambda}_1 & \mathbf{A}^* \\ \mathbf{B}^* & -\mathbf{\Lambda}_2 \end{bmatrix}, \quad \mathbf{A}^* = \{u_i^* a_{ij}\}, \quad \mathbf{B}^* = \{v_i^* b_{ij}\}, \quad \mathbf{\Lambda}_1 = \{d_i^A \lambda_k \delta_{ij}\}, \quad \mathbf{\Lambda}_2 = \{d_i^B \lambda_k \delta_{ij}\}.$$

For its trace, we have $\mathrm{tr}(\mathbf{J}) = -\lambda_k \sum_i (d_i^A + d_i^B) < 0$. Hence, equilibrium position $(\mathbf{u}^*, \mathbf{v}^*)$ is stable.

3.1. Game Dynamics of Distributed Replicator Equations

Definition 2. Pair $(\hat{\mathbf{w}}^A(x), \hat{\mathbf{w}}^B(x)) \in S_n(D) \times S_n(D)$ is the distributed Nash equilibrium if the following conditions are met:

$$\int_D (\mathbf{u}(x,t), \mathbf{A}\hat{\mathbf{w}}^B(x))dx \le \int_D (\hat{\mathbf{w}}^A(x), \mathbf{A}\hat{\mathbf{w}}^B(x))dx,$$

$$\int_D (\mathbf{v}(x,t), \mathbf{B}\hat{\mathbf{w}}^A(x))dx \le \int_D (\hat{\mathbf{w}}^B(x), \mathbf{B}\hat{\mathbf{w}}^A(x))dx,$$

$$\forall (\mathbf{u}(x,t), \mathbf{v}(x,t)) \in S_n(D) \times S_n(D) : \quad \mathbf{u} \ne \hat{\mathbf{w}}^A, \quad \mathbf{v} \ne \hat{\mathbf{w}}^B.$$

Note that if $(\hat{\mathbf{w}}^A, \hat{\mathbf{w}}^B)$ is the distributed Nash equilibrium, it is also the Nash equilibrium in the classical sense:

$$\int_D (\mathbf{u}(x,t), \mathbf{A}\hat{\mathbf{w}}^B(x))dx = (\overline{\mathbf{u}}(t), \mathbf{A}\hat{\mathbf{w}}^B), \quad \overline{\mathbf{u}}(t) = \int_D u_i(x,t)dx,$$

$$\int_D (\mathbf{v}(x,t), \mathbf{B}\hat{\mathbf{w}}^A(x))dx = (\overline{\mathbf{v}}(t), \mathbf{B}\hat{\mathbf{w}}^A), \quad \overline{\mathbf{v}}(t) = \int_D v_i(x,t)dx.$$

For any t, we have

$$\sum_{i=1}^n \int_D u_i(x,t)dx = 1, \quad \sum_{i=1}^n \int_D v_i(x,t)dx = 1,$$

hence, $\overline{\mathbf{u}}(t), \overline{\mathbf{v}}(t) \in S_n$. Consequently, we have

$$(\overline{\mathbf{u}}, \mathbf{A}\hat{\mathbf{w}}^B) \le (\hat{\mathbf{w}}^A, \mathbf{A}\hat{\mathbf{w}}^B), \quad (\overline{\mathbf{v}}, \mathbf{B}\hat{\mathbf{w}}^A) \le (\hat{\mathbf{w}}^B, \mathbf{B}\hat{\mathbf{w}}^A).$$

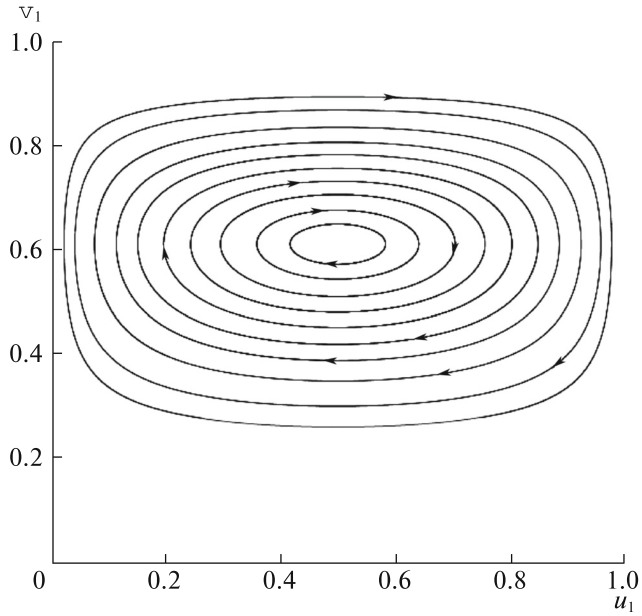

Fig. 1. Phase portrait of a battle-of-the-sexes system with no diffusion.

Theorem 2. *If* $(\hat{\mathbf{w}}^A(x), \hat{\mathbf{w}}^B(x)) \in \mathrm{int}(Sn \times S_n)$ *is a Lyapunov stable solution to system* (5), *then* $(\hat{\mathbf{w}}^A(x))$, $\hat{\mathbf{w}}^B(x)$ *is the distributed Nash equilibrium*.

The proof is analogous to the one for symmetric replicator systems that was given in [13].

4. REPLICATOR SYSTEMS WITH 2-BY-2 MATRICES: THE PARENTAL CONTRIBUTION

Let us consider systems (4) with two strategies; these can be divided into two classes according to the type of equilibrium position [2]: a center or a saddle. We shall discuss the problem known as the parental contribution or the "battle of the sexes." In its initial formulation [16], the problem considers the contribution from individuals of the two sexes to rearing their common descendants with a set of two strategies ("protect" or "leave") and the following parameters: the probability of the descendants surviving with different pairs of strategies, the probability of producing descendants with another female (for males), and the number of the female's descendants [17]. The contribution to rearing common descendants is also estimated in the alternative interpretation [18], but here other parameters and types of behavior are taken into account [2, 17].

Two types of strategies are taken for each sex: A female can use "slow" and "fast" strategies (v_1, v_2); a male can also use two types of strategies, "fickle" and "loyal" (u_1, u_2). We introduce constants that characterize payoffs: g signifies a successful rearing of descendants (raising the fitness of both sexes), $-c$ corresponds to one individual (a female) rearing descendants alone, $-c/2$ denotes individuals rearing descendants equally, and $-e$ represents the costs of prolonged courting.

Payoff matrices **A** (male) and **B** (female) are defined as

$$\mathbf{A} = \begin{bmatrix} 0 & g \\ g - c/2 - e & g - c/2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & g - c/2 - e \\ g - c & g - c/2 \end{bmatrix},$$

where $0 < e < g < c < 2(g - e)$. The equilibrium position of the system (the center) is

$$F = \left(\frac{e}{c - g + e}; \quad \frac{g - c}{g - c - e}; \quad \frac{c}{2(g - e)}; \quad \frac{g - c/2 - e}{g - e}\right).$$

This simplified model of interaction between the sexes is therefore an example of a natural biological oscillator.

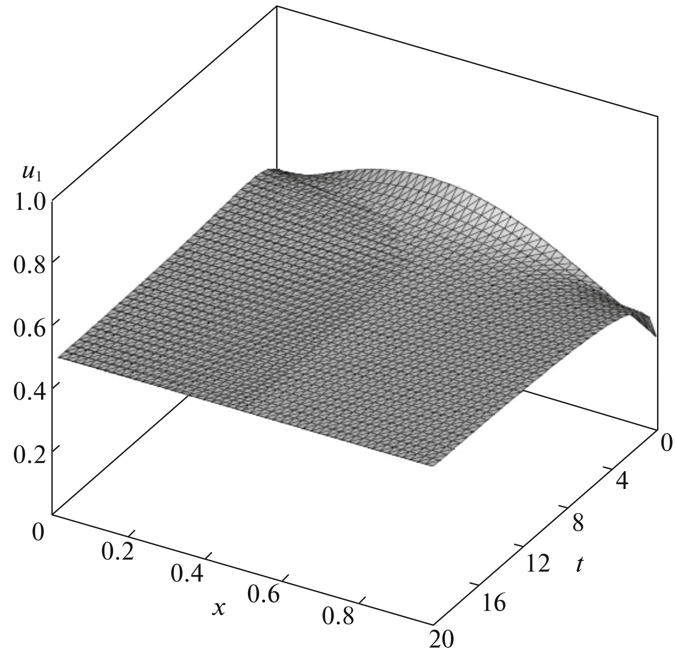

Fig. 2. Solution to a distributed replicator battle-of-the-sexes system, depending on time.

The problem is solved via numerical integration using a Euler explicit scheme of the first order; derivatives are approximated by central differences. For purposes of numerical simulation, we took the following values of costs: $g = 1.0$, $c = 1.1$, and $e = 0.1$. Figure 1 shows the phase portrait of this system. When the mechanism of diffusion is activated, spatial inhomogeneity is gradually eliminated. Figure 2 illustrates the change in $u_1(t, x)$ for diffusion coefficients $d_i^A = d_i^B = 0.02$ and the spatially inhomogeneous initial distribution.

5. REPLICATOR SYSTEMS WITH 2-BY-2 MATRICES: HAWKS AND DOVES

Let us consider one more classical example: Two individuals (or two species) compete for a territory or a useful resource. Each player can choose the "hawk" strategy or the "dove" strategy. The names of these strategies are conditional and specify only two types of behavior: launching a war of aggression or retreating. In the asymmetric form of the game, we assume that the losses of the players differ if they choose different strategies.

We assume that the first and second players are the "natives" and the "invader," respectively. If both players choose aggressive behavior, we assume that the losses are identical and equal to a; if both players retreat, we assume that the losses are zero. With an attack by the invader, the losses are $e \leqslant c$; with aggressive behavior by the natives, they are $-b \leqslant d$. Here, $a \leqslant c \leqslant e$ and $a \leqslant d \leqslant b$. Matrices **A** and **B** have the form

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a & d \\ e & 0 \end{bmatrix}.$$

This class of problems (but with different interpretations) is often encountered. The prisoner's dilemma and the coordination game are analogous to it in game theory. In any version, if there exists an inner stationary point, it is a saddle.

The problem in question was also studied numerically for the following parameters: $a = 1$, $b = 3$, $c = 4$, $d = 5$, and $e = 3$. The phase portrait of the system is presented in Fig. 3. When all diffusion coefficients equal 0.02, the spatially inhomogeneous initial conditions become spatially homogeneous with the pas-

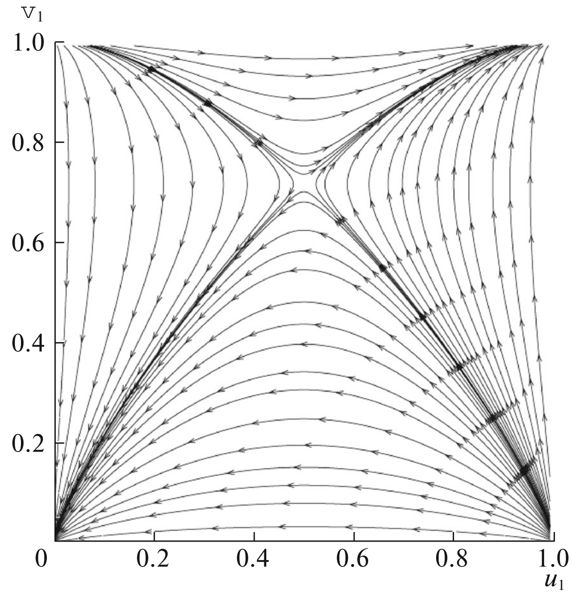

Fig. 3. Phase portrait of a natives–invader system with no diffusion.

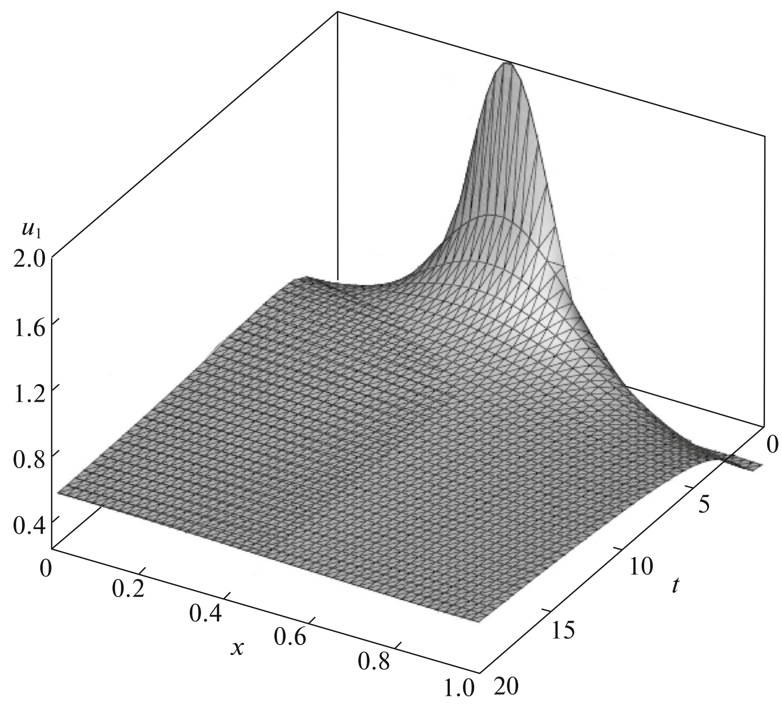

Fig. 4. Stabilizing effect in a distributed replicator natives–invader system.

sage of time (Fig. 4). Here, the stability of the attained equilibrium position is a result of the stabilizing effect of diffusion (normally, this is an unstable equilibrium). The stabilizing effect is observed only for sufficiently high diffusion coefficients.

REFERENCES

1. R. Cressman, *Evolutionary Dynamics and Extensive Form Games* (MIT Press, Cambridge, 2003).

2. K. Sigmund and J. Hofbauer, *Evolutionary Games and Population Dynamics* (Cambridge Univ. Press, Cambridge, 1998).
3. A. Bratus, A. Novozhilov, and V. Posvyanskii, "Existence and stability of stationary solutions to spatially extended autocatalytic and hypercyclic systems under global regulation and with nonlinear growth rates," Nonlinear Anal.: Real World Appl. **11** (3), 1897–1917 (2010).
4. M. Eigen and P. Schuster, "The hypercycle. A principle of natural self-organization. Part A: Emergence of the hypercycle," Naturwissenschaften **64** (11), 541–565 (1977).
5. M. Doebeli and C. Hauert, "Models of cooperation based on the Prisoner's Dilemma and the Snowdrift game," Ecol. Lett. **8**, 748–766 (2005).
6. M. A. Nowak, A. Sasaki, C. Taylor, and D. Fudenberg, "Emergence of cooperation and evolutionary stability in finite populations," Nature **428**, 646–650 (2004).
7. K. Sigmund and J. Hofbauer, "Evolutionary game dynamics," Bull. Am. Math. Soc. **40** (4), 479–519 (2003).
8. K. Brauchli, T. Killingback, and M. Doebeli, "Evolution of cooperation in spatially structured populations," J. Theor. Biol. **200**, 405–417 (1999).
9. C. Hauert and G. Szabo, "Game theory and physics," Am. J. Phys. **73**, 405–414 (2005).
10. H. Ohtsuki and M. Nowak, "The replicator equation on graphs," J. Theor. Biol. **243**, 86–97 (2006).
11. R. A. Fisher, "The wave of advance of advantageous genes," Ann. Eugen. **7**, 353–369 (1938).
12. K. P. Hadeler, "Diffusion in Fisher's population model," Rocky Mountain J. Math. **11**, 39–45 (1981).
13. A. Bratus, A. Novozhilov, and V. Posvyanskii, "A note on the replicator equation with explicit space and global regulation," Math. Biosci. **8** (3), 659–676 (2011).
14. A. S. Bratus', A. S. Novozhilov, and A. P. Platonov, *Dynamic Systems and Models in Biology* (Fizmatlit, Moscow, 2010) [in Russian].
15. L. Collatz, *Eigenwertaufgaben mit technischen Anwendungen* (Verlagsgesellschaft Geest & Portig, Leipzig, 1963; Nauka, Moscow, 1968).
16. J. M. Smith, "Parental investment: a prospective analysis," Anim. Behav. **25**, 1–9 (1977).
17. J. M. Smith, *Evolution and the Theory of Games* (Cambridge Univ. Press, Cambridge, 1982).
18. R. Dawkins and T. R. Carlisle, "Parental investment, mate desertion and a fallacy," Nature **262**, 131–133 (1976).

Translated by L. Kartvelishvili