

Gelfand–Zetlin Polytopes and Flag Varieties

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I construct a correspondence between the Schubert cycles on the variety of complete flags in \mathbb{C}^n and some faces of the Gelfand–Zetlin polytope associated with the irreducible representation of $SL_n(\mathbb{C})$ with a strictly dominant highest weight. The construction is motivated by the geometric presentation of Schubert cells using Demazure modules due to Bernstein–Gelfand–Gelfand [3]. The correspondence between the Schubert cycles and faces is then used to interpret the classical Chevalley formula in Schubert calculus in terms of the Gelfand–Zetlin polytopes. The whole picture resembles the picture for toric varieties and their polytopes.

1 Introduction

Let G be the group $SL_n(\mathbb{C})$ and $X = G/B$ the flag variety for G (here, $B \subset G$ denotes a Borel subgroup). The main goal of this paper is to translate to the flag variety some of the rich interplay that exists between geometry of toric varieties and combinatorics of convex polytopes. As in the case of toric varieties, there is a polytope P_H , namely *Gelfand–Zetlin polytope*, naturally associated with each very ample divisor H on X . For a toric variety, an analogous polytope associated with a divisor H gives information about torus orbits in the toric variety and their intersection products with H . For the flag variety, I will show how to extract similar information about *Schubert cycles* in X and their intersection products with H using the Gelfand–Zetlin polytope P_H . In particular, the classical

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Chevalley formula can be reformulated nicely in terms of Gelfand–Zetlin polytopes (see Theorem 1.1 and Theorem 5.5). Toric and flag varieties are the most studied examples of *spherical varieties*. A further objective would be to use the relation between the geometry of flag varieties and Gelfand–Zetlin polytopes to get new insights into geometry of more general spherical varieties as outlined in [10].

Recall that a *Schubert or Bruhat cell* is defined as an orbit of B in X under the left action, and *Schubert cycles* are the cycles in the Chow ring of X represented by the closures of Schubert cells. Schubert cycles provide a basis in the Chow ring of X , and the latter is isomorphic to the cohomology ring $H^*(X, \mathbb{Z})$ of X (see, e.g., [4, 1.3]). On the other hand, the cohomology ring of the flag variety is generated by the degree 2 classes (see, for instance [15, Theorem 3.6.15]). The group $H^2(X, \mathbb{Z})$ is isomorphic to the Picard group of X and can be identified with the weight lattice of G so that very ample divisors are identified with strictly dominant weights (see [4, 1.4.3]). Recall that the weight lattice of G is by definition the character lattice \mathbb{Z}^{n-1} of a maximal torus in G . A central formula in Schubert calculus is the Chevalley formula for the intersection product of a Schubert cycle with a divisor (see Subsection 2.3 for more details). The Chevalley formula was proved independently by Bernstein–Gelfand–Gelfand [3] and Demazure [7] and was already contained in a manuscript of Chevalley [6], which for many years remained unpublished. This formula allows to express Schubert cycles in terms of divisors, thus relating two different descriptions of the cohomology ring of the flag variety [3].

Fix the upper triangular Borel subgroup B^+ . Let λ be a strictly dominant (with respect to B^+) weight and H_λ the divisor corresponding to λ . We now assign to H_λ a convex polytope Q_λ . Recall that with each strictly dominant weight λ , one can associate the Gelfand–Zetlin polytope Q_λ (note that Zetlin is sometimes also transliterated as Cetlin or Tsetlin). This is a convex polytope in \mathbb{R}^d whose vertices lie in the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ (see Subsection 2.1 for the definition). Here, $d = n(n-1)/2$ denotes the dimension of X . Let T be the diagonal maximal torus. The integral points inside and at the boundary of Q_λ parameterize a natural basis of T -eigenvectors (introduced in [9]) in the irreducible representation V_λ of G with the highest weight λ .

I will assign to each Schubert cycle in X a face of the Gelfand–Zetlin polytope (see Section 3). My construction depends on a choice of a Borel subgroup B containing the maximal torus T (so in fact, I provide $n!$ different correspondences between Schubert cycles and faces). For each choice of B , we first construct a correspondence between B -orbits and faces and then use the one-to-one correspondence between Schubert cycles and B -orbits. The correspondence between B -orbits and faces preserves dimensions. The faces obtained for a given B correspond to *Demazure B -modules* in the representa-

tion space V_λ . The freedom in the choice of a Borel subgroup allowed by this construction is very useful. In many cases, it allows us to choose a face whose combinatorics captures geometry of a given Schubert cycle especially well (see Theorem 1.1 below). It might also lead to an interesting realization of Schubert calculus in terms of Gelfand–Zetlin polytopes (this is work in progress with Evgeny Smirnov and Vladlen Timorin). See Section 4 for an example of such calculus in the case $G = SL_3(\mathbb{C})$.

For a special choice of a Borel subgroup, namely for the lower triangular Borel subgroup B^- , my construction gives the correspondence between some of the Schubert cells and faces of the Gelfand–Zetlin polytope constructed by Kogan using the moment map $X \rightarrow Q_\lambda$ [13] (see Section 3 for more details). In [14], Kogan and Miller extended this correspondence to all Schubert cycles: they assigned to each Schubert cycle a union of faces using Caldero’s toric degenerations of flag varieties [5]. Both approaches (with moment map and toric degenerations) work with B^- -orbits, that is, there is only one way to assign a face or a union of faces to a given Schubert cycle.

For some of the faces of the Gelfand–Zetlin polytope that correspond to the Schubert cycles, the Chevalley formula for the intersection product of a Schubert cycle with the divisor H_λ admits the following interpretation in terms of the respective face (cf. Theorem 3.3). We fix a Borel subgroup B containing T and, hence, fix a correspondence between Schubert cycles and faces. Denote by \mathcal{O}_Γ the B -orbit corresponding to a face Γ and by Z_Γ the Schubert cycle defined by \mathcal{O}_Γ . In what follows, we only consider those faces that do correspond to Schubert cycles. We say that a face Γ is *admissible* if for each codimension one orbit \mathcal{O}_Δ in the closure of the orbit \mathcal{O}_Γ , the face Γ contains the face Δ . In other words, the *Bruhat order* on Schubert cycles agrees with the natural order on faces given by inclusion.

Theorem 1.1. For any admissible face Γ we have

$$H_\lambda Z_\Gamma = \sum_{\Delta \subset \Gamma} d(v, \Delta) Z_\Delta,$$

where the sum is taken over the facets Δ of Γ (that correspond to the Schubert cells \mathcal{O}_Δ of codimension 1 at the boundary of \mathcal{O}_Γ). Here, v is a fixed vertex of the face Γ and $d(v, \Delta)$ denotes the integral distance from v to the face Δ (see Section 2.2 for the definition). □

Note that in this form, the Chevalley formula is completely analogous to the well-known formula for toric varieties (e.g., see [8, Section 3.4] or [10]). There is a generalization of Theorem 1.1 that holds for all faces (see Theorem 5.5).

Many Schubert cycles can be represented by an admissible face for different choices of B but not all of them, for example, for $G = SL_3$, all Schubert cycles can be represented by admissible faces (and all Schubert cycles are smooth). For $G = SL_4$, there are two non-smooth Schubert cycles, and none of them can be represented by an admissible face. There might be some relation between smoothness of Schubert cycles and their representability by admissible faces. Note that if we only take B^- (as in [13, 14]), then already for SL_3 there will be a Schubert cycle such that the corresponding face is not admissible (see Remark 4.1).

It might be possible to extend the correspondence between Schubert cycles and faces constructed in this paper to the complete flag varieties for other reductive groups by replacing the Gelfand–Zetlin polytope with appropriate *string polytopes* (see [1, Definition 1.2] for a definition of string polytopes).

This paper is organized as follows. In Section 2, we recall the definition of the Gelfand–Zetlin polytope and the notion of integral distance. We also state the classical Chevalley formula. Section 3 contains the main results: the construction of correspondences between faces of the Gelfand–Zetlin polytope and Schubert cycles and the Chevalley formula in terms of the Gelfand–Zetlin polytope (Theorem 3.3). In Section 4, we consider in detail the example $G = SL_3$. In Section 5, we study combinatorics and geometry of the Gelfand–Zetlin polytope and prove Theorem 3.3. We also formulate and prove an extension of Theorem 3.3 to non-admissible faces (Theorem 5.5).

2 Gelfand–Zetlin Polytopes and Chevalley Formula

In this section, we recall the definition of the Gelfand–Zetlin polytope and the Chevalley formula for the intersection product of a Schubert cycle with a divisor. We also discuss the notion of integral distance.

2.1 Gelfand–Zetlin polytope

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a strictly increasing collection of n integer numbers. To each such collection, we assign the irreducible representation

$$\pi_\lambda : G \rightarrow GL(V_\lambda)$$

with the strictly dominant highest weight $(\lambda_2 - \lambda_1)\omega_1 + \dots + (\lambda_n - \lambda_{n-1})\omega_{n-1}$ (which will also be denoted by λ), where $\omega_1, \dots, \omega_{n-1}$ are the fundamental weights of G . To define the fundamental weights, we fix the diagonal maximal torus T and the upper triangular Borel subgroup B^+ . The Gelfand–Zetlin polytope Q_λ associated with λ is a convex polytope in \mathbb{R}^d (recall that $d = n(n-1)/2$) defined by the inequalities

$$\begin{array}{ccccccc}
 \lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\
 & x_{1,1} & & x_{1,2} & & \dots & & x_{1,n-1} & \\
 & & x_{2,1} & & \dots & & x_{2,n-2} & & \\
 & & & \ddots & & \dots & & & \\
 & & & & x_{n-2,1} & & x_{n-2,2} & & \\
 & & & & & & & x_{n-1,1} &
 \end{array}$$

where $(x_{1,1}, \dots, x_{1,n-1}; x_{2,1}, \dots, x_{2,n-2}; \dots; x_{n-2,1}, x_{n-2,2}; x_{n-1,1})$ are coordinates in \mathbb{R}^d and the notation

$$\begin{array}{cc}
 a & b \\
 & c
 \end{array}$$

means $a \leq c \leq b$. See Figure 1 for a picture of the Gelfand–Zetlin polytope for $G = SL_3$.

There is a T -eigenbasis in V_λ such that its vectors are in one-to-one correspondence with the integral points inside Q_λ (see for instance [14, Section 5] for the description of this basis). We will denote by the same letter v an integral point in Q_λ and the corresponding basis vector in V_λ . There is a natural map p that assigns to each integral point v the weight of the corresponding basis vector $v \in V_\lambda$. Let us extend this map by linearity to the map $p: \mathbb{R}^d \rightarrow \mathbb{R}^{n-1}$. Denote by $P_\lambda \subset \mathbb{R}^{n-1}$ the weight polytope of the representation V_λ . The map p sends the Gelfand–Zetlin polytope Q_λ to the weight polytope P_λ and can be written in coordinates as follows [13, 2.1.2]. Let $\alpha_1, \dots, \alpha_{n-1}$ be the simple roots of G (so they form a basis in \mathbb{R}^{n-1} dual with respect to the Cartan–Killing form to the basis of the fundamental weights $\omega_1, \dots, \omega_{n-1}$). Then we have

$$p: (x_{ij}) \rightarrow \left(\sum_{i=1}^{n-1} x_{1,i} \right) \alpha_1 + \left(\sum_{i=1}^{n-2} x_{2,i} \right) \alpha_2 + \dots + (x_{n-2,1} + x_{n-2,2}) \alpha_{n-2} + x_{n-1,1} \alpha_{n-1} + \text{constant vector}.$$

Remark 2.1. Note that for any two strictly dominant weights λ and μ , the corresponding Gelfand–Zetlin polytopes Q_λ and Q_μ are *analogous*, that is, have the same normal fan. In particular, there is a bijective correspondence between their faces. This is similar to the toric case, where polytopes corresponding to any two very ample divisors are analogous. \square

2.2 Integral distance

Below, we recall the notion of *integral distance*. Consider the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ in the affine space \mathbb{R}^d . Let H be a hyperplane spanned by lattice vectors and $v \in \mathbb{Z}^d$ an integral point. Then the *integral distance* $d(v, H)$ from v to the hyperplane H is the index in \mathbb{Z}^d of the subgroup spanned by the vectors $v - u$ for all $u \in H$. By definition, the integral distance is invariant under unimodular linear transformations of \mathbb{R}^d . To compute the integral distance, we first find a primitive integral equation $f(x) = 0$ defining H , that is, $f(x) = a_0 + a_1 x_1 + \dots + a_d x_d$ where $a_i \in \mathbb{Z}$ and the greatest common divisor of a_0, \dots, a_d is 1. It is then easy to check that the integral distance between v and H is equal to the absolute value of $f(v)$. In particular, if H is parallel to a coordinate hyperplane, then the integral distance coincides with the Euclidean distance.

Example 2.2. In what follows, we will be interested in the case where H is a hyperplane containing a facet of the Gelfand–Zetlin polytope Q_λ . A primitive integral equation for H is then either $x_{i,j} - x_{i-1,j} = 0$ or $x_{i,j} - x_{i-1,j+1} = 0$ for some $i = 1, \dots, n-1, j = 1, \dots, n-i$ (we put $x_{0,j} = \lambda_j$). Hence, the integral distance from a point $v = (v_{i,j})$ to H is equal to $v_{i,j} - v_{i-1,j}$ or to $v_{i-1,j+1} - v_{i,j}$, respectively. \square

In the sequel, we will use the notion of integral distance in the following setting. Let P be a convex lattice polytope of dimension d in \mathbb{R}^d . Recall that a vertex u of P is called *simple* if exactly d facets intersect in u (or equivalently, exactly d edges meet at u). In other words, in the neighborhood of u , the polytope P looks like a d -dimensional simplex. Let $\Gamma \subset P$ be a face of P , and $\Delta \subset \Gamma$ a facet of Γ that contains at least one simple vertex of P . This ensures that there is a unique hyperplane H such that $H \cap P$ is a facet of P and $H \cap \Gamma = \Delta$. For any integral point $v \in \Gamma$, we can now define the integral distance $d(v, \Delta)$ as the integral distance from v to the hyperplane H . Such distances arise naturally in toric geometry when one computes products of toric orbits with divisors.

2.3 Bruhat order and Chevalley formula

Fix a strictly dominant weight λ . Recall that V_λ denotes the irreducible representation with the highest weight λ . We assume that $X = G/B$ is embedded into the projective space $\mathbb{P}(V_\lambda)$ as the G -orbit of the line spanned by a highest weight vector $v \in V_\lambda$. Denote by H_λ the divisor of hyperplane section on X (this is one of the equivalent ways to identify strictly dominant weights with very ample divisors [4, 1.4]). For each Schubert cell \mathcal{O} in X , the Chevalley formula computes explicitly the intersection of H_λ with the closure of \mathcal{O} as a linear combination of the closures of Schubert cells at the boundary of \mathcal{O} . We will now state this formula.

First, recall that the choice of a Borel subgroup B in G defines a one-to-one correspondence between the Schubert cells in X and elements of the Weyl group W of G . We identify the Weyl group with $N(T)/T$, where $N(T)$ is the normalizer of T in G . Then the Schubert cell \mathcal{O}_w is the B -orbit of the line spanned by $wv \in V_\lambda$. Note that the length $l(w)$ (defined as the minimal number of simple reflections in a decomposition of w) is equal to the dimension of \mathcal{O}_w . Recall that there is a natural partial order on Schubert cells called *Bruhat order*. We say that $\mathcal{O}_{w'}$ *precedes* \mathcal{O}_w with respect to the Bruhat order if $\mathcal{O}_{w'}$ is contained in the closure of \mathcal{O}_w and $\dim \mathcal{O}_{w'} = \dim \mathcal{O}_w - 1$. In other words, $\overline{\mathcal{O}_{w'}}$ is a boundary divisor in $\overline{\mathcal{O}_w}$. The Bruhat order can also be defined in terms of the Weyl group as follows. Denote by s_α the reflection in the hyperplane perpendicular to a root α . Then $\mathcal{O}_{w'}$ precedes \mathcal{O}_w if and only if $w' = ws_\alpha$ for some root α and $l(w') = l(w) - 1$ (see, e.g., [3, Theorem 2.11] or [15, Proposition 3.6.4]).

For each root α , define the linear function (\cdot, α) (that is, the *coroot*) on the weight lattice of G by the property $s_\alpha \lambda = \lambda - (\lambda, \alpha)\alpha$ for all weights λ (the pairing (a, b) is often denoted by $\langle a, b^\vee \rangle$ or by $\langle a, b \rangle$). Denote by Z_w the Schubert cycle represented by the closure of the orbit \mathcal{O}_w . The following result is proved in [3, Proposition 4.1] and [7, Proposition 4.4]:

$$H_\lambda Z_w = \sum_{\alpha} (\lambda, \alpha) Z_{ws_\alpha},$$

where the sum is taken over all positive roots α of G such that $l(ws_\alpha) = l(w) - 1$. In particular, the coefficients (λ, α) are always nonnegative.

One of our goals is to interpret this formula in terms of the Gelfand–Zetlin polytope Q_λ . In what follows we will use the following equivalent formulation:

$$H_\lambda Z_w = \sum_{\alpha} (w\lambda, \alpha) Z_{s_\alpha w}, \tag{2.1}$$

where the sum is taken over all roots α such that $w^{-1}\alpha$ is positive and $l(s_\alpha w) = l(w) - 1$.

3 Correspondence between the Schubert Cells and the Faces of the Gelfand–Zetlin Polytope

In this section, we will construct a correspondence between Schubert cycles and some of the faces of the Gelfand–Zetlin polytope Q_λ (by Remark 2.1, it does not matter which weight λ we choose).

3.1 Schubert cells

Fix once and for all the diagonal maximal torus $T \subset G$. Everything below (weight vectors, Borel subalgebras, etc.) are assumed to be compatible with T . As before, we assume that X is embedded into the projectivization $\mathbb{P}(V_\lambda)$ of the irreducible representation V_λ as the G -orbit of the line spanned by a highest weight vector.

We will use the following description of the Schubert cells from [3]. Let $v \in V_\lambda$ be a non-zero weight vector with an *extremal weight* (recall that a weight is extremal if it is of the form $w\lambda$ for some element w in the Weyl group of G). Extremal weights are exactly the vertices of the weight polytope of V_λ , and their weight spaces are always one dimensional. In what follows, we will not distinguish between non-zero proportional vectors with the same extremal weight. Let B be a Borel subgroup in G containing T and \mathfrak{b} its Lie algebra. Note that all such Borel subgroups lie in the same orbit under the action of the Weyl group W , and there are exactly $|W|$ of them. Denote by $U(\mathfrak{b})$ the universal enveloping algebra of \mathfrak{b} . Then the pair (v, B) defines the Schubert cell $\mathcal{O}(v, B)$, which is the B -orbit of v and the closure of this cell in the flag variety can be realized as follows [3, Lemma 2.12]:

$$\overline{\mathcal{O}(v, B)} = X \cap \mathbb{P}(U(\mathfrak{b})v).$$

Note that $U(\mathfrak{b})v$ is a B -invariant vector subspace in V_λ (called *Demazure module*). It would be natural to assign to the cell $\mathcal{O}(v, n)$ a face of the Gelfand–Zetlin polytope Q_λ by taking the convex hull of all basis vectors in the Gelfand–Zetlin basis that lie in the subspace $U(\mathfrak{b})v$ (we identify the basis vectors in the Gelfand–Zetlin basis with the integral points in Q_λ). Unfortunately, it might happen that the convex hull is not a face or has wrong dimension. However, this approach still works after some modification (see Subsection 3.2).

Two Schubert cells $\mathcal{O}(v, B)$ and $\mathcal{O}(v', B')$ are conjugate by the action of the Weyl group (and hence represent the same cohomology class) if and only if $B' = wBw^{-1}$ and v' is proportional to wv for some element $w \in W$. If we fix a Borel subgroup B , then the Schubert cells $\mathcal{O}(wv, B)$ for all $w \in W$ give the full set of B -orbits in the flag variety. In particular,

$$X = \sqcup_{w \in W} \mathcal{O}(wv, B).$$

For all possible choices of v and B , we get $|W|^2$ Schubert cells forming $|W|$ orbits under the action of the Weyl group (so they can be identified with $W \times W$ where $w \in W$ acts by sending (w_1, w_2) to (ww_1, ww_2)). Note that there is no canonical identification (i.e., independent of the choice of a Borel subgroup containing the torus T) between the cohomology classes of the cells $\mathcal{O}(v, B)$ and the elements of the Weyl group. By different choices of a Borel subgroup, we assign to each cohomology class $[\overline{\mathcal{O}(v, B)}]$ different elements of the Weyl group that are conjugate to each other. Since we use simultaneously the cells $\mathcal{O}(v, B)$ for different choices of B , we will not identify the Schubert cells with elements of the Weyl group.

3.2 Faces of the Gelfand–Zetlin polytope.

To each cell $\mathcal{O}(v, B)$ of dimension l , we now assign an l -dimensional face of the Gelfand–Zetlin polytope Q_λ . Recall that to each extremal vector v , there corresponds a unique vertex of the Gelfand–Zetlin polytope, which we also denote by v . The vertex v is a unique preimage of a vertex of the weight polytope P_λ under the map $p: Q_\lambda \rightarrow P_\lambda$. It is easy to show that the vertex v is simple [13, 2.2.2]. The edges coming out of v are in one-to-one correspondence with the roots α of G such that the root space \mathfrak{g}_α does not annihilate the extremal weight vector v [13, 2.2.3]. For each such root α denote by $e(v, \alpha)$ the edge corresponding to α . The edge $e(v, \alpha)$ is uniquely defined by the property that its projection under the map $p: Q_\lambda \rightarrow P_\lambda$ is parallel to the root α (see Section 5 for more details on simple vertices v and edges $e(v, \alpha)$).

Denote by $R(v, B)$ the set of roots such that the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is contained in \mathfrak{b} and does not annihilate v . The cardinality of $R(v, B)$ is equal to the dimension of the cell $\mathcal{O}(v, B)$ [3, Lemma 2.2]. Let $\{\beta_1, \dots, \beta_l\}$ be all roots in the set $R(v, B)$. Assign to the Schubert cell $\mathcal{O}(v, B)$ the l -dimensional face $\Gamma(v, B)$ of the Gelfand–Zetlin polytope spanned by the edges $e(v, \beta_1), \dots, e(v, \beta_l)$. There is a unique such face since the vertex v is simple. This face can be thought of as a lifting of the Demazure module $U(\mathfrak{b})v$ to the Gelfand–Zetlin polytope.

Remark 3.1. There is an alternative description of the face $\Gamma(v, B)$ using the *Morse theory* on polytopes (such analog of the Morse theory was introduced in [12]). Namely, choose a linear function f_B on \mathbb{R}^{n-1} that takes positive values on all roots α whose root spaces \mathfrak{g}_α are contained in \mathfrak{b} . Then the composition $f_B \circ p$ is a linear function on the Gelfand–Zetlin polytope. The face $\Gamma(v, B)$ is precisely the *upper separatrix face* for the function $f_B \circ p$ at the vertex v . The *upper separatrix face* is by definition the face spanned by all edges at v going upward with respect to the function $f_B \circ p$ (that is, $f_B \circ p$ increases along these edges).

Note that the Bruhat cells $\mathcal{O}(v, B)$ can be defined in an analogous way. Namely, there is a Morse function on X given by the composition of the moment map $X \rightarrow P_\lambda$ with the same function f_B , and the cells $\mathcal{O}(v, B)$ are the upper separatrix manifolds for this Morse function (see [2, Section 4]). \square

We now compare the Bruhat order on the cells $\mathcal{O}(v, B)$ with the inclusion order on the faces $\Gamma(v, B)$. It is easy to see that if $\Gamma(u, B)$ is a facet in $\Gamma(v, B)$, then $\mathcal{O}(u, B)$ precedes $\mathcal{O}(v, B)$ with respect to the Bruhat order. The converse is wrong, that is, it happens already for $G = SL_3$ that $\mathcal{O}(u, B)$ lies at the boundary of $\mathcal{O}(v, B)$ in the flag variety, but the face $\Gamma(u, B)$ does not belong to the face $\Gamma(v, B)$ (see Section 4). We say that the face $\Gamma(v, B)$ is *admissible* if it contains all faces $\Gamma(u, B)$ such that the Schubert cell $\mathcal{O}(u, B)$ precedes the Schubert cell $\mathcal{O}(v, B)$ with respect to the Bruhat order.

Denote by B^- the lower triangular Borel subgroup, that is, B^- is opposite to the Borel subgroup used to construct the Gelfand–Zetlin polytope. If we fix $B = B^-$ and only vary v , then my correspondence between Schubert cycles and faces reduces to the correspondence defined in [13] (see Remark 5.1). Note that the collection of faces assigned to $\mathcal{O}(v, B^-)$ in [14, Section 4] always contains $\Gamma(v, B^-)$. In particular, if this collection consists of just one face (that is, of $\Gamma(v, B^-)$), then the corresponding Schubert cycle is a Kempf variety [13, Proposition 2.3.2], which is a very restrictive condition (see [13, Proposition 2.2.1] for a characterization of such Schubert cycles). Equivalently, the corresponding Schubert polynomial consists of a single monomial. In particular, it is easy to check that in this case, $\Gamma(v, B^-)$ must be admissible. An advantage of my construction is that the freedom in the choice of B allows us to represent more general Schubert cycles by a single admissible face of the Gelfand–Zetlin polytope (see Remark 4.1).

An interesting problem is to describe all admissible faces as well as the corresponding Schubert cycles. It is not true that for each Schubert cycle, there exists a representative $\mathcal{O}(v, B)$ such that the face $\Gamma(v, B)$ is admissible. There is a counterexample for the flag variety X_4 of $SL_4(\mathbb{C})$. Namely, if the closure of a Schubert cell \mathcal{O} in X_4

is not smooth, then none of the faces corresponding to the cohomology class of $\overline{\mathcal{O}}$ is admissible (there are two such Schubert cells in X_4). There is also one smooth Schubert cycle in $SL_4(\mathbb{C})$ which cannot be represented by an admissible face. For the flag variety X_3 of $SL_3(\mathbb{C})$, the closure of each Schubert cell is smooth, and every Schubert cycle can be represented by an admissible face (see Section 4).

We now state the Chevalley formula in terms of the Gelfand–Zetlin polytope. For a weight vector u , denote by $p(u)$ the weight of u .

Proposition 3.2. If $\Gamma(u, B)$ is a facet of $\Gamma(v, B)$ (in particular, $p(v) = s_\alpha p(u)$ for some root α), then

$$|(p(v), \alpha)| = d(v, \Gamma(u, B)),$$

where $d(v, \Gamma(u, B))$ is the integral distance from v to the face $\Gamma(u, B)$ as defined in Section 2.2. □

This proposition will be proved in Section 5. If we apply it to formula (2.1), we immediately get the following Chevalley formula for the admissible faces.

Theorem 3.3. If the face $\Gamma(v, B)$ is admissible, then the Chevalley formula for the Schubert variety $\overline{\mathcal{O}(v, B)}$ and the divisor H_λ can be written as

$$H_\lambda \overline{\mathcal{O}(v, B)} = \sum d(v, \Gamma(u, B)) \overline{\mathcal{O}(u, B)},$$

where the sum is taken over all Schubert cells $\mathcal{O}(u, B)$ that precede $\mathcal{O}(v, B)$. □

4 Example: Flag Variety for $SL_3(\mathbb{C})$

Figure 1 shows the Gelfand–Zetlin polytope Q_λ for the irreducible representation of $SL_3(\mathbb{C})$ with the highest weight $\lambda = a\omega_1 + b\omega_2$. This is a polytope in \mathbb{R}^3 (with coordinates x , y , and z) defined by the following six inequalities:

$$0 \leq x \leq a; \quad a \leq y \leq b; \quad x \leq z \leq y.$$

The weight polytope P_λ is a hexagon in \mathbb{R}^2 . The polytope Q_λ has six simple vertices which are mapped bijectively to the vertices of the weight polytope P_λ under the map p . This

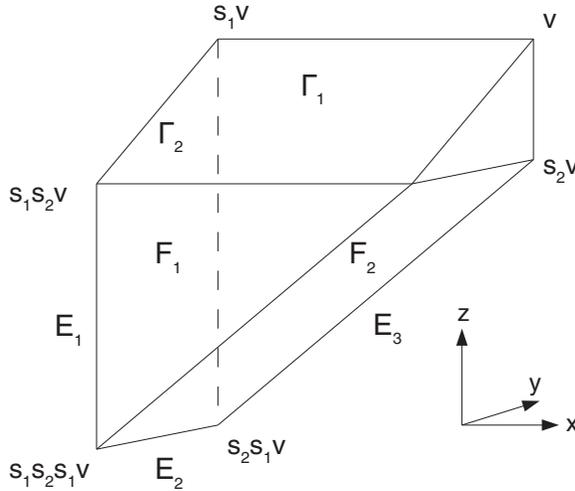


Fig. 1. The Gelfand–Zetlin polytope Q_λ for the irreducible representation of $SL_3(\mathbb{C})$.

bijection is used to label the simple vertices of Q_λ . Namely, we label by v the vertex that goes to the highest weight λ . A simple vertex u is then labeled by wv if $p(u) = wp(v)$ for some element w from the Weyl group. Put $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$. We denote by $[u_1, u_2]$ the edge of the Gelfand–Zetlin polytope connecting vertices u_1 and u_2 .

All faces of Q_λ except for a unique non-simple vertex can be represented as $\Gamma(v, B)$ for some choice of a simple vertex v and a Borel subgroup B , for example, if $B = B^+$, then $\Gamma(v, B^+) = v$, $\Gamma(s_1 v, B^+)$ is the edge $[s_1 v, v]$, and $\Gamma(s_2 s_1 v, B^+)$ is the face $\{y = b\} \cap Q_\lambda$. If $B = B^-$, then $\Gamma(v, B^-) = Q_\lambda$, $\Gamma(s_1 v, B^-)$ is the face $\{x = 0\} \cap Q_\lambda$, and $\Gamma(s_2 s_1 v, B^-)$ is the edge $[s_2 s_1 v, s_1 s_2 s_1 v]$.

All faces of Q_λ that do not contain the non-simple vertex are admissible. In particular, there are two two-dimensional admissible faces $\Gamma_1 = \Gamma(s_2 s_1 v, B^+)$ and $\Gamma_2 = \Gamma(s_1 v, B^-)$ corresponding to the cells $\mathcal{O}(s_2 s_1 v, B^+)$ and $\mathcal{O}(s_1 v, B^-)$. It is easy to check that these two cells represent different Schubert cycles. Denote these cycles by Z_{21} and Z_{12} , respectively (that is, we label the cohomology class of $\mathcal{O}(wv, B^+)$ by Z_w and encode $w = s_1 s_2$ by 12, etc). There are also six admissible edges that connect simple vertices of Q_λ . These correspond to two Schubert cycles of dimension 1. Namely, the edges $[v, s_1 v]$, $[s_1 s_2 v, s_1 s_2 s_1 v]$, and $[s_2 s_1 v, s_2 v]$ correspond to Z_1 and the other three edges correspond to Z_2 . Then Theorem 3.3 together with Example 2.2 applied to the two-dimensional admissible faces tells that

$$H_\lambda Z_{12} = bZ_1 + (a + b)Z_2; \quad H_\lambda Z_{21} = (a + b)Z_1 + aZ_2.$$

Remark 4.1. Note that if we only considered faces $\Gamma(u, B^-)$ for the lower triangular Borel subgroup B^- (that is, proceeded as in [13, 14]), then we would not be able to represent the Schubert cycle Z_{21} by a single admissible face. Instead, we would get the union of two faces: the rectangular one $\{x = z\}$ and the triangular one $\{y = a\}$. The union of these two faces looks like the admissible face Γ_1 (corresponding to Z_{21} by my construction) broken into two pieces.

This example extends to arbitrary n . Namely, the Schubert divisor with the Schubert polynomial $x_1 + \dots + x_{n-1}$ corresponds to the union of $n - 1$ facets of type $\Gamma(v, B^-)$ according to [14]. On the other hand, it can be represented by a single admissible face of type $\Gamma(v, B^+)$. \square

We now describe heuristic Schubert calculus on the faces of Q_λ . We can represent Schubert cycle Z_{21} by faces in two different ways: as Γ_1 and as $F_1 + F_2$, where F_1 and F_2 denote the faces given by the equations $y = a$ and $x = z$, respectively. The latter representation comes from [14]. We also represent Z_{12} by Γ_2 . Finally, we represent the one-dimensional Schubert cycle Z_1 in two ways, by the edge $E_1 = [s_1 s_2 v, s_1 s_2 s_1 v]$ and the edge $E_3 = [s_2 v, s_2 s_1 v]$, and represent Z_2 by the edge $E_2 = [s_2 s_1 v, s_1 s_2 s_1 v]$ (see Figure 1). We can now compute $Z_{21} Z_{12}$ and Z_{12}^2 by intersecting the corresponding faces:

$$(F_1 + F_2) \cap \Gamma_2 = E_1 + E_2,$$

which is exactly the identity $Z_{21} Z_{12} = Z_1 + Z_2$. Similarly,

$$(F_1 + F_2) \cap \Gamma_1 = E_3$$

gives the identity $Z_{21}^2 = Z_1$. We can also get the identities $Z_1 Z_{12} = Z_2 Z_{21} = [pt]$ and $Z_1 Z_{21} = Z_2 Z_{12} = 0$ by choosing the edges representing Z_1 and Z_2 so that they have transverse intersection with Γ_1 or Γ_2 , for example, to find $Z_1 Z_{12}$, we represent Z_1 by E_3 and Z_{12} by Γ_2 and get that $\Gamma_2 \cap E_3 = pt$. Similarly, to find $Z_1 Z_{21}$, we represent Z_1 by E_1 and Z_{21} by Γ_1 , which yields $\Gamma_1 \cap E_1 = \emptyset$.

An analogous Schubert calculus on the Gelfand–Zetlin polytope can be done for arbitrary n [11]. It can be rigorously justified using the concept of the *polytope ring* associated with the *volume polynomial* of the Gelfand–Zetlin polytope. The elements of the polytope ring can be naturally identified with linear combinations of faces.

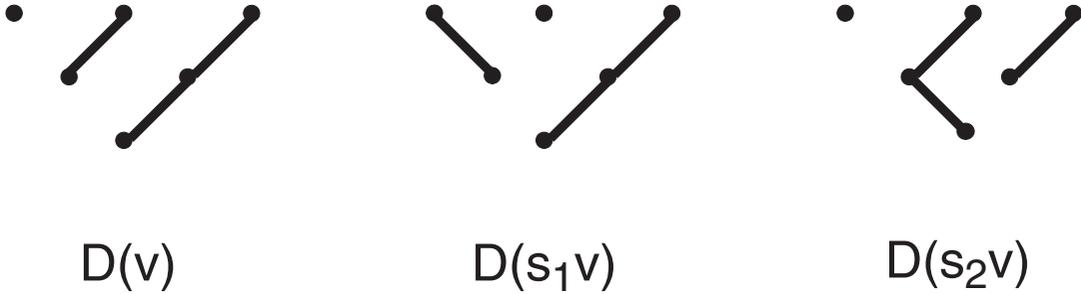


Fig. 2. The diagrams for the vertices v , s_1v , and s_2v of the Gelfand–Zetlin polytope for SL_3 .

5 Geometry and Combinatorics of the Gelfand–Zetlin Polytope

To prove Proposition 3.2, we have to study the faces of the Gelfand–Zetlin polytope Q_λ . First, we describe explicitly the simple vertices of Q_λ and the edges going out of simple vertices. We mostly follow [13]. Brief explanations are provided for the reader’s convenience, for more details see [13, 2.1–2.3]. Next, we will find out under which conditions two simple vertices are connected by the edge (see Lemma 5.2). Finally, we prove Proposition 3.2 and formulate and prove a Chevalley formula for arbitrary faces $\Gamma(v, B)$ (see Theorem 5.5).

We describe the faces of Q_λ by triangular *diagrams* following [13]. Put $x_{0,i} := \lambda_i$ for $i = 1, \dots, n$. It is easy to see that each face of Q_λ is defined by the equations of the form $x_{i,j} = x_{i-1,j}$ or $x_{i,j} = x_{i-1,j+1}$ for some $i = 1, \dots, n-1, j = 1, \dots, n-i$. For a face Γ , encode all the equations defining Γ by the following graph $D(\Gamma)$. Draw n rows indexed by $1, \dots, n$ with $n-i+1$ points $p_{i,1}, \dots, p_{i,n-i+1}$ in the i th row. These are the vertices of the graph $D(\Gamma)$ (each vertex $p_{i,j}$ corresponds to the coordinate $x_{i-1,j}$). For each equality $x_{i,j} = x_{i-1,j}$ and $x_{i,j} = x_{i-1,j+1}$ defining the face Γ , we draw the edge $e_{i+1,j}^L$ of *type L* between the vertices $p_{i+1,j}$ and $p_{i,j}$ and the edge $e_{i+1,j}^R$ of *type R* between $p_{i+1,j}$ and $p_{i,j+1}$, respectively. The resulting graph is the *diagram* of the face Γ . Figure 2 shows the diagrams for the vertices v, s_1v , and s_2v of the Gelfand–Zetlin polytope for SL_3 considered in Section 4.

5.1 Simple vertices

It is easy to show that v is a simple vertex of the Gelfand–Zetlin polytope if and only if the corresponding diagram $D(v)$ has exactly $n-i$ edges starting at the i th row and ending at the $(i+1)$ -st row (for all positive $i < n$) and two such edges never start or end at the same

point. In other words, the graph $D(v)$ is the disjoint union of n simple trees $T_1(v), \dots, T_n(v)$. Each tree $T_i(v)$ starts at the first row of $D(v)$ and ends at the i th row (that is, each T_i looks like the Dynkin diagram A_i). The vertex of $T_i(v)$ in the first row will be called the *starting point* of $T_i(v)$. Note that the coordinates $x_{i,j}$ and $x_{k,l}$ of the vertex v are equal if and only if the vertices $p_{i+1,j}$ and $p_{k+1,l}$ belong to the same tree. The diagram $D(v)$ can also be thought of as an *RC-graph* or a *pipe dream* (see [13, 14] for details on connection between pipe dreams and faces of the Gelfand–Zetlin polytope). Let us call the diagram of a simple vertex also *simple*. There is a different way to characterize simple diagrams (see [13, 2.2.2]). Namely, the diagram $D(v)$ is simple if for all $i = 2, \dots, n$ exactly $n - i + 1$ edges end at the i th row of $D(v)$, and the edges $e_{i,j}^L$ are strictly to the left of the edges $e_{i,j}^R$.

Each simple diagram $D(v)$ defines a permutation σ_v of elements $1, \dots, n$ as follows: the vertex $p_{1,i}$ is the starting point of the tree $T_{\sigma_v(i)}$. It is easy to check that this gives a bijective correspondence between simple vertices of Q_λ and elements of the symmetric group S_n , which is isomorphic to the Weyl group of G (we choose the isomorphism which sends the elementary transposition $(i \ i + 1)$ to the simple reflection s_{α_i}). This bijection is compatible with the bijection between the vertices of the weight polytope P_λ and elements of the Weyl group, that is, $p(v) = \sigma_v \lambda$. Indeed, using the formula for the projection $p: Q_\lambda \rightarrow P_\lambda$ from Section 2.1, we get that if $p(u) = s_{\alpha_i} p(v)$ (and thus $p(u) = p(v) - (p(v), \alpha_i) \alpha_i$), then the sums of coordinates $\sum_{k=1}^{n-j} x_{j,k}$ for the vertices v and u only differ for $j = i$. This is only possible if the trees $T_j(v)$ and $T_j(u)$ have the same starting points for all $j \neq i, (i + 1)$, which implies $\sigma_v = s_{\alpha_i} \sigma_u$.

5.2 Edges

We now describe the edges of the Gelfand–Zetlin polytope. Let u and v be two vertices of the Gelfand–Zetlin polytope. We say that the diagram $D(u)$ is obtained from the diagram $D(v)$ by *switching* the edge $e_{i,j}^L$, if the diagrams have the same set of edges with one exception: instead of the edge $e_{i,j}^L$, the diagram $D(v)$ has the edge $e_{i,j}^R$. Switching of $e_{i,j}^R$ is defined in the same way, for example, the diagrams $D(s_1 v)$ and $D(s_2 v)$ on Figure 2 are obtained from the diagram $D(v)$ by switching the edges $e_{2,1}^R$ and $e_{3,1}^R$, respectively. It is easy to see that two vertices u and v are connected by an edge of the Gelfand–Zetlin polytope if and only if their diagrams can be obtained from each other by switching the edge $e_{i,j}^L$ or $e_{i,j}^R$ for some i and j . If $D(u)$ is obtained from $D(v)$ by switching the edge $e_{i,j}^L$, then the diagram $D([u, v])$ of the edge of the Gelfand–Zetlin polytope connecting u and v is obtained from $D(v)$ by deleting the edge $e_{i,j}^L$.

We now focus on the edges going out of a given simple vertex v . Their diagrams are obtained by deleting one of the edges of the diagram $D(v)$. Denote by e the i th edge

of the tree T_j for $i = 1, \dots, j - 1$ (that is the edge of the tree $T_j(v)$ starting at the i th row of the diagram $D(v)$ and ending at the $(i + 1)$ -st row). Recall that we denoted by $e(v, \alpha)$ the edge of the Gelfand–Zetlin polytope whose projection $p(e(v, \alpha))$ is parallel to the root α . It is easy to check using again the formula for the projection $p : Q_\lambda \rightarrow P_\lambda$ (see Section 2.1) that if we delete the edge e from the diagram $D(v)$, we get the diagram of the edge $e(v, \alpha)$, where $\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$ if e is of type L and $\alpha = -\alpha_i - \alpha_{i+1} - \dots - \alpha_{j-1}$ if e is of type R . Indeed, let $p_{i,s}$ and $p_{i+1,s}$ be the vertices of the edge e . Then switching e only changes coordinates of v corresponding to the vertices of the tree $T_j(v)$ lying strictly below $p_{i,s}$. These coordinates increase by the same number $x_{i-1,s+1}(v) - x_{i-1,s}(v)$. Hence, the sums of coordinates $\sum_{k=1}^{n-r} x_{r,k}$ increase by the same number for $r = i, \dots, j - 1$ and stay the same for all other r . In particular, for each simple root α_i , the diagram of the edge $e(v, \pm\alpha_i)$ is obtained from $D(v)$ by deleting the lowest edge (that is, the i th edge) of the tree $T_{i+1}(v)$, and the sign in $\pm\alpha$ is determined by the slope of the lowest edge. Thus, we get an explicit one-to-one correspondence between the edges $e(v, \alpha)$ of the Gelfand–Zetlin polytope and the edges of the diagram $D(v)$.

5.3 Faces $\Gamma(v, B)$ and proof of Proposition 3.2

It is now easy to describe the diagrams of the faces $\Gamma(v, B)$ in terms of the diagram for v . Namely, we should delete all edges in $D(v)$ that correspond to the roots in $R(v, B)$ under the above correspondence.

Remark 5.1. If $B = B^-$ is the lower triangular, the diagram of $\Gamma(v, B^-)$ is obtained from the diagram $D(v)$ by deleting all edges of type R . The faces $\Gamma(v, B^-)$ are exactly the so-called *Gelfand–Zetlin faces* considered in [13, Subsection 2.2.1]. Note that the notation in [13] is different: my $x_{i,j}$ is his $\lambda_{i+j,i}$ and my σ_v is his w_v^{-1} . □

We now determine under which conditions two simple vertices u and v of the Gelfand–Zetlin polytope are connected by an edge. The necessary condition $p(u) = s_\alpha p(v)$ for some root α is obviously not sufficient (e.g., the vertices s_2v and s_1s_2v on Figure 1 are not connected by the edge though $p(s_1s_2v) = s_1p(s_2v)$).

Lemma 5.2. Let u and v be two simple vertices of the Gelfand–Zetlin polytope such that the weights $p(u)$ and $p(v)$ can be obtained from each other by the reflection s_α with respect to some root α . Then u and v are connected by the edge if and only if the diagram

$D(u)$ can be obtained from the diagram $D(v)$ by switching the edge of $D(u)$ corresponding to the root α . \square

Proof. Choose α so that $(p(v), \alpha) < (p(u), \alpha)$. Then the vertices v and u can only be connected by the edge $e(v, \alpha)$ (which will then coincide with the edge $e(u, -\alpha)$), and the lemma immediately follows from the description of edges in the Gelfand–Zetlin polytope. \blacksquare

To prove Proposition 3.2, we will need the following two lemmas.

Lemma 5.3. If $\Gamma(u, B)$ is a facet of $\Gamma(v, B)$, then the vertices v and u are connected by the edge. \square

Note that for $B = B^-$ (as well as for $B = B^+$), the statement of the lemma is straightforward. Indeed, by Remark 5.1, the diagrams of the faces $\Gamma(u, B^-)$ and $\Gamma(v, B^-)$ are obtained from the diagrams of u and v , respectively, by deleting all edges of type R . On the other hand, the diagram $D(\Gamma(v, B^-))$ is obtained from the diagram $D(\Gamma(u, B^-))$ by deleting one edge (since $\Gamma(u, B)$ is a facet of $\Gamma(v, B)$). Hence, $D(v)$ is obtained from $D(u)$ by switching this edge. We now prove the lemma for arbitrary B .

Proof. First, note that the assumptions of the lemma imply that $\mathcal{O}(u, B)$ precedes $\mathcal{O}(v, B)$ with respect to the Bruhat order. Hence, $p(u) = s_\alpha p(v)$ for some root $\alpha \in R(v, B)$. Let $(i\ j)$ be the transposition corresponding to s_α and $e_{i+1,s}^R$ the edge of the diagram $D(v)$ corresponding to the root α (we assume that this edge is of type R ; type L case is completely analogous). We now compare the edges starting at the i th rows of the diagrams $D(v)$ and $D(u)$. Let $e_{i+1,s-k}^L$ be the last edge of type L (when going from left to right) starting from the i th row of the diagram $D(v)$. We want to show that $k = 1$ so that $e_{i+1,s}^R$ can be switched and the resulting diagram remains simple. Consider all edges of $D(v)$ between $e_{i+1,s-k}^L$ and $e_{i+1,s}^R$, that is, the edges $e_{i+1,s-k+1}^R, \dots, e_{i+1,s-1}^R$. The above explicit correspondence between simple vertices v and permutations σ_v implies that the trees $T_l(v)$ and $T_l(u)$ have the same starting points unless $l = i, j$. From this, it is easy to deduce that the diagram $D(u)$ contains the edges $e_{i+1,s-k+1}^L, \dots, e_{i+1,s}^L$. Moreover, if $e_{i+1,s-k+l}^R$ in $D(v)$ for $l = 1, \dots, k-1$ corresponds to a root β , then $e_{i+1,s-k+l+1}^L$ in $D(u)$ corresponds to $-\beta$. Finally, $e_{i+1,s-k+1}^L$ corresponds to the root $-\alpha$. Hence, the diagrams $D(\Gamma(v, B))$ and $D(\Gamma(u, B))$ will differ in at least k edges. Indeed, whenever the diagram $D(\Gamma(v, B))$ con-

tains (or does not contain) the edge $e_{i+1,s-k+l}^R$, the diagram $D(\Gamma(u, B))$ does not contain (or contains) the edge $e_{i+1,s-k+l+1}^L$ for $l = 1, \dots, k - 1$. Also, $D(\Gamma(v, B))$ does not contain the edge $e_{i+1,s-k+1}^L$, while $D(\Gamma(u, B))$ does. It remains to note that the diagram of $\Gamma(v, B)$ is obtained from the diagram of $\Gamma(u, n)$ by deleting exactly one edge (since $\Gamma(u, B)$ is a facet in $\Gamma(v, n)$). Hence, $k = 1$. ■

Lemma 5.4. If v and u are two simple vertices of the Gelfand–Zetlin polytope such that $p(v) = s_\alpha p(u)$ for the root $\alpha = \alpha_i + \dots + \alpha_{j-1}$, then

$$|(p(v), \alpha)| = |\lambda_r - \lambda_s|,$$

where $s = \sigma_v^{-1}(i)$ and $r = \sigma_v^{-1}(j)$ (that is, $p_{1,s}$ and $p_{1,r}$ are the starting points of the trees $T_i(v)$ and $T_j(v)$, respectively). □

Proof. Since $p(v) = \sigma_v \lambda$, we have $(p(v), \alpha) = (\sigma_v \lambda, \alpha) = (\lambda, \sigma_v^{-1} \alpha)$. Note that the reflection defined by the root $\sigma_v^{-1} \alpha$ corresponds to the transposition $(\sigma_v^{-1}(i) \sigma_v^{-1}(j)) = (s r)$. Hence, $|(\lambda, \sigma_v^{-1} \alpha)| = |\lambda_r - \lambda_s|$. ■

We now prove Proposition 3.2. Let $\Gamma(u, B)$ be a facet of $\Gamma(v, B)$. By Lemma 5.3, the vertices v and u are connected by the edge. We have $p(u) = s_\alpha p(v)$ for some root α . Suppose that $\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$, where $0 < i < j < n$. By Lemma 5.4, we have that $|(p(v), \alpha)| = |\lambda_r - \lambda_s|$, where $p_{1,s}$ and $p_{1,r}$ are the starting points of the trees $T_i(v)$ and $T_j(v)$, respectively. We now show that $|\lambda_r - \lambda_s| = d(v, \Gamma(u, B))$. Denote by e the i th edge of the tree $T_j(v)$. Since u and v are connected by the edge, we get by Lemma 5.2 that the diagram $D(u)$ is obtained from $D(v)$ by switching the edge e . We again consider the case where e is of type R since the proof for the other case is completely the same. Let $p_{i,l+1}$ and $p_{i+1,l}$ be the vertices of the edge e . Denote by F the facet of the Gelfand–Zetlin polytope given by the equation $x_{i-1,l} = x_{i,l}$. It is easy to check that $\Gamma(u, B) = F \cap \Gamma(v, B)$. Hence, the integral distance $d(v, \Gamma(u, B))$ is by definition equal to the distance $d(v, F)$. To compute the latter, we note that the equation $x_{i-1,l} = x_{i,l}$ defining F is already primitive. Since $p_{i,l}$ belongs to $T_i(v)$ and $p_{i+1,l}$ to $T_j(v)$, we get that the $x_{i-1,l}$ coordinate of v is equal to λ_s and the $x_{i,l}$ coordinate to λ_r . Hence, $d(v, F) = \lambda_r - \lambda_s$.

5.4 Chevalley formula for arbitrary faces $\Gamma(v, B)$

The same arguments as in the proof of Proposition 3.2 allow us to prove a more general Chevalley-type formula for the faces of the Gelfand–Zetlin polytope. Let $\Gamma(v, B)$ be any

(not necessarily) admissible face and $\Gamma(u, B)$ a face such that $\mathcal{O}(u, B)$ precedes $\mathcal{O}(v, B)$ (but we no longer require that $\Gamma(u, B) \subset \Gamma(v, B)$). Let $\alpha = \alpha_i + \dots + \alpha_{j-1}$ be the root such that $p(v) = s_\alpha p(u)$. Consider the edges e_1, \dots, e_k that end at the $(i+1)$ -st row of the diagram $D(u)$ and differ in the slope from the corresponding edges at the $(i+1)$ -st row of $D(v)$. Each such edge considered alone gives the diagram of a facet in the Gelfand–Zetlin polytope. Denote by F_i the facet defined by the edge e_i . Put $d(v, u) := d(v, F_1) + \dots + d(v, F_k)$.

Theorem 5.5. Let $\Gamma(v, B)$ be any (not necessarily) admissible face. Then

$$H_\lambda \overline{\mathcal{O}(v, B)} = \sum d(v, u) \overline{\mathcal{O}(u, B)},$$

where the sum is taken over all Schubert cells $\mathcal{O}(u, B)$ that precede $\mathcal{O}(v, B)$. □

Note that for admissible faces, Theorem 5.5 reduces to Theorem 3.3 (since we have $k = 1$ by Lemma 5.3 and $\Gamma(u, n) = F_1 \cap \Gamma(v, n)$). Theorem 5.5 might be important for a realization of Schubert cycles by unions of faces of the Gelfand–Zetlin polytope [11].

Proof. The proof is almost the same as for admissible faces. We have $|(p(v), \alpha)| = |\lambda_r - \lambda_s|$ by Lemma 5.4. Assume that $r > s$. We can also write $\lambda_r - \lambda_s$ as $(\lambda_r - \lambda_{i_{k-1}}) + (\lambda_{i_{k-1}} - \lambda_{i_{k-2}}) + \dots + (\lambda_{i_1} - \lambda_s)$, where $\lambda_{i_1}, \dots, \lambda_{i_{k-1}}$ correspond to the starting points of the trees in $D(u)$ containing the edges e_1, \dots, e_{k-1} , respectively. It is easy to check that $(\lambda_{i_l} - \lambda_{i_{l-1}}) = d(v, F_l)$ using the same argument as in the proof of Proposition 3.2. ■

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