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Pseudo-Riemannian Foliations and Their Graphs

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Abstract—We prove that a foliation (M, F) of codimension q on a n-dimensional pseudo-Riemannian manifold with induced metrics on leaves is pseudo-Riemannian if and only if any geodesic that is orthogonal at one point to a leaf is orthogonal to every leaf it intersects. We show that on the graph $G = G(F)$ of a pseudo-Riemannian foliation there exists a unique pseudo-Riemannian metric such that canonical projections are pseudo-Riemannian submersions and the fibers of different projections are orthogonal at common points. Relatively this metric the induced foliation $(G, \mathbb{F})$ on the graph is pseudo-Riemannian and the structure of the leaves of $(G, \mathbb{F})$ is described. Special attention is given to the structure of graphs of transversally (geodesically) complete pseudo-Riemannian foliations which are totally geodesic pseudo-Riemannian ones.

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1. INTRODUCTION

Let (M, F) be a smooth foliation. Recall that a pseudo-Riemannian metric g on the manifold M is transversally projectable if the Lie derivative $L_X g$ along X is zero for any vector field X tangent to this foliation.

Definition 1. *A foliation (M, F) on a pseudo-Riemannian manifold (M, g) is referred to as a pseudo-Riemannian foliation if every leaf L with induced metric $g|_L$ is a pseudo-Riemannian manifold and g is transversally projectable.*

A pseudo-Riemannian submersion (see [1]) is a smooth map $p : M \to B$ which is onto and satisfies the following three axioms:

(a) the differential $p_{*x} : T_x M \to T_{p(x)} B$ is onto for all $x \in M$;

(b) the fibers $p^{-1}(b)$, $b \in B$, are pseudo-Riemannian submanifolds of M;

(c) the differential p_* preserves scalar products of vectors normal to fibers.

Definition 1 is equivalent to the fact that a foliation (M, F) is given locally by pseudo-Riemannian submersions, i.e. it is equivalent to the following definition.

Definition 2. *A foliation (M, F) on a pseudo-Riemannian manifold (M, g) is said to be a pseudo-Riemannian foliation if at any point there exists an adapted neighborhood U and a Riemannian metric g^V on the leaf space $V = U/F_U$ such that the canonical projection $f : U \to V$ is a pseudo-Riemannian submersion of $(U, g|_U)$ onto (V, g^V).*

Further a pseudo-Riemannian manifold (M, g) is considered with the Levi-Civita connection ∇.

Definition 3. *Let (M, ∇) be a manifold M with a linear connection ∇. A smooth distribution D on the manifold M is called geodesically invariant if for any point $x \in M$ and each vector $X \in D_x$*

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the geodesic $\gamma = \gamma(s)$ of (M, ∇) such that $\gamma(0) = x$ and $\dot\gamma(0) = X$ has the property $\dot\gamma(s) \in D_{\gamma(s)}$ for every s of the domain of γ. A foliation (M, F) on (M, ∇) is called geodesically invariant or totally geodesic if its tangent distribution TF is geodesically invariant.

First we prove the following criterion of a pseudo-Riemannian nature for a foliation of a pseudo-Riemannian manifold.

Theorem 1. *Let (M, F) be a foliation of codimension q on an n-dimensional pseudo-Riemannian manifold (M, g), $0 < q < n$. Then (M, F) is a pseudo-Riemannian foliation if and only if the q-dimensional distribution D, orthogonal to TF, is geodesically invariant and the metric on the leaves is non-degenerate.*

Corollary 1. *Let M and B be pseudo-Riemannian manifolds and $p : M \to B$ be surjective submersion. Then this submersion is pseudo-Riemannian iff there are induced pseudo-Riemannian metrics on the fibers and any geodesic orthogonal to the fiber at one point also is orthogonal to every fiber it intersects.*

For Riemannian foliations a similar result was proposed by B. Reinhardt [2], and it was proven by P. Molino ([3], Propositions 3.5 and 6.1). We emphasize that Molino's proof bears on the property of a geodesic to be a local extremum of the length functional that does not have an analogue in pseudo-Riemannian geometry.

In the proof of Theorem 1 we essentially use the result of A. D. Lewis [4] on geodesical invariance of distributions on manifolds with affine connection.

Let $\mathfrak{Fol}$ be the category of foliations where morphisms are smooth maps transforming leaves into leaves. Since a pseudo-Riemannian foliation may be considered as a foliation with transverse linear connection, according to ([5], Theorem 1.1) the group of all automorphisms in the category $\mathfrak{Fol}$ of a pseudo-Riemannian foliation admits a structure of an infinite dimensional Lie group modelled on LF-spaces .

By a holonomy group $\Gamma(L)$ of a leaf L of a foliation (M, F) we mean a germinal holonomy group of L usually used in the foliation theory. If $\Gamma(L) = 0$ the leaf L is said to be a leaf without holonomy.

Construction of the holonomy groupoid of a foliation was presented by S. Ehresmann. Another equivalent construction was given by H. Winkelnkemper [6] and named by him *the graph* of a foliation. The graph $G(F)$ contains all information about the foliation (M, F) and its holonomy groups. C^*-algebras of complex valued functions of foliations (M, F) are determined on $G(F)$ and are one of the fundamental concepts in K-theory of foliations.

In the general case the graph of a smooth foliation (M, F) of codimension q on an n-dimensional smooth manifold M is a non-Hausdorff smooth $(2n - q)$-manifold (the precise definition is given in Subsection 3.3.1).

For the graph $G(F)$ of a pseudo-Riemannian foliation (M, F) we prove the following statement.

Theorem 2. *Let (M, F) be a smooth pseudo-Riemannian foliation of codimension q on an n-dimensional pseudo-Riemannian manifold (M, g). Let $G(F)$ be its graph with the canonical projections $p_i : G(F) \to M$, $i = 1, 2$. Then*:

1. *The graph $G(F)$ of a foliation (M, F) is a Hausdorff $(2n - q)$-dimensional manifold with the induced foliation $\mathbb{F} = \{\mathbb{L}_\alpha = p_i^{-1}(L_\alpha) | L_\alpha \in F\}$, $i = 1, 2$. Moreover, the germinal holonomy groups $\Gamma(L_\alpha)$ and $\Gamma(\mathbb{L}_\alpha)$ of the appropriate leaves L_α* and *$\mathbb{L}_\alpha$ are isomorphic.*

2. *On the graph $G(F)$ there exists a unique pseudo-Riemannian metric d with respect to which $(G(F), \mathbb{F})$ is a pseudo-Riemannian foliation and p_i are pseudo-Riemannian submersions. In this case fibers of p_1 are orthogonal to the fibers of p_2 at common points.*

3. *Every leaf $\mathbb{L}_\alpha = p_i^{-1}(L_\alpha) \in \mathbb{F}$ is a reducible pseudo-Riemannian manifold that is isometric to the quotient manifold $(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha$ of the pseudo-Riemannian product $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ of the pseudo-Riemannian holonomy covering space $\mathcal{L}_\alpha$ of L_α by the isometry group Ψ_α, and $\Psi_\alpha \cong \Gamma(L_\alpha) \cong \Gamma(\mathbb{L}_\alpha)$.*

Definition 4. *A pseudo-Riemannian metric d on the graph $G(F)$ satisfying Theorem 2 is called the induced metric.*

Corollary 2. *There exists a dense saturated G_δ-subset of $G(F)$ any leaf $(\mathbb{L}_\alpha, d)$ of which is isometric to the direct product $L_\alpha \times L_\alpha$ of pseudo-Riemannian manifolds (L_α, g).*

Remark 1. *We prove property 3 in Theorem 2 without the application of the well known Wu's theorem* [7]. *This theorem is not applicable here because the completeness of the pseudo-Riemannian metric d is not assumed.*

By a geodesic we mean a piecewise geodesic.

Definition 5. *A pseudo-Riemannian foliation (M, F) is called transversally complete if the canonical parameter on every maximal orthogonal geodesic is defined on the whole real line.*

Under the additional assumption of the transversal completeness of a pseudo-Riemannian foliation we prove the following statement.

Theorem 3. *Let (M, F) be a transversally complete pseudo-Riemannian foliation on a pseudo-Riemannian manifold (M, g) and d be the induced pseudo-Riemannian metric on its graph $G(F)$. Then I:*

1. *The induced foliation $(G(F), \mathbb{F})$ is a transversally complete pseudo-Riemannian foliation.*

2. *The orthogonal q-dimensional distributions $\mathfrak{M}$ and $\mathfrak{N}$ are Ehresmann connections for the foliations (M, F) and $(G(F), \mathbb{F})$ respectively, and, for any $L_\alpha \in F$, $\mathbb{L}_\alpha = p_i^{-1}(L_\alpha)$, the following holonomy groups $\Gamma(L_\alpha)$, $H_{\mathfrak{M}}(L_\alpha)$, $\Gamma(\mathbb{L}_\alpha)$ and $H_{\mathfrak{N}}(\mathbb{L}_\alpha)$ are isomorphic.*

3. *The canonical projections $p_i : G(F) \to M$, $i = 1, 2$, form locally trivial fibrations with the same standard fiber L_0, and L_0 is diffeomorphic to any leaf without holonomy of (M, F).*

4. *Every leaf $\mathbb{L}_\alpha = p_i^{-1}(L_\alpha) \in \mathbb{F}$ is a reducible pseudo-Riemannian manifold that is isometric to the quotient manifold $(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)/\Psi_\alpha$ of the pseudo-Riemannian product $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ of the pseudo-Riemannian holonomy covering space $\mathcal{L}_\alpha$* for *L_α by an isometry group Ψ_α, where $\Psi_\alpha \cong \Gamma(L_\alpha) \cong H_{\mathfrak{M}}(L_\alpha) \cong \Gamma(\mathbb{L}_\alpha) \cong H_{\mathfrak{N}}(\mathbb{L}_\alpha)$, and every $\mathcal{L}_\alpha$ is diffeomorphic to L_0.*

II. *If, moreover, the foliation (M, F) is also geodesically invariant, then:*

(i) *Each foliation $\mathbb{F}, F^{(i)} := \left\{p_i^{-1}(x) | x \in M\right\}, i = 1, 2$, is geodesically invariant and pseudo-Riemannian.*

(ii) *Any leaf without holonomy of (M, F) is isometric to any fibers of submersions p_1 and p_2 with respect to corresponding induced metrics.*

(iii) *Every leaf $\mathbb{L}$ without holonomy of the foliation $(G(F), \mathbb{F})$ is isometric to the pseudo-Riemannian product $L_0 \times L_0$ and any other leaf $\mathbb{L}_\alpha$ is isometric to the pseudo-Riemannian quotient manifold $(L_0 \times L_0)/\Psi_\alpha$, where $\Psi_\alpha \cong \Gamma(\mathbb{L}_\alpha)$.*

We emphasize that the notion of an Ehresmann connection of a foliation proposed by R.A. Blumenthal and J.J. Hebda [8] essentially are used in the proofs of Theorems 2 and 3. The results of the second author on graphs of foliations with Ehresmann connection [9, 10] and some other statements (see Sections 4—5) are also applied.

As example we describe the structure of the graphs of suspended algebraic Lorentzian foliations of codimension 2 on closed 3-manifolds in the Subsection 6.6.2.

Notations. Further smoothness is understood to mean C^∞. We denote by $\mathfrak{X}(N)$ the set of all smooth vector fields on a manifold N. Put $\mathfrak{X}_{\mathfrak{M}}(M) := \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \ \forall u \in M\}$ for a smooth distribution $\mathfrak{M}$ on M. For a foliation (M, F) we denote $\mathfrak{X}_{TF}(M)$ also by $\mathfrak{X}_F(M)$. Let us denote the leaf of foliation (M, F) passing through a point $x \in M$ by $L(x)$.

Let $\cong$ be the symbol of a group isomorphism and of a manifold diffeomorphism as well.

2. A CRITERION OF PSEUDO-RIEMANNIANCE OF A FOLIATION

2.1. Foliate and Transversal Vector Fields

Let (M, F) be a smooth foliation of codimension q of a smooth n-dimensional manifold M. A function $f \in \mathfrak{F}(M)$ is called *basic* if it is constant on every leaf L of this foliation. A vector field $X \in \mathfrak{X}(M)$ is called *foliate* if for any $Y \in \mathfrak{X}_F(M)$ the vector field $[X, Y]$ belongs to $\mathfrak{X}_F(M)$. Following Molino [3] we denote by $L(M, F)$ the set of all foliate vector fields. In this case $L(M, F)$ is a normalizer of the Lie subalgebra $\mathfrak{X}_F(M)$ in the Lie algebra of vector fields $\mathfrak{X}(M)$. Therefore $L(M, F)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

A q-dimensional smooth distribution $\mathfrak{M}$ on the manifold M is called *transversal* to the foliation (M, F) if the equality $T_xM = T_xF \oplus \mathfrak{M}_x$ holds for any $x \in M$, where $\oplus$ stands for the direct sum of vector spaces. Let us identify the vector quotient bundle TM/TF with a transversal distribution $\mathfrak{M}$. Every $X \in \mathfrak{X}(M)$ may be uniquely represented in the form $X = X^F + X^{\mathfrak{M}}$ where $X^F \in \mathfrak{X}_F(M)$ and $X^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$. In particular, if X is a foliate vector field, then $X^{\mathfrak{M}}$ is called *the transverse vector field* associated to X. Let $l(M, F)$ be the set of transverse vector fields. The projection $L(M, F) \to l(M, F) : X \mapsto X^{\mathfrak{M}}$ is well defined, with kernel is equal to $\mathfrak{X}_F(M)$. Therefore, there exists for the foliation (M, F) an exact sequence of vector spaces

$$0 \to \mathfrak{X}_F(M) \to L(M, F) \to l(M, F) \to 0. \tag{1}$$

2.2. Lewis's Criterion

Further we use the following criterion.

Lewis's Theorem [4]. *A smooth distribution $\mathfrak{M}$ on a manifold of an affine connection (M, ∇) is geodesically invariant if and only if $\nabla_X Y + \nabla_Y X \in \mathfrak{X}_{\mathfrak{M}}(M)$ for any vector fields X, Y belonging to $\mathfrak{X}_{\mathfrak{M}}(M)$.*

2.3. Proof of Theorem 1

Let (M, F) be a smooth foliation of codimension q on a pseudo-Riemannian manifold (M, g) such that the induced metrics on leaves are non-degenerate. Suppose now that the q-dimensional distribution $\mathfrak{M}$ orthogonal to (M, F) is geodesically invariant. Let us consider any foliate vector fields $X^{\mathfrak{M}}, Y^{\mathfrak{M}} \in l(M, F)$ and an arbitrary vector field $Z^F \in \mathfrak{X}_F(M)$. Observe that the metric g is transversally projectable with respect to the foliation (M, F) if and only if $Z^F \cdot g(X^{\mathfrak{M}}, Y^{\mathfrak{M}}) = 0$. Recall that the equality $\nabla_X g = 0$ is equivalent to

$$Z \cdot g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \ \ \forall X, Y, Z \in \mathfrak{X}(M). \tag{2}$$

According to (2) for $Z = Z^F \in \mathfrak{X}_F(M)$ and $X = X^{\mathfrak{M}}$, $Y = Y^{\mathfrak{M}} \in l(M, F)$ using the identity $g(X^{\mathfrak{M}}, Z^F) = 0$ we have

$$Y^{\mathfrak{M}} \cdot g(X^{\mathfrak{M}}, Z^F) = g(\nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}}, Z^F) + g(X^{\mathfrak{M}}, \nabla_{Y^{\mathfrak{M}}} Z^F) = 0. \tag{3}$$

By analogy, changing $Y^{\mathfrak{M}}$ and $X^{\mathfrak{M}}$ we get

$$X^{\mathfrak{M}} \cdot g(Y^{\mathfrak{M}}, Z^F) = g(\nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}}, Z^F) + g(Y^{\mathfrak{M}}, \nabla_{X^{\mathfrak{M}}} Z^F) = 0. \tag{4}$$

Add (3) to (4), then apply the bilinearity of the pseudo-Riemannian metric g and obtain

$$g(\nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} + \nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}}, Z^F) + g(X^{\mathfrak{M}}, \nabla_{Y^{\mathfrak{M}}} Z^F) + g(Y^{\mathfrak{M}}, \nabla_{X^{\mathfrak{M}}} Z^F) = 0. \tag{5}$$

Due to the geodesical invariance of the distribution $\mathfrak{M}$ the Lewis's criterion implies $\nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}} + \nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$. Therefore the first term in (5) was equal to zero. Since ∇ is the Levi-Civita connection, it is torsion free and

$$\nabla_{Y^{\mathfrak{M}}} Z^F = \nabla_{Z^F} Y^{\mathfrak{M}} + [Y^{\mathfrak{M}}, Z^F], \quad \nabla_{X^{\mathfrak{M}}} Z^F = \nabla_{Z^F} X^{\mathfrak{M}} + [X^{\mathfrak{M}}, Z^F]. \tag{6}$$

Putting (6) into (5) we obtain

$$g(X^{\mathfrak{M}}, \nabla_{Z^F} Y^{\mathfrak{M}}) + g(Y^{\mathfrak{M}}, \nabla_{Z^F} X^{\mathfrak{M}}) + g(X^{\mathfrak{M}}, [Y^{\mathfrak{M}}, Z^F]) + g(Y^{\mathfrak{M}}, [X^{\mathfrak{M}}, Z^F]) = 0.$$

In concordance with the choice, the vector fields $X^{\mathfrak{M}}$ and $Y^{\mathfrak{M}}$ are foliate, so $[Z^{\mathfrak{M}}, Y^F]$ and $[X^{\mathfrak{M}}, Y^F]$ belong to $\mathfrak{X}_F(M)$. Hence the third and fourth terms in the previous equation vanish. Therefore

$$g(X^{\mathfrak{M}}, \nabla_{Z^F} Y^{\mathfrak{M}}) + g(Y^{\mathfrak{M}}, \nabla_{Z^F} X^{\mathfrak{M}}) = 0. \quad (7)$$

Let in (2) $Z = Z^F \in \mathfrak{X}_F(M)$, $X = X^{\mathfrak{M}}$, $Y = Y^{\mathfrak{M}} \in l(M, F)$ and using the relation (7) we have the following

$$Z^F \cdot g(X^{\mathfrak{M}}, Y^{\mathfrak{M}}) = g(\nabla_{Z^F} X^{\mathfrak{M}}, Y^{\mathfrak{M}}) + g(X^{\mathfrak{M}}, \nabla_{Z^F} Y^{\mathfrak{M}}) = 0. \quad (8)$$

The equality (8) implies the pseudo-Riemanniance of the foliation (M, F).

Converse. Let (M, F) be a pseudo-Riemannian foliation of codimension q on a pseudo-Riemannian manifold (M, g), hence by Definition 1 the restriction of this metric on leaves is non-degenerate. Denote by $\mathfrak{M}$ the orthogonal q-dimensional distribution to this foliation. The pseudo-Riemanniance of the foliation (M, F) implies that equalities (8) and (7) hold for an arbitrary vector field $Z^F \in \mathfrak{X}_F(M)$ and any foliate vector fields $X^{\mathfrak{M}}, Y^{\mathfrak{M}} \in l(M, F)$. From (6) we find

$$\nabla_{Z^F} Y^{\mathfrak{M}} = \nabla_{Y^{\mathfrak{M}}} Z^F - [Y^{\mathfrak{M}}, Z^F], \quad \nabla_{Z^F} X^{\mathfrak{M}} = \nabla_{X^{\mathfrak{M}}} Z^F - [X^{\mathfrak{M}}, Z^F]. \quad (9)$$

Putting (9) in (7) and taking into account that $[Z^{\mathfrak{M}}, Y^F]$ and $[X^{\mathfrak{M}}, Y^F]$ belong to $\mathfrak{X}_F(M)$ we get

$$g(X^{\mathfrak{M}}, \nabla_{Y^{\mathfrak{M}}} Z^F) + g(Y^{\mathfrak{M}}, \nabla_{X^{\mathfrak{M}}} Z^F) = 0. \quad (10)$$

Recall that (5) is obtained using only the condition $\nabla g = 0$ and the orthogonality of the distributions $\mathfrak{M}$ and TF. Therefore we may apply (5). Thus we have $g(\nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} + \nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}}, Z^F) = 0$ for any vector field Z^F tangent to the foliation (M, F). Non-degeneracy of the restriction of g to the leaves of (M, F) implies $\nabla_{Y^{\mathfrak{M}}} X^{\mathfrak{M}} + \nabla_{X^{\mathfrak{M}}} Y^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$ for every $X^{\mathfrak{M}}, Y^{\mathfrak{M}} \in l(M, F)$. It is easy to check that this implies

$$\nabla_Y X + \nabla_X Y \in \mathfrak{X}_{\mathfrak{M}}(M) \; \forall X, Y \in \mathfrak{X}_{\mathfrak{M}}(M). \quad (11)$$

According to Lewis's theorem mentioned above the relation (11) guarantees that the distribution $\mathfrak{M}$ is geodesically invariant.

3. GRAPHS OF PSEUDO-RIEMANNIAN FOLIATIONS

3.1. The Graph of a Smooth Foliation

Let (M, F) be a smooth foliation of codimension q of an n-dimensional manifold M and $\mathfrak{M}$ be transversal q-dimension distribution on M. Denote by $A(x, y)$ the set of all piecewise smooth paths from x to y on the same leaf L. Two paths $h_1, h_2 \in A(x, y)$ are called equivalent and are denoted by $h_1 \sim h_2$ if and only if they define the same germ at the point x of the holonomy diffeomorphisms $D_x^q \to D_y^q$ from D_x^q to D_y^q, where D_x^q and D_y^q are q-dimensional disks transversal to this foliation. Denote $\langle h \rangle$ the equivalence class containing h.

Definition 6. *The set* $G(F) := \{(x, \langle h \rangle, y) | x \in M, y \in L(x), h \in A(x, y)\}$ *is called the graph of the foliation* (M, F) [6].

Suppose that h and g are paths such that $h(1) = g(0)$. Denote by $h \cdot g$ the product of the path h and g. The equality $(x, \langle h \rangle, v) \circ (v, \langle g \rangle, y) := (x, \langle h \cdot g \rangle, y)$ where $h(1) = g(0)$, defines a partial multiplication $\circ$ in the graph $G(F)$ with respect to which $G(F)$ is a groupoid named *the holonomy groupoid* of the foliation (M, F). By a natural way the graph $G(F)$ is provided by a structure of a $(2n - q)$-dimensional smooth manifold which is non-Hausdorff, in general [6].

Definition 7. *A pseudogroup* $\mathcal{H}$ *of local holonomy diffeomorphisms of a manifold* N *is called quasi-analytic if the existence of an open connected subset* V *in* N *such that* $h|_V = id_V$ *for an element* $h \in \mathcal{H}$ *implies that* $h = id_{D(h)}$ *in the whole connected domain* $D(h)$ *of* h *that contains* V.

According to ([10], Proposition 2), Winkelnkemper's criterion of the property of the graph $G(F)$ to be Hausdorff can be reformulated as follows:

Proposition 1. *The topological space of the graph* $G(F)$ *of a foliation* (M, F) *is Hausdorff iff the holonomy pseudogroup of this foliation is quasi-analytic.*

The mappings

$$p_1 : G(F) \to M : (x, \langle h \rangle, y) \mapsto x, \quad p_2 : G(F) \to M : (x, \langle h \rangle, y) \mapsto y$$

are referred to as *canonical projections,* and p_1 and p_2 are submersions onto M.

Definition 8. *A foliation $\mathbb{F} = \{\mathbb{L}_\alpha = p_1^{-1}(L_\alpha) = p_2^{-1}(L_\alpha) | L_\alpha \in F\}$ is defined on the graph $G(F)$ and is called the induced foliation.*

3.2. An Ehresmann Connection for a Smooth Foliation

Recall the notion of an Ehresmann connection which was introduced by R.A. Blumenthal and J.J. Hebda [8]. We use the term *a vertical-horizontal homotopy* introduced previously by R. Hermann. All mappings are supposed to be piecewise smooth.

Let (M, F) be a foliation of an arbitrary codimension $q \geq 1$. Let $\mathfrak{M}$ be a q-dimensional transverse distribution on M, then for any $x \in M$ the equality $T_xM = T_xF \oplus \mathfrak{M}_x$ holds. Vectors from $\mathfrak{M}_x$, $x \in M$, are called *horizontal*. A piecewise smooth curve σ is *horizontal* (or $\mathfrak{M}$-horizontal) if each of its smooth segments is an integral curve of the distribution $\mathfrak{M}$. The distribution TF tangent to the leaves of the foliation (M, F) is called *vertical*. One says that a curve h is *vertical* if h is contained in some leaf of a foliation (M, F).

A *vertical-horizontal homotopy* is a piecewise smooth map $H : I_1 \times I_2 \to M$, where $I_1 = I_2 = [0, 1]$, such that for any $(s, t) \in I_1 \times I_2$ the curve $H|_{I_1 \times \{t\}}$ is horizontal and the curve $H|_{\{s\} \times I_2}$ is vertical. The pair of curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called *a base of the vertical-horizontal homotopy H*. Two paths (σ, h) with common origin $\sigma(0) = h(0)$, where σ is a horizontal path and h is a vertical one, are called an *admissible pair of paths*.

A q-dimensional distribution $\mathfrak{M}$ transversal to a foliation (M, F) of codimension q is called an *Ehresmann connection for* (M, F) if for any admissible pair of paths (σ, h) there exists a vertical-horizontal homotopy with the base (σ, h).

Let $\mathfrak{M}$ be an Ehresmann connection for a foliation (M, F). Then for any admissible pair of paths (σ, h) there exists a unique vertical-horizontal homotopy H with the base (σ, h). We say that $\widetilde{\sigma} := H|_{I_1 \times \{1\}}$ is the result of *the transfer of the path σ along h with respect to the Ehresmann connection* $\mathfrak{M}$.

3.3. Proof of Theorem 2

Let (M, F) be a pseudo-Riemannian foliation on a pseudo-Riemannian manifold (M, g). As for as every pseudogroup of local isometries of a pseudo-Riemannian manifold is quasi-analytic, according to Winkelnkemper's criterion (Proposition 1) the graph $G(F)$ is Hausdorff. Observe that the definition of the inducted foliation $(G(F), \mathbb{F})$ implies that both foliations (M, F) and $(G(F), \mathbb{F})$ are given by the same holonomy pseudogroup $\mathcal{H}$. Recall that the germinal holonomy group of any foliation is interpreted as a group of germs at the relevant point v of the local transformations φ from the holonomy pseudogroup $\mathcal{H}$ such that $\varphi(v) = v$. This implies that the holonomy groups $\Gamma(L, x)$ and $\Gamma(\mathbb{L}, z)$, where $\mathbb{L} = p_1^{-1}(L)$, $z = (x, \langle h \rangle, y) \in G(F)$, are isomorphic. Thus the assertion 1 of Theorem 2 is valid.

Let $\mathbb{F}$ be the induced foliation and $F^{(i)} := \left\{p_i^{-1}(x) | x \in M\right\}, i = 1, 2,$ be two simple foliations on the graph $G(F)$. Define a special pseudo-Riemannian metric on the graph $G(F)$. Denote by $\mathfrak{M}$ the q-dimensional distribution on M orthogonal to the pseudo-Riemannian foliation (M, F). Since the pseudo-Riemannian metric g is non-degenerate on the leaves of this foliation, there exists a decomposition of the tangent space $T_xM = T_xF \oplus \mathfrak{M}_x$, $x \in M$, of M into the orthogonal sum of vector subspaces. For any $z = (x, \langle h \rangle, y) \in G(F)$ put $\mathfrak{N}_z := \{X \in T_zG(F) | p_{1*z}X \in \mathfrak{M}_x, p_{2*z}(X) \in \mathfrak{M}_y\}$. Emphasize that there exists a bijective mapping of the a intersection $p_1^{-1}(x) \cap p_2^{-1}(y)$ to the holonomy group $\Gamma(L, x)$. Therefore $\mathfrak{N} = \{\mathfrak{N}_z | z \in G(F)\}$ is a smooth q-dimensional distribution on the graph $G(F)$ and for any $z \in G(F)$, the tangent vector space $T_zG(F)$ admits the following decomposition into a direct sum of vector subspaces

$$T_z(G(F)) = T_z(F^{(1)}) \oplus \mathfrak{N}_z \oplus T_z(F^{(2)}). \tag{12}$$

According to the decomposition (12), any vector field $X \in \mathfrak{X}(G(F))$ admits a unique representation in the form

$$X = X^{(1)} + X^{\mathfrak{N}} + X^{(2)}, \tag{13}$$

where $X^{(i)} \in \mathfrak{X}_{F^{(i)}}G(F)$ and $X^{\mathfrak{N}} \in \mathfrak{X}_{\mathfrak{N}}G(F)$. Let us define the pseudo-Riemannian metric d on $G(F)$ by the equality

$$d(X,Y) := g(p_{1*}X, p_{1*}Y) + g(p_{2*}X^{(1)}, p_{2*}Y^{(1)}), \tag{14}$$

where X, Y are represented in the form (13).

Note that the foliations $F^{(1)}, F^{(2)}$ and distribution $\mathfrak{N}$ are pairwise orthogonal in the pseudo-Riemannian manifold $(G(F), d)$. Moreover, the restriction of d onto any leaf of each foliation $\mathbb{F}, F^{(1)}$ and $F^{(2)}$ on the graph $G(F)$ is non-degenerate, and, for any $X, Y \in \mathfrak{X}_{T\mathbb{F}}(G)$ we have $d(X,Y) := g(p_{1*}X^{(2)}, p_{1*}Y^{(2)}) + g(p_{2*}X^{(1)}, p_{2*}Y^{(1)})$. This means that the induced pseudo-Riemannian metric $d|_{\mathbb{L}}$ on a leaf $\mathbb{L}$ is locally the direct product of the pseudo-Riemannian metric induced on leaves of foliations $F^{(1)}|_{\mathbb{L}}$ and $F^{(2)}|_{\mathbb{L}}$. Therefore (see, for example [7]) the distributions $TF^{(1)}$ and $TF^{(2)}$ are orthogonal and parallel on $(\mathbb{L}, d|_{\mathbb{L}})$. Hence $(\mathbb{L}, d|_{\mathbb{L}})$ is a non-degenerately reducible pseudo-Riemannian manifold.

Further the restriction of d (respectively, g or p_{i*}) onto the corresponding submanifold of $G(F)$ (or vector subspaces) will be denoted as well by d (respectively, g or p_{i*}).

The canonical projection $p_1 : G(F) \to M$ is pseudo-Riemannian submersion because $p_{1*z} : T_zF^{(2)} \oplus \mathfrak{N}_z \to T_xM$ is an isomorphism of the pseudo-Euclidean vector spaces $(T_zF^{(2)} \oplus \mathfrak{N}_z, d)$ and $(T_xF, g|_{T_xF})$ by the definition of d.

Show that the canonical projection p_2 is also a pseudo-Riemannian submersion. Take any point $z = (x, \langle h\rangle, y)$ in $G(F)$. The pseudo-Riemanniance of (M, F) implies the existence of a linear isomorphism $\phi_{xy} : \mathfrak{M}_x \to \mathfrak{M}_y$ induced by the local holonomy isometry along h of the pseudo-Euclidean vector spaces $(\mathfrak{M}_x, g)$ and $(\mathfrak{M}_y, g)$. According to the definition of d the restriction $p_{2*z} : T_zF^{(1)} \to T_yF$ is an isomorphism of pseudo-Euclidean spaces. Observe that the restriction $p_{2*z} : T_zF^{(1)} \oplus \mathfrak{N}_z \to T_yM = T_yF \oplus \mathfrak{M}_y$ is equal to $p_{2*z} = (p_{2*z}|_{T_zF^{(1)}}, \phi_{xy} \circ p_{1*z}|_{\mathfrak{N}_z})$. This implies that p_2 is also a pseudo-Riemannian submersion.

Take any vector $X \in \mathfrak{N}_z$. Let $\gamma = \gamma(s)$ be the geodesic of $(G(F), d)$ passing through the point z in the direction of the vector X, i.e. $\gamma(0) = z$, $\dot\gamma(0) = X$. Therefore γ is the geodesic orthogonal to the leaves $L^{(1)}(z)$ and $L^{(2)}(z)$ of the foliations $F^{(1)}$ and $F^{(2)}$. As for as $L^{(1)}$ and $L^{(2)}$ are the fibers of the pseudo-Riemannian submersions p_1 and p_2, according to Corollary 1 at the any point of $\gamma(s)$ the tangent vector $\dot\gamma(s)$ is orthogonal to the both fibers $L^{(1)}(\gamma(s))$ and $L^{(2)}(\gamma(s))$. Since $T_{\gamma(s)}\mathbb{L} = T_{\gamma(s)}L^{(1)} \oplus T_{\gamma(s)}L^{(2)}$, then the tangent vector $\dot\gamma(s)$ is orthogonal to $T_{\gamma(s)}\mathbb{L}$. Thus geodesic $\gamma = \gamma(s)$ of the pseudo-Riemannian manifold $(G(F), d)$ orthogonal to leaves of the foliation $(G(F), \mathbb{F})$ at the one its point is orthogonal to the foliation $(G(F), \mathbb{F})$ at each its point.

According to Theorem 1, due to non-degeneracy of a pseudo-Riemannian metric on leaves of this foliation, $(G(F), \mathbb{F})$ is a pseudo-Riemannian foliation.

Assume that there exists another pseudo-Riemannian metric $\hat d$ on $G(F)$ satisfying the second statement of Theorem 2. Let $\hat{\mathfrak{N}}$ be the q-dimensional distribution that is orthogonal to the foliation $\mathbb{F}$ in $(G(F), \hat d)$. Therefore $p_{1*}\hat{\mathfrak{N}}_z = \mathfrak{M}_x$ and $p_{2*}\hat{\mathfrak{N}}_z = \mathfrak{M}_y$ for every point $z = (x, \langle h\rangle, y)$ in $G(F)$. Consequently $\hat{\mathfrak{N}} = \mathfrak{N}$ and $\hat d(X,Y) = d(X,Y) = g(p_{1*}X, p_{1*}Y)$ for any $X, Y \in \mathfrak{X}_{\mathfrak{N}}G(F)$. According to our assumption, in relation to both metrics d and $\hat d$ the foliations $F^{(1)}$, $F^{(2)}$ are orthogonal, with p_i, $i = 1, 2$, are pseudo-Riemannian submersions. Due to the decomposition (13) and bilinearity of d and $\hat d$ it is necessary that $d = \hat d$.

Thus the statements 1 and 2 of Theorem 2 are proven. The statement 3 of Theorem 2 is proved by analogy with ([11], Proposition 5).

4. TWO GRAPHS OF A FOLIATION WITH AN EHRESMANN CONNECTION

4.1. Holonomy Groups of Foliations with Ehresmann Connections

Let (M, F) be a foliation with an Ehresmann connection $\mathfrak{M}$. Take any point $x \in M$. Denote by Ω_x the set of horizontal curves with the origin at x. An action of the fundamental group $\pi_1(L, x)$ of the leaf $L = L(x)$ on the set Ω_x is defined in the following way: $\Phi_x : \pi_1(L, x) \times \Omega_x \to \Omega_x : ([h], \sigma) \mapsto \widetilde{\sigma}$, where $[h] \in \pi_1(L, x)$ and $\widetilde{\sigma}$ is the result of the transfer of $\sigma \in \Omega_x$ along h with respect to $\mathfrak{M}$. Let $K_{\mathfrak{M}}(L, x) := \{\alpha \in \pi_1(L, x) | \alpha(\sigma) = \sigma \ \forall \sigma \in \Omega_x\}$ be the kernel of the action Φ_x. The quotient group $H_{\mathfrak{M}}(L, x) = \pi_1(L, x)/K_{\mathfrak{M}}(L, x)$ is *the $\mathfrak{M}$-holonomy group of the leaf L*, see [8]. Due to the pathwise connectedness of the leaves, the $\mathfrak{M}$-holonomy groups at different points on the same leaf are isomorphic.

Let $\Gamma(L, x)$ be a germinal holonomy group of a leaf L. Then there exists a unique group epimorphism $\chi : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ satisfying the equality $\chi \circ \mu = \nu$, where $\mu : \pi_1(L, x) \to H_{\mathfrak{M}}(L, x) : [h] \mapsto [h] \cdot K_{\mathfrak{M}}(L, x)$ is the quotient map and $\nu : \pi_1(L, x) \to \Gamma(L, x) : [h] \mapsto \langle h \rangle$, where $\langle h \rangle$ is the germ of the holonomy diffeomorphism of a transverse q-dimensional disk along the loop h at the point x.

Emphasize that the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$ has *a global character* unlike the germinal holonomy group $\Gamma(L, x)$ having a local-global character: global along the leaves and local along the transverse directions.

4.2. The Graph $G_{\mathfrak{M}}(F)$

The graph $G_{\mathfrak{M}}(F)$ of a foliation (M, F) with an Ehresmann connection was introduced by the second author in [9] (see also [10]).

Let (M, F) be a foliation of an arbitrary dimension k on an n-manifold M and $q = n - k$. Suppose that the foliation (M, F) admits an Ehresmann connection $\mathfrak{M}$. Take any points x and y in a leaf L of (M, F). Introduce an equivalence relation ρ on the set $A(x, y)$ of vertical paths in L connecting x with y. Paths h and f in $A(x, y)$ are called ρ-equivalent if they define the same transfers of $\mathfrak{M}$-horizontal curves from Ω_x to Ω_y with respect to the Ehresmann connection $\mathfrak{M}$.

The set of ordered triplets $(x, \{h\}, y)$, where x and y are any points in a leaf L of the foliation (M, F) and $\{h\}$ is a class of paths connecting x and y which are ρ-equivalent to h, is called *the graph of the foliation (M, F) with an Ehresmann connection* $\mathfrak{M}$ and is denoted by $G_{\mathfrak{M}}(F)$. The following maps $p_1 : G_{\mathfrak{M}}(F) \to M : (x, \{h\}, y) \mapsto x$, $p_2 : G_{\mathfrak{M}}(F) \to M : (x, \{h\}, y) \mapsto y$ are called the *canonical projections*.

The graph $G_{\mathfrak{M}}(F)$ is equipped with a smooth structure and the binary operation $(y, \{h_1\}, z) * (x, \{h_2\}, y) := (x, \{h_1 \cdot h_2\}, z)$ becomes a smooth $\mathfrak{M}$-holonomy groupoid.

5. THE PROOF OF THEOREM 3

I. We will use the following lemma which is easy proved.

Lemma 1. *Let (M, F) be any transversally complete pseudo-Riemannian foliation of a codimension q. Then the orthogonal q-dimension distribution $\mathfrak{M}$ is an Ehresmann connection for (M, F).*

Let (M, F) be a transversally complete pseudo-Riemannian foliation. According to Lemma 1, the distribution $\mathfrak{M}$ is an Ehresmann connection for (M, F). Therefore two graphs $G(F)$ and $G_{\mathfrak{M}}(F)$ are defined. Since the graph $G(F)$ is Hausdorff, according to ([9], Theorem 2), we may identify the graph $G(F)$ with the graph $G_{\mathfrak{M}}(F)$. In this case $\Gamma(L, x) \cong H_{\mathfrak{M}}(L, x)$ for any $x \in M$. In accordance with ([9], Theorem 1), the canonical projections $p_i : G(F) \to M$, $i = 1, 2$, define locally trivial bundles with the same standard fiber Y. It is easy to see that $Y = L_0$ is diffeomorphic to any leaf L_α without the holonomy of the foliation (M, F).

Observe that the q-dimensional distribution $\mathfrak{N}$ is an Ehresmann connection for the induced foliation $(G(F), \mathbb{F})$. It implies that the distribution $\mathfrak{K} = \mathfrak{N} \oplus TF^{(2)}$ is an Ehresmann connection for the submersion $p_1 : G(F) \to M$. Moreover, a $\mathfrak{K}$-lift of any $\mathfrak{M}$-curve in M is a $\mathfrak{N}$-curve in $G(F)$. Since p_1 is a pseudo-Riemannian submersion, any $\mathfrak{N}$-lift of a $\mathfrak{M}$-geodesic γ in (M, g) to any point $z \in p_1^{-1}(\gamma(0))$ is a $\mathfrak{N}$-geodesic in $(G(F), d)$ and the projection $p_1 \circ \hat{\gamma}$ of every $\mathfrak{N}$-geodesic $\hat{\gamma}$ in $(G(F), d)$ is the $\mathfrak{M}$-geodesic in (M, g). These facts and the transversal completeness of (M, F) imply the transversal completeness of $(G(F), \mathbb{F})$.

Since the holonomy pseudogroup $\mathcal{H}(G(F), \mathbb{F})$ is quasi-analytical, the holonomy groups $\Gamma(\mathbb{L}_\alpha, z)$ and $H_{\mathfrak{N}}(\mathbb{L}_\alpha, z)$ are isomorphic. In accordance with the first statement of Theorem 2 the holonomy groups $\Gamma(\mathbb{L}_\alpha, z)$ and $\Gamma(L_\alpha, x)$, $x = p_1(z)$, are isomorphic. Therefore $\Gamma(L_\alpha) \cong H_{\mathfrak{M}}(L_\alpha) \cong \Gamma(\mathbb{L}_\alpha) \cong H_{\mathfrak{N}}(\mathbb{L}_\alpha)$.

Thus, three statements of Theorem 3 are proved.

Remark that statement 4 follows from the proven statements 2 and 3 of Theorem 2.

II. Assume that a pseudo-Riemannian foliation (M, F) is geodesically invariant.

It is easy to show that for any two leaves L_0 and L there exists a piecewise smooth horizontal geodesic $\sigma : [0, 1] \to M$ such that $a_0 = \sigma(0) \in L_0$ and $a = \sigma(1) \in L$. Let L_0 be a fixed leaf without holonomy and L be any other leaf of the foliation (M, F). Take any point $x \in L_0$. Connect a_0 with x by a vertical path $h : [0, 1] \to L_0$, $h(0) = a_0$, $h(1) = x$. As (M, F) is a transversally complete pseudo-Riemannian foliation according to Lemma 1 $\mathfrak{M}$ is an Ehresmann connection for foliation (M, F). Then for the admissible pair (σ, h) there exists the vertical-horizontal homotopy $H : I_1 \times I_2 \to M$ with the base (σ, h). Let $\tilde{\sigma} := H|_{I_1 \times \{1\}}$. In this case by ([10], Lemma 1) the following map $f_\sigma : L_0 \to L : x \mapsto \tilde{\sigma}(1)$ is well defined and is a regular covering with the deck transformation group isomorphic to $H_{\mathfrak{M}}(L, x) \cong \Gamma(L, x)$.

According to our assumption, the foliation (M, F) is geodesically invariant, then Proposition 2.7 from [12] implies that the covering $f_\sigma : L_0 \to L$ is local isometry with respect to the induced pseudo-Riemannian metrics $g|_{L_0}$ and $g|_L$ on L_0 and L. Therefore for any leaf L without holonomy the map $f_\sigma : L_0 \to L$ is an isometry.

Using ([12], Proposition 2.7) and considering that p_1 and p_2 are pseudo-Riemannian submersions we get that $\mathbb{F}$, $F^{(1)}$ and $F^{(2)}$ are geodesically invariant and pseudo-Riemannian foliations on the pseudo-Riemannian manifold $(G(F), d)$, i.e. (3) is true.

Now it easy to check the fulfilment of statements (ii) and (iii) in II.

6. LORENTZIAN FOLIATIONS OF CODIMENSION 2 ON CLOSED 3-MANIFOLDS

6.1. Theorem of C. Boubel, P. Mounoud, C. Tarquini [13]

Definition. *The algebraic Anosov flows of codimension 2 on closed 3-manifolds, up to finite coverings and finite quotients, are the following:*

1) *The geodesic flows of the unit tangent bundle of hyperbolic compact surfaces*;

2) *The flows defined by the suspensions of linear hyperbolic diffeomorphisms of the* 2*-torus.*

An application of Molino's theory of Riemannian foliations on compact manifolds [3] and the classification of the Lorentzian Anosov flows given by E. Ghys in [14] allowed C. Boubel, P. Mounoud, C. Tarquini ([13], Theorem 4.1) to describe the topological structure of transversally complete Lorentzian foliations of codimension 2 on closed 3-manifolds in the following way:

Theorem 4. *Up to finite coverings, a 1-dimensional transversally complete Lorentzian foliation on a compact closed 3-manifold is either smoothly equivalent to a foliation generated by an algebraic Anosov flow or a Riemannian foliation.*

The structure of suspended algebraic Lorentzian foliations of codimension 2 on closed 3-manifolds and their graphs is described in the following Subsection 6.6.2.

6.2. Example

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ be a matrix of integers such that $ad - bc = 1$ and $a + d > 2$. Such matrix A induces an Anosov automorphism f_A of a torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ conserving its orientation.

Let us consider the action of the group of integers $\mathbb{Z}$ by the formula

$$\Phi_A := \mathbb{T}^2 \times \mathbb{R}^1 \times \mathbb{Z} \to \mathbb{T}^2 \times \mathbb{R}^1 : (u, t, n) \mapsto (f_A^n(u), f + n), n \in \mathbb{Z}. \quad (15)$$

Then the quotient manifold $M := \mathbb{T}^2 \times_{\mathbb{Z}} \mathbb{R}^1$ is defined, and M is a closed 3-manifold. Let $\varphi : \mathbb{T}^2 \times \mathbb{R} \to M$ be the quotient mapping. Since Φ_A preserves the trivial foliation $F_{tr} = \{\{u\} \times \mathbb{R}^1 | u \in \mathbb{T}^2\}$, the foliation (M, F) of codimension 2 is defined.

There exists the Lorentzian metric $g = \eta \begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}$, $\eta \in R \setminus \{0\}$, on the plane $\mathbb{R}^2$ which is invariant in relation to A ([15], Theorem 1). This metric induces the flat Lorentzian metric on $\mathbb{T}^2$ which is denoted also by g. Therefore $\tilde{g} := \begin{pmatrix} -2c & a-d & 0 \\ a-d & 2b & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a flat Lorentzian metric on the manifold M, and $(M, \tilde{g})$ is a locally pseudo-Euclidean manifold with the Lorentzian totally geodesic foliation on (M, F). Emphasize that (M, F) is not a Riemannian foliation.

Let $k : \tilde{M} \to M$ be the smooth universal covering map for M and $\tilde{F} := k^* F$ be the induced foliation $\tilde{M}$. Then $\tilde{M} = \mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}^1$ and $\tilde{F} = \{\{v\} \times \mathbb{R}^1 | v \in \mathbb{R}^2\}$. Let $pr : \tilde{M} = \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ be the projection onto the first multiplier. The group $\Psi := \langle A \rangle \cong \mathbb{Z}$ is the global holonomy group of the foliation (M, F) that is covered by the trivial fiber bundle $pr : \tilde{M} = \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$. The restriction $k|_{\tilde{L}} : \tilde{L} \to L$ onto an arbitrary leaf $\tilde{L} \cong \mathbb{R}^1$ of $(\tilde{M}, \tilde{F})$ is the holonomy covering map and a local isometry.

The graph $G(F)$ is a 4-dimensional manifold with the induced foliation $\mathbb{F}$ of codimension 2. The generic leaf L of (M, F) has a trivial holonomy group and $L \cong \mathbb{R}^1$. Hence the generic leaf $\mathbb{L} \cong L \times L \cong \mathbb{R}^2$. The holonomy group of any other leaf L_α of (M, F) is isomorphic to $\mathbb{Z}$ and $L_\alpha \cong S^1$. In this case the leaf $\mathbb{L}_\alpha := p_1^{-1}(L_\alpha)$ is isometric to the Euclidean cylinder $\mathbb{R}^1 \times_{\mathbb{Z}} \mathbb{R}^1$.

Let $\tilde{k} : \tilde{G} \to G(F)$ be the universal covering map, in this case $\tilde{G}$ is diffeomorphic to $\mathbb{R}^4$ and it is provided by the induced foliation $\tilde{\mathbb{F}} := \{\{u\} \times \mathbb{R}^2 | u \in \mathbb{R}^2\}$, where $\tilde{G} = \tilde{G}(\tilde{F})$ is the graph of the foliation $(\tilde{M}, \tilde{F})$ and the following diagram

$$\begin{array}{ccccc} G(F) & \xleftarrow{\tilde{k}} & \tilde{G}(\tilde{F}) \cong \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\tilde{pr}} & \mathbb{R}^2 \\ \downarrow p_1 & & \downarrow \tilde{p}_1 & & \downarrow id \\ M & \xleftarrow{k} & \tilde{M} \cong \mathbb{R}^2 \times \mathbb{R}^1 & \xrightarrow{pr} & \mathbb{R}^2 \end{array}$$

is commutative, where $p_1 : G(F) \to M$ and $\tilde{p}_1 : G(\tilde{F}) \to \tilde{M}$ are the canonical projections, $\tilde{pr} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the first multiplier.

Denote by $f : \mathbb{R}^3 \to \mathbb{T}^2 \times \mathbb{R}^1$ the universal covering map. Let $y_0 := f(0_3)$ where 0_3 is zero in $\mathbb{R}^3$ and $x_0 := \varphi(y_0)$. Compute the fundamental group $\pi_1(M, x_0)$. The regular covering map $\varphi : \mathbb{T}^2 \times \mathbb{R}^1 \to M$ induces a group monomorphism $\widehat{\varphi} : \pi_1(\mathbb{T}^2 \times \mathbb{R}^1, y_0) \to \pi_1(M, x_0)$ onto a normal subgroup $N \cong \mathbb{Z}^2$ of $\pi_1(M, x_0)$, and, for fixed point y_0, the quotient group $\pi_1(M, x_0)/N$ is isomorphic to the deck transformation group $\widehat{G} \cong \mathbb{Z}$ with a generator $\Phi_A|_{\mathbb{T}^2 \times \mathbb{R}^1 \times \{1\}}$. Let us consider the leaf $L_0 = L_0(x_0)$ which is diffeomorphic to the circle. Observe that the inclusion $j : L_0 \to M$ induces a group monomorphism $\widehat{j} : \pi_1(L_0, x_0) \to \pi_1(M, x_0)$ onto $H := Im(\widehat{j}) \cong \mathbb{Z}$, and the deck transformation group induced by H is equal $\widehat{G}$. Therefore ([16], Proposition 1.3.1) the fundamental group $\pi_1(M, x_0)$ is the semi-direct product $H \ltimes N \cong \mathbb{Z} \ltimes \mathbb{Z}^2$.

Emphasize that foliations (M, F) and $(G(F), \mathbb{F})$ are both totally geodesic and Lorentzian, with the manifolds M and $G(F)$ are the Eilenberg–MacLane spaces of the type $K(H \ltimes N, 1)$, i.e. $\pi_n(M) = \pi_n(G(M)) = 0\ \forall n \geq 2,\ \pi_1(M) = \pi_1(G(F)) \cong H \ltimes N$.

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REFERENCES

1. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity* (Academic, New York, London, 1983).
2. B. Reinhart, "Foliated manifolds with bundle-like metrics," Ann. Math. **69**, 119–132 (1958).
3. P. Molino, *Riemannian Foliations,* Vol. 73 of *Progress in Mathematics* (Birkhauser, Boston, 1988).
4. A. D. Lewis, "Affine connections and distributions," Rep. Math. Phys. **42**, 135–164 (1998).
5. N. I. Zhukova and A. Y. Dolgonosova, "The automorphism groups of foliations with transverse linear connection," Cent. Eur. J. Math. **11**, 2076–2088 (2013).
6. H. E. Winkelnkemper, "The graph of a foliation," Ann. Glob. Anal. Geom. **1** (3), 51–75 (1983).
7. H. Wu, "On the de Rham decomposition theorem," Illinois J. Math. **8**, 291–311 (1964).
8. R. A. Blumenthal and J. J. Hebda, "Ehresmann connections for foliations," Indiana Univ. Math. J. **33**, 597–611 (1984).
9. N. I. Zhukova, "The graph of a foliation with Ehresmann connection and stability of leaves," Russ. Math. **38**, 76–79 (1994).
10. N. I. Zhukova, "Local and global stability of compact leaves and foliations," J. Math. Phys., Anal. Geom. **9**, 400–420 (2013).
11. N. I. Zhukova, "Singular foliations with Ehresmann connections and their holonomy groupoids," Banach Center Publ. **76**, 471–490 (2007).
12. K. Yokumoto, "Mutual exclusiveness along spacelike, timelike, and lightlike leaves in totally geodesic foliations of lightlike complete Lorentzian two-dimensional tori," Hokkaido Math. J. **31**, 643–663 (2000).
13. C. Boubel, P. Mounoud, and C. Tarquini, "Lorentzian foliations on 3-manifolds," Ergodic Theory Dynam. System **26**, 1339–1362 (2006).
14. E. Ghys, "Deformations de flots d'Anosov et de groupes fuchsiens," Ann. Inst. Fourier **42**, 209–247 (1992).
15. N. I. Zhukova and E. A. Rogozhina, "Classification of compact Lorentzian 2-orbifolds with non-compact full isometry groups," Sib. Math. J. **53**, 1037–1050 (2012).
16. D. Bump, Group Representation Theory. http://sporadic.stanford.edu/bump/group/. Accessed 2010.