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Classes of Subcubic Planar Graphs for Which the Independent Set Problem Is Polynomially Solvable

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Abstract—We prove the polynomial solvability of the independent set problem for some family of classes of the planar subcubic graphs.

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INTRODUCTION

The present article is an extension of the series of works [1, 2, 4] in which we studied the problem of the efficient solvability of the independent set problem in a family of hereditary subclasses of the class of planar graphs. Recall that a subset of pairwise nonadjacent vertices in a graph is called an *independent set* (i.s.). An independent set in a graph is called *maximum* if it contains the largest possible number of vertices; the size of a maximum independent set (m.i.s.) is called the *independence number* and denoted by $\alpha(G)$ for a graph G.

The *independent set problem* (Problem IS) for a given graph consists in finding a maximum independence set of the graph. It is well known that this problem is polynomially equivalent to the computation of the independence number.

A class of graphs is called *hereditary* if it is closed under removal of vertices. Each hereditary (and only hereditary) class $\mathcal{X}$ is determined by a set of forbidden induced subgraphs $\mathcal{Y}$; in this case, we write

$$\mathcal{X} = \mathcal{F}ree(\mathcal{Y}).$$

If in addition $\mathcal{Y}$ is finite then $\mathcal{X}$ is called *finitely defined*.

A hereditary class of graphs $\mathcal{X}$ is called IS-*easy* if there exists an algorithm solving Problem IS for the graphs of this class in polynomial time. Otherwise, $\mathcal{X}$ is called IS-*hard*. Throughout the sequel, we assume that P≠NP, and this condition is not explicitly included in the statements of the assertions, for example, in the following: if Problem IS is NP-complete in a hereditary class $\mathcal{X}$ then $\mathcal{X}$ is IS-hard. It is well known that the class $\mathcal{P}$ of all planar graphs and the class $\mathcal{D}(k)$ of the graphs with vertex degrees at most k ($k > 2$) are IS-hard.

The goal of this article is to prove the IS-easiness of some subsets in $\mathcal{P}$.

In [3], we introduced the notion of IS-*boundary class* of graphs—an inclusion minimal class which is an intersection of a decreasing sequence of IS-hard classes—and proved that a finitely defined class is IS-hard if and only if it contains some IS-boundary class. Thus, the knowledge of all IS-boundary classes would make it possible to fully characterize all finitely defined IS-hard classes.

It is also proved in [3] that the class $\mathcal{T}$ of all graphs whose each connected component is a tree with at most three leaves is boundary. In other words, each connected component is a triode where by a *triode*

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$T_{i,j,k}$ we mean a tree obtained by joining one vertex to three other vertices by simple paths of lengths $i \geq 0$, $j \geq 0$, and $k \geq 0$.

It is yet unknown whether there exist other IS-boundary classes. Their existence is equivalent to the existence of a graph $G \in \mathcal{T}$ for which the class $\mathcal{F}ree(\{G\})$ is IS-hard [3]. The difficulty of the problem is manifested in that up to now the complexity status of Problem IS for the class $\mathcal{F}ree(\{P_5\})$ is unknown. At the same time, considering not the whole family of hereditary classes but a part thereof, we can hope for an exhaustive answer to the question. For example, in [3], we proved that the class $\mathcal{T}$ is the only IS-boundary class among (in the family of) strongly hereditary classes, i.e., of graph classes closed under vertex and edge removal.

In considering some restrictions of the set of all graphs, there naturally appears the notion of a relative boundary class which extends the notion of a boundary class. Let $\mathcal{Y}$ be an IS-hard class. Call a hereditary class of graphs $\mathcal{X}$ IS-*boundary relative to a class* $\mathcal{Y}$ if there exists a sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \dots$ of IS-hard subsets in $\mathcal{Y}$ such that

$$\mathcal{X} = \bigcap_{i=1}^{\infty} \mathcal{X}_i,$$

and $\mathcal{X}$ is minimal with this property. The following holds for relative IS-boundary classes: If $\mathcal{S}$ is a finite subset in $\mathcal{Y}$ then the class $\mathcal{Y} \cap \mathcal{F}ree(\mathcal{S})$ is IS-hard if and only if it contains an IS-boundary relative to $\mathcal{Y}$ class. It can be proved in almost the same manner as the corresponding assertion in [3].

The class $\mathcal{T}$ is IS-boundary relative to $\mathcal{Y} = \mathcal{P}$ and $\mathcal{Y} = \mathcal{D}(k)$ for $k > 2$. The proofs of these facts repeat almost verbatim the corresponding proofs in [3]. Though we have not yet established whether there are some other IS-boundary classes relative to $\mathcal{Y} = \mathcal{P}$ or $\mathcal{Y} = \mathcal{D}(3)$ (for this, we need to find a graph $G \in \mathcal{T}$ for which the class $\mathcal{Y} \cap \mathcal{F}ree(\{G\})$ is IS-hard), there is a more significant progress in moving towards this aim than for simply boundary classes. Namely, we can prove that the class $\mathcal{Y} \cap \mathcal{F}ree(\{T_{1,i,j}\})$ is IS-easy for all i, j and such $\mathcal{Y}$ [4, 5]. For $G = T_{2,2,2}$, the question on complexity of Problem IS in the class $\mathcal{Y} \cap \mathcal{F}ree(\{G\})$ is open.

There appears a natural idea of considering the intersection $\mathcal{P}(3)$ of $\mathcal{D}(3)$ and $\mathcal{P}$. The class $\mathcal{T}$ is IS-boundary relative to $\mathcal{P}(3)$. It is possible that $\mathcal{T}$ is the only such class. Some boundary classes relative to $\mathcal{P}(3)$ different from $\mathcal{T}$ exist if and only if among the classes $\mathcal{P}(3) \cap \mathcal{F}ree(\{G\})$, $G \in \mathcal{T}$, there are IS-hard classes. For $\mathcal{Y} = \mathcal{P}(3)$, we indeed manage to obtain a deeper advance towards proving the uniqueness of $\mathcal{T}$ than for $\mathcal{Y} = \mathcal{D}(3)$ and $\mathcal{P}$. Namely, we will prove that the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,i}\})$ is IS-easy for each fixed i.

We adopt the following notation:

◇ graphs C_k and P_k are defined by counting their vertices, for instance, $C_k = (x_1, x_2, \dots, x_k)$;

◇ an *apple* A_k is a graph obtained from the cycle $(a_1, a_2, a_3, \dots, a_k)$ by adding some vertex a_0 adjacent to a_1;

◇ a *glider* is a graph obtained by adding the vertices x_6 and x_7 as well as the edges (x_1, x_6) and (x_3, x_7) to the graph $K_{2,3}$ with parts x_1, x_2, x_3 and x_4, x_5;

◇ a *tire* is a graph obtained from two simple cycles $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_k)$ by adding the edges $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$;

◇ a *punctured tire* is a graph obtained by removal of an edge (x_{k-1}, x_k) from a tire;

◇ by $G \setminus V$ we denote the graph obtained from G by removal of all vertices belonging to $V \subseteq V(G)$.

1. SOME DEFINITIONS AND AUXILIARY RESULTS

A *separating clique in a graph* is a set of its vertices generating a complete subgraph whose removal leads to increase of the number of connected components. Given a graph, a C-*block* is an inclusion maximal connected induced subgraph having no separating clique. There is a polynomial algorithm for finding all C-blocks in an input graph [3]. It is also known that, for every hereditary class, Problem IS is polynomially reduced to its C-blocks [3].

For proving the main result, we need the two following auxiliary assertions:

Lemma 1. *Suppose that a graph $G \in \mathcal{P}(3)$ contains a glider and is a C-block. Then*

$$\alpha(G) = \alpha(G \setminus \{x_2, x_4\}) + 1.$$

Proof. Prove that the vertex x_2 is necessarily of degree 2. Suppose the contrary. Then $\deg(x_2) = 3$. Since G is a planar C-block (and hence is a two-connected graph) with degrees of all vertices at most three; therefore, there exists a path connecting x_2 with x_1 (or with x_3) containing neither x_4 nor x_5 and containing x_6 (or x_7). This is easy to show by considering both possible ways of location for x_2 in the planar packing of G (inside the cycle (x_1, x_4, x_3, x_5) and outside it) and noticing that otherwise $\{x_2\}$ is a separating clique of G. Then, due to the planarity of G and the constraints on the vertex degrees in the graph, the set $\{x_3\}$ ($\{x_1\}$) is a separating clique. We obtain a contradiction to the assumption. Hence,

$$\deg(x_2) = 2.$$

Prove now that there is a m.i.s. in G that contains either x_4 and x_5 or x_1, x_2, and x_3. Clearly, every m.i.s. in G either contains x_4 and x_5 simultaneously or contains neither x_4 nor x_5. At the same time, if x_4 and x_5 do not belong to some m.i.s. in G then x_2 and exactly one vertex from each of the pairs (x_1, x_6) and (x_3, x_7) belongs to this m.i.s. But then, from the given m.i.s., we can remove vertices from each of the pairs and add x_1 and x_3; in result, we again obtain a m.i.s. in G. Thus, we have proved the existence of a m.i.s. for G with the desired properties. It is easy to verify that there exists a m.i.s. of the graph $G \setminus \{x_2, x_4\}$ which contains either x_1 and x_3 or x_5.

Thus, given some m.i.s. of G, we can remove a vertex (x_2 or x_4) so that, in result, we obtain an i.s. of $G \setminus \{x_2, x_4\}$. Therefore, $\alpha(G) \le \alpha(G \setminus \{x_2, x_4\}) + 1$. On the other hand, to each m.i.s. of $G \setminus \{x_2, x_4\}$, we can add a vertex (x_2 or x_4) so that we obtain an i.s. in G. Therefore, $\alpha(G) \ge \alpha(G \setminus \{x_2, x_4\}) + 1$.

Lemma 1 is proved. □

Lemma 2. *Suppose that H is an arbitrary graph containing a cycle $(x_1, x_2, x_3, x_4, x_5)$ (which is not necessary induced), while (x_2, x_4) and (x_3, x_5) do not belong to $E(H)$. If the degrees $\deg(x_2) = \deg(x_5)$ are equal to 2 or 3 and (x_2, x_5) belongs to $E(H)$ then*

$$\alpha(H) = \alpha(H \setminus \{x_1\}).$$

Proof. Let IS be a m.i.s. of H. If $x_1 \notin IS$ then $\alpha(H) = \alpha(H \setminus \{x_1\})$.

Suppose now that $x_1 \in IS$, then x_2 and x_5 do not lie in IS. Among x_3 and x_4, at most one vertex belongs to IS. Consequently, there exists a vertex $x \in \{x_2, x_5\}$ adjacent to a vertex in $\{x_3, x_4\}$ not belonging to IS. Then $IS \cup \{x\} \setminus \{x_1\}$ is a m.i.s. in H. Therefore, $\alpha(H) = \alpha(H \setminus \{x_1\})$.

Lemma 2 is proved. □

2. ADJOINT CYCLES AND THEIR DESTRUCTION

Let G be a C-block of the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ containing some apple A_k ($k \ge 10$). Call an induced cycle $(x_1, x_2, x_3, x_4, x_5)$ in G *adjoint* (*to the apple A_k*) if

$$a_0 = x_5, \qquad a_1 = x_1, \qquad a_2 = x_2, \qquad a_3 = x_3.$$

Clearly, x_4 can be adjacent to none of the vertices $a_6, a_7, \dots, a_{k-1}$ since otherwise G would contain an induced subgraph $T_{2,2,2}$. Therefore, among the vertices of $V(A_k)$, the vertex x_4 can be adjacent only to a_k, a_4, and a_5.

A scrupulous analysis of the neighborhoods of some vertices reveals some properties of a m.i.s. in G. Namely, it is always possible to indicate a vertex of the adjoint cycle (starting from the composition of the neighborhoods) not lying in some m.i.s. of G. Removing this vertex does not change the independence number, and the mechanism of destruction of adjoint cycles is an important component of the algorithm for solving Problem IS in the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$. The above-mentioned analysis is the contents of the proofs of Lemmas 3–5.

Lemma 3. *Let $\deg(x_2) = 2$. If $\deg(x_4) = 2$ or $\deg(x_5) = 2$ or x_4 is adjacent only to the vertices of an apple A_k then $\alpha(G) = \alpha(G \setminus \{x_1\})$. In all other cases, $\alpha(G) = \alpha(G \setminus \{x_4\})$.*

Proof. We may assume that there exists a vertex b, with $(b, x_5) \in E(G)$, $b \neq x_1$ and $b \neq x_4$ (otherwise, $\alpha(G) = \alpha(G \setminus \{x_1\})$ by Lemma 2). The vertex b must be adjacent to either a_k or a_{k-1} (since the contrary leads to the subgraph $T_{2,2,2}$ in G induced by $a_1, a_0, b, a_k, a_{k-1}, a_2, a_3$).

Suppose first that x_4 is adjacent only to the vertices of A_k or that $\deg(x_4) = 2$. Prove that the vertices x_4 and a_5 are necessarily nonadjacent (therefore, either $\deg(x_4) = 2$, or $(x_4, a_4) \in E(G)$, or $(x_4, a_k) \in E(G)$). Suppose the contrary. The vertices b and a_6 must be adjacent because otherwise G contains the subgraph $T_{2,2,2}$ induced by the vertices x_4, x_5, b, x_3, x_2, a_5, and a_6. Hence, b is adjacent to neither a_7 nor a_8 (because $k \geq 10$). But then G contains the subgraph $T_{2,2,2}$ induced by a_6, a_5, a_4, b, a_0, a_7, and a_8; a contradiction.

Demonstrate now that there exists a m.i.s. in G that does not contain x_1. This will imply the first part of the lemma. Consider IS, an arbitrary m.i.s. in G. We may assume that $x_1 \in IS$ (otherwise, $\alpha(G) = \alpha(G \setminus \{x_1\})$). If $x_3 \notin IS$ then $IS \cup \{x_2\} \setminus \{x_1\}$ is a m.i.s. in G. If $x_3 \in IS$ then the set of vertices $IS \cup \{x_2, x_4\} \setminus \{x_1, x_3\}$ is also a m.i.s. in G (since either $\deg(x_4) = 2$, or $(x_4, a_k) \in E(G)$, or $(x_4, a_4) \in E(G)$).

In what follows, we assume that x_4 is adjacent to $x \notin V(A_k)$. The vertices b and x_4 are nonadjacent (and hence $b \neq x$) since otherwise $b = x$ and either the vertices x_1, a_k, a_{k-1}, x_5, x, x_2, and x_3 or the vertices x_3, x_2, x_1, x_4, x, a_4, and a_5 generate a subgraph $T_{2,2,2}$ in G. Prove that there exists a m.i.s. in G not containing x_4. This implies the second part of the lemma. Let IS' be some m.i.s. containing x_4 (if there is no such set then the assertion is obvious). If non of the vertices b and x_1 belongs to IS' then $IS' \cup \{x_5\} \setminus \{x_4\}$ is a m.i.s. in G. If exactly one vertex $y \in \{x_1, b\}$ belongs to IS' then the set $IS' \cup \{x_5, x_2\} \setminus \{x_4, y\}$ is also a m.i.s. for G. Therefore, we may assume that b and x_1 belong to IS'. Finally, we may assume that $a_4 \in IS'$ (therefore, $(b, a_4) \notin E(G)$); otherwise, $IS' \cup \{x_3\} \setminus \{x_4\}$ is a m.i.s. for G.

Recall that b must be adjacent to at least one of the vertices a_{k-1} and a_k. Denote by a_i the vertex in $\{a_{k-1}, a_k\}$ adjacent to b and having minimal index (among these vertices, both can be adjacent to b simultaneously). The vertex x is necessarily adjacent to one of the vertices a_4 and a_5 (otherwise G would contain the subgraph $T_{2,2,2}$ induced by the set of vertices $\{x_3, x_2, x_1, x_4, x, a_4, a_5\}$). Show that b and x are nonadjacent. Suppose the contrary. Then b is adjacent to exactly one vertex of $\{a_{k-1}, a_k\}$. Clearly, x must be adjacent to a_{i-1} or a_{i-2} (otherwise G contains the subgraph $T_{2,2,2}$ induced by a_i, a_{i-1}, a_{i-2}, b, x, a_{i+1}, and a_{i+2}, where the subscripts of the vertices are understood modulo k). This is a contradiction since x must be adjacent to x_4, b, a_4 or a_5, a_{i-1} or a_{i-2}. Indeed, x is not adjacent to a_5 (and hence must be adjacent to a_4) since otherwise G would contain the subgraph $T_{2,2,2}$ induced by x_4, x_5, b, x_3, x_2, x, and a_5.

Note that if x is adjacent to some $y \notin \{a_4, x_4\}$ then b and y are also adjacent; otherwise the vertices x_4, x_5, b, x, y, x_3, and x_2 would induce a subgraph $T_{2,2,2}$. Therefore, $IS' \cup \{x_3, x\} \setminus \{a_4, x_4\}$ is a m.i.s. for G.

Lemma 3 is proved. □

Lemma 4. *Suppose that* $\deg(x_5) = 2$ *and there exists* $c \notin V(A_k)$ *adjacent to* x_2*. If the set* $\{x_1, x_4, a_4, c\}$ *is independent then* $\alpha(G) = \alpha(G \setminus \{x_3\})$*; otherwise,* $\alpha(G) = \alpha(G \setminus \{x_4\})$.

Proof. Suppose that $\{x_1, x_4, a_4, c\}$ is not independent and prove that $\alpha(G) = \alpha(G \setminus \{x_4\})$. Let IS be a m.i.s. for G containing x_4 (if there is no such m.i.s. then, obviously, $\alpha(G) = \alpha(G \setminus \{x_4\})$). If $\{x_1, x_4, a_4, c\}$ is not independent then at least one of the vertices x_1, c, and a_4 does not belong to IS. This implies that at least one of the sets

$$IS \cup \{x_5\} \setminus \{x_4\}, \qquad IS \cup \{x_2, x_5\} \setminus \{x_1, x_4\}, \qquad IS \cup \{x_3, x_5\} \setminus \{x_1, x_4\}$$

is a m.i.s. for G. Therefore, we may assume that the set $\{x_1, x_4, a_4, c\}$ is independent.

Suppose that there is some $a' \notin V(A_k)$ adjacent to a_4. Prove that a' must be adjacent to c. Suppose the contrary. Then a' must be adjacent to x_4 since otherwise the vertices a_3, a_2, c, x_4, x_5, a_4, and a' would generate a subgraph $T_{2,2,2}$. Clearly, c must be adjacent to either a_k or a_{k-1} since otherwise the vertices x_1, x_5, x_4, x_2, c, a_k, and a_{k-1} would generate some $T_{2,2,2}$. Moreover, c cannot be adjacent to a_{k-1} (therefore, it is adjacent to a_k) since otherwise a_2, c, a_{k-1}, a_1, a_0, a_3, and a_4 would generate

$T_{2,2,2}$. The vertex c must be adjacent to a_5, since otherwise x_3, x_4, x_5, x_2, c, a_4, and a_5 would generate some $T_{2,2,2}$. But then the vertices a_k, c, a_5, a_{k-1}, a_{k-2}, a_1, and a_0 generate a subgraph isomorphic to $T_{2,2,2}$; a contradiction.

The two alternatives are open: either the vertices x_4 and a_5 are nonadjacent or adjacent.

Case 1. Let the vertices x_4 and a_5 be nonadjacent. The vertex c must be adjacent to a_5 since otherwise G would contain the subgraph $T_{2,2,2}$ induced by the set of vertices $\{x_3, x_2, c, x_4, x_5, a_4, a_5\}$. The vertex c cannot be adjacent to a_k since otherwise G would contain $T_{2,2,2}$ induced by a_k, a_1, a_0, c, a_5, a_{k-1}, and a_{k-2}. The vertex x_4 must be adjacent to a_k since otherwise x_2, c, a_5, x_3, x_4, a_1, and a_k would generate a subgraph $T_{2,2,2}$.

Prove that $\alpha(G) = \alpha(G \setminus \{x_3\})$. Let IS' be a m.i.s. containing x_3 (if there is no such a set then the desired equality is obvious). If none of the vertices x_1 and c belongs to IS' then $IS' \cup \{x_2\} \setminus \{x_3\}$ is a m.i.s. for G. If $x_1 \in IS'$ and $c \notin IS'$ then $IS' \cup \{x_2, x_5\} \setminus \{x_1, x_3\}$ is a m.i.s. for G. If $x_1 \notin IS'$ and $c \in IS'$ then $IS' \cup \{x_2, a_4\} \setminus \{c, x_3\}$ is a m.i.s. for G. Finally, if $c \in IS'$ and $x_1 \in IS'$ then the set $IS' \cup \{x_2, x_5, a_4\} \setminus \{x_1, x_3, c\}$ is a m.i.s. for G. The equality $\alpha(G) = \alpha(G \setminus \{x_3\})$ is proved.

Case 2. Let the vertices x_4 and a_5 be adjacent. Prove that $\alpha(G) = \alpha(G \setminus \{x_3\})$. Clearly, c must be adjacent to at least one of the vertices a_{k-1} and a_k; otherwise the vertices a_1, a_k, a_{k-1}, x_5, x_4, a_2, and c would generate a subgraph isomorphic to $T_{2,2,2}$ in G. At the same time, it cannot be adjacent to a_{k-1} (and hence it cannot be adjacent to a_k) since otherwise there would be the subgraph $T_{2,2,2}$ induced by a_2, c, a_{k-1}, a_1, a_0, a_3, and a_4.

Let IS'' be a m.i.s. containing x_3. If $c \notin IS''$ then

$$\text{either} \quad IS'' \cup \{x_2\} \setminus \{x_3\} \quad \text{or} \quad IS'' \cup \{x_2, x_5\} \setminus \{x_1, x_3\}$$

is a m.i.s. for G. Therefore, we can assume that $c \in IS''$, and hence $a_k \notin IS''$. If $a_5 \notin IS''$ then $IS'' \cup \{a_4\} \setminus \{x_3\}$ is a m.i.s. for G. Hence, we can assume that $c \in IS''$, $x_3 \in IS''$, $a_5 \in IS''$. If $x_5 \notin IS''$ then $IS'' \cup \{a_4, x_4\} \setminus \{x_3, a_5\}$ is a m.i.s. for G; if $x_5 \in IS''$ then $IS'' \cup \{a_4, x_1, x_4\} \setminus \{x_3, x_5, a_5\}$ is a m.i.s. for G. Thus, there always exists a m.i.s. for G not containing x_3; and hence $\alpha(G) = \alpha(G \setminus \{x_3\})$.

Lemma 4 is proved. □

Lemma 5. *Suppose that the vertex x_5 is adjacent to $b \notin V(A_k) \cup \{x_4\}$ and x_2 is adjacent to $c \notin V(A_k)$. If $b = c$ and x_4 is adjacent to a_4 or x_4 adjacent to $x \notin V(A_k)$ and x is adjacent to a_5 and is not adjacent to a_4 then $\alpha(G) = \alpha(G \setminus \{x_3\})$. If $b \neq c$ and b is adjacent to a_k then $\alpha(G) = \alpha(G \setminus \{x_4\})$. In all other cases, $\alpha(G) = \alpha(G \setminus \{x_5\})$.*

Proof. Consider the two cases: $b \neq c$ and $b = c$.

Case 1. Let $b \neq c$. In proving Lemma 3, we have showed that, for $\deg(x_5) > 2$, the vertices x_4 and a_5 are nonadjacent (and the degree of x_2 does not influence the validity of this fact). The vertex b must be adjacent to at least one of the vertices a_{k-1}, a_k. Clearly, x_4 cannot adjacent to a_k; otherwise the planarity of G would create an obstacle to the adjacency of b and a_{k-1}. Since G is a planar graph, either x_4 and c are nonadjacent or $(x_4, c) \in E(G)$ and $(a_{k-1}, c) \notin E(G)$, $(a_k, c) \notin E(G)$. Prove that in fact x_4 and c are adjacent.

Suppose the contrary. The vertex c must be adjacent to at least one of the vertices in $\{a_{k-1}, a_k\}$ (otherwise there would be the subgraph $T_{2,2,2}$ in G induced by x_1, x_2, c, x_5, x_4, a_k, and a_{k-1}). Show that $(a_{k-1}, c) \notin E(G)$ (consequently, $(a_k, c) \in E(G)$ and $(a_{k-1}, b) \in E(G)$). Let $(a_{k-1}, c) \in E(G)$; hence, $(a_k, b) \in E(G)$. If $(c, a_4) \notin E(G)$ then the vertices x_2, x_1, x_5, c, a_{k-1}, a_3, and a_4 generate a subgraph $T_{2,2,2}$ in G and if $(c, a_4) \in E(G)$ then the vertices c, x_2, x_1, a_{k-1}, a_{k-2}, a_4, and a_5 generate a subgraph $T_{2,2,2}$ in G. Neither of the vertices b and c can be adjacent to a_{k-2} or a_{k-3}. If b is adjacent to $a \in \{a_{k-3}, a_{k-2}\}$ then G contains the subgraph $T_{2,2,2}$ induced by the vertices x_5, b, a, a_1, a_k, x_4, and x_3. If c is adjacent to $a \in \{a_{k-3}, a_{k-2}\}$ then G contains the subgraph $T_{2,2,2}$ induced by the vertices x_2, c, a, x_1, x_5, a_3, and a_4. But then the vertices a_{k-1}, a_{k-2}, a_{k-3}, b, x_5, a_k, and c induce a subgraph $T_{2,2,2}$ in G. Thus, x_4 and c are adjacent. Then $\deg(c) = 2$; otherwise (by the planarity of G) the set $\{c\}$ would be a separating clique.

Demonstrate that if b is adjacent to a_k then there exists a m.i.s. for G not containing x_4. Let IS be a m.i.s. for G. We may assume that $x_4 \in IS$. If at least one of the vertices x_2 and b does not lie in IS then either $IS \cup \{c\} \setminus \{x_4\}$ or $IS \cup \{x_5\} \setminus \{x_4\}$ is a m.i.s. for G. If x_2, x_4, b belong to IS then $IS \cup \{x_1, c\} \setminus \{x_2, x_4\}$ is a m.i.s. for G and this set does not contain x_4.

Prove that if b is adjacent to a_{k-1} then there exists a m.i.s. without x_5. If there exists a vertex x adjacent to a_k (to b) and not belonging to $V(A_k)$ then it must be adjacent to b (to a_k). Indeed, if $(x, a_k) \in E(G)$ and $(x, b) \notin E(G)$ then G contains the subgraph $T_{2,2,2}$ induced by the vertices a_1, a_0, b, a_k, x, a_2, and a_3. If (x, b) belongs to $E(G)$ and $(x, a_k) \notin E(G)$ then G contains the subgraph $T_{2,2,2}$ induced by the vertices a_0, b, x, a_1, a_k, x_4, and x_3. Let IS' be a m.i.s. containing x_5. If $a_k \in IS'$ then $IS' \cup \{b\} \setminus \{x_5\}$ is a m.i.s. for G. If $x_2 \in IS'$ and $a_k \notin IS'$ then $IS' \cup \{x_4\} \setminus \{x_5\}$ is a m.i.s. for G. If neither a_k nor x_2 belongs to IS' then $IS' \cup \{x_1\} \setminus \{x_5\}$ is a m.i.s. for G.

Case 2. Suppose now that $b = c$. The vertex x_4 cannot be adjacent to a_k since the contrary leads to the subgraph $T_{2,2,2}$ induced by x_4, a_k, a_{k-1}, x_5, b, a_3, and a_4. Prove that if x_4 is adjacent to a_4 then there exists a m.i.s. for G not containing x_3. Clearly, $\deg(b) = 2$ since otherwise the set $\{b\}$ would be a separating clique. Let IS'' be a m.i.s. for G containing x_3. If $x_5 \notin IS''$ then $IS'' \cup \{x_4\} \setminus \{x_3\}$ is a m.i.s. for G. If $x_5 \in IS''$ then $IS'' \cup \{x_2\} \setminus \{x_3\}$ is a m.i.s. for G. Thus, in both cases, there exists a m.i.s. not containing x_3. Prove also that if x_4 is adjacent to $x \notin V(A_k)$ and x adjacent to a_5 and nonadjacent to a_4 then there exists a m.i.s. not containing x_3. Let IS'' be a m.i.s. for G containing x_3. If there is a vertex y adjacent to a_4 and not belonging to $V(A_k)$ then it must be adjacent also to x; otherwise, the vertices x_3, x_4, x, a_4, y, a_2, and a_1 induce a subgraph $T_{2,2,2}$. This implies that if $x \in IS''$ then $IS'' \cup \{a_4\} \setminus \{x_3\}$ is a m.i.s. for G. Therefore, we may assume that $x \notin IS''$. We may also assume that $x_5 \in IS''$ since otherwise $IS'' \cup \{x_4\} \setminus \{x_3\}$ is a m.i.s. for G. But then $IS'' \cup \{x_2\} \setminus \{x_3\}$ is a m.i.s. for G. Hence, in the case under consideration, there is a m.i.s. not containing x_3.

Show that, in all other cases, there exists a m.i.s. for G not containing x_5. Let IS''' be a m.i.s. containing x_5. If $(x_4, x) \in E(G)$ and $x \notin V(A_k) \cup \{b\}$ then x is adjacent to a_4 or to a_5, since otherwise G would contain the subgraph $T_{2,2,2}$ induced by the vertices a_3, a_2, a_1, x_4, x, a_4, and a_5. Only the following situations are possible:

(a) The vertex b is adjacent to x_4. If $x_3 \in IS'''$ then $IS''' \cup \{b\} \setminus \{x_5\}$ is a m.i.s. for G. If $x_3 \notin IS'''$ then $IS''' \cup \{x_4\} \setminus \{x_5\}$ is a m.i.s. for G.

(b) The degree of x_4 is equal to 2 or x_4 is adjacent to a_5. If none of the vertices in a neighborhood of x_4 (except x_5) belongs to IS''' then $IS''' \cup \{x_4\} \setminus \{x_5\}$ is a m.i.s. for G. If $x_3 \in IS'''$ then $IS''' \cup \{b\} \setminus \{x_5\}$ is a m.i.s. for G. If $(x_4, a_5) \in E(G)$ and $x_3 \notin IS'''$ while $a_5 \in IS'''$ then $x_2 \in IS'''$ (by the maximality of IS'''), and hence $IS''' \cup \{x_3, b\} \setminus \{x_2, x_5\}$ is a m.i.s. for G.

(c) The vertex x_4 is adjacent to $x \notin V(A_k)$ that is adjacent to a_4. We may assume that $x_3 \notin IS'''$ and $x \in IS'''$, $x_2 \in IS'''$ (otherwise, one of the sets

$$IS''' \cup \{b\} \setminus \{x_5\}, \qquad IS''' \cup \{x_4\} \setminus \{x_5\}$$

would be a m.i.s. for G or IS''' would be maximal). Then the set $IS''' \cup \{x_3, b\} \setminus \{x_2, x_5\}$ is a m.i.s. for G.

Lemma 5 is proved. □

We say that a hereditary class of graphs $\mathcal{X}$ is a *class without adjoint cycles* if, for its arbitrary graph and every apple A_k ($k \geq 10$) in this graph, there is no cycle adjoint to this graph. The main result of this section is

Theorem 1. *Problem* IS *for graphs of the class* $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ *is polynomially reduced to the same problem for its hereditary part without adjoint cycles.*

Proof. Problem IS for the graphs in $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ is polynomially reduced to its C-blocks. Let G be a connected graph without separating cliques of class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ and let x be its vertex. In time polynomial in the number of vertices of G, we can check whether G contains the apple A_k as an induced subgraph for which $k \geq 10$, the vertex x has degree 3 in this apple, and there is a cycle c adjoint to A_k. To this end, by direct exhaustion, we can find all cycles of length 5 in G (their number is linear with respect to the number of vertices) and choose among these those containing x.

Let $C' = (x, y_1, y_2, y_3, y_4)$ be one of such cycles. Remove from G the vertices y_1, y_3, y_4 and their neighborhoods (y_1, y_2, y_4 and their neighborhoods) and obtain a graph G_1 (G_2). Clearly, in G, the cycle C' is adjoint to some apple if and only if in G_1 there is an induced path of length at least 8 joining x and y_2 or in G_2 there is an induced path of such length joining x and y_3. The check of the existence of such a path can be done as follows: For definiteness, consider G_1. Let $v \in V(G_1)$; denote by $N_k(v)$ the set of the vertices in G_1 situated at distance $k \geq 0$ from v. Clearly,

$$|N_0(v)| = 1, \qquad |N_k(v)| \leq 3 \cdot 2^{k-1},$$

$$\left|\bigcup_{i=0}^{k} N_i(v)\right| \leq 1 + 3 + 3 \cdot 2 + 3 \cdot 2^2 + \dots + 3 \cdot 2^{k-1} = 3 \cdot 2^k - 2$$

for each $k \geq 1$. Consider the subgraph G_1 induced by the set of vertices

$$\bigcup_{k=0}^{4} (N_k(x) \cup N_k(y_2)).$$

This graph contains at most $2 \cdot (3 \cdot 2^4 - 2) = 92$ vertices. Therefore, in time $O(1)$, we can check in this graph the existence of an induced path from x to y_2 of length at least 8. Suppose that the graph has no such path. Then, in G_1, the induced path between x and y_2 is of length at least 8 exists if and only if there exist induced paths P_1 from x to $z_1 \in N_3(x)$ and P_2 from y_2 to $z_2 \in N_3(y_2)$ such that, after removing the vertices of the paths $P_1 \setminus \{z_1\}$ and $P_2 \setminus \{z_2\}$ and their neighborhoods, z_1 and z_2 are in one connected component. The cartesian product $N_3(x)$ and $N_3(y_2)$ contains at most 22^2 elements; therefore, the search of P_1 and P_2 is executed in polynomial time. This and earlier arguments imply that the check of the existence of an adjoint cycle C (and its finding if it exists) are executed in polynomial time.

It follows from Lemmas 3–5 that there exists a vertex $y \in V(C)$ such that y does not belong to some m.i.s. for G and $\alpha(G) = \alpha(G \setminus \{y\})$. At the same time, the search of y (starting from the statements of Lemmas 3–5) is executed in polynomial time. Having implemented (in polynomial time) the destruction of the adjoint cycles and reduction to connected graphs without separating cliques sufficiently many times, we obtain a system of induced subgraphs $\{G_1, G_2, \dots, G_s\}$ (without separating cliques) in G every apple A_k ($k \geq 10$) in each of which has no cycle adjoint to this apple. Every such graph contains at least two vertices. Each vertex in G belongs to at most three graphs in $\{G_1, G_2, \dots, G_s\}$. Therefore, $2s \leq 3|V(G)|$. This implies that the above-indicated reduction holds.

Theorem 1 is proved. ☐

3. ACCORDIONS AND THEIR DESTRUCTION

Recall that, by Lemmas 1 and 2, Problem IS for the graphs in $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ is polynomially reduced to graphs in this class not containing induced glider and subgraph complementary to a simple path with five vertices. At the same time, by Theorem 1, for such graphs, the problem is polynomially reduced to C-blocks for which a sufficiently large induced apple has no adjoint cycles.

Note that the independence number of a tire with $2n$ vertices is equal to n (if n is even) or $n - 1$ (if n is odd) and the check if a graph is a tire is done in polynomial time. Note also that the independence number of a punctured tire with $2n$ vertices is equal to n and the check if a given graph is a punctured tire is executed in polynomial time. Thus, Problem IS for graphs in $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ is polynomially reduced to *simplest graphs*; i.e., to graphs of this class different from a tire and a punctured tire that are C-blocks themselves, do not contain induced subgraphs of the form indicated at the beginning of this paragraph, and for which each induced apple A_k ($k \geq 10$) has no adjoint cycles.

Lemma 6. *Let G be a simplest graph containing a graph A_k ($k \geq 10$). Then the neighborhood of a_0 without a_1 is nonempty, and, among the vertices of A_k, each vertex of this truncated neighborhood is adjacent to a_2 or to a_k but not to both the vertices simultaneously.*

Proof. Since G is a simplest graph, it is a C-block and contains no cycles adjoint to A_k. Hence, there is at least one vertex $x \neq a_1$ adjacent to a_0. The vertex x cannot be adjacent to a_3 and a_{k-1}; otherwise, either there is an adjoint cycle to A_k or the vertices x, a_1, a_0, a_{k-1}, a_{k-2}, a_3, and a_4 generate a subgraph $T_{2,2,2}$ in G. If x is adjacent neither to a_2 nor to a_k then G contains the subgraph $T_{2,2,2}$ induced by the vertices a_1, a_0, x, a_2, a_3, a_k, and a_{k-1}. Therefore, x must be adjacent to a_2 or to a_k but not to both; otherwise G would contain the glider induced by the vertices a_{k-1}, a_k, a_1, a_0, x, a_2, and a_3.

Lemma 6 is proved. □

In a simplest graph G, consider a cycle C_k of the apple A_k, $k \geq 10$. We will consider the sets $\{x_1, \dots, x_p\}$ $(p \geq 2)$ of consecutive vertices of this cycle containing a_1 and having the following properties:

(i) each vertex x_i is adjacent to $y_i \notin V(C_k)$, where all vertices y_i are different;

(ii) for each $i \in \overline{2, p-1}$, the set of vertices $V(C_k) \cup \{y_i\}$ induces an apple in G;

(iii) the graph G contains the edges $(y_1, y_2), (y_2, y_3), \dots, (y_{p-1}, y_p)$.

The family of sets with these properties is nonempty since it contains either $\{a_k, a_1\}$ or $\{a_1, a_2\}$. This follows from Lemma 6. As $\{x_1, \dots, x_p\}$, consider a set in the family under consideration with the greatest number of elements. The sets $\{x_1, \dots, x_p\}$ and $\{a_1, a_2, a_3, \dots, a_k\}$ do not coincide since G is different form a tire or a punctured tire. The maximality of the set $\{x_1, \dots, x_p\}$, Lemma 6, and the prohibition in G of $T_{2,2,2}$, a glider, and the complement to a path with 5 vertices as induced subgraphs implies that, for each $i \in \{1, p\}$, either $\deg(y_i) = 2$ or y_i is adjacent to a vertex in $V(A_k)$ situated at distance 2 from x_i. Call the subgraph of G an *accordion* if it is induced by the set $\{x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p\}$.

The vertex x_p is adjacent to a vertex $x_{p+1} \in V(C_k)$, $x_{p+1} \neq x_{p-1}$; x_{p+1} is adjacent to a vertex $x_{p+2} \in V(C_k)$, $x_{p+2} \neq x_p$; ..., the vertex x_{k-1} is adjacent to a vertex $x_k \in V(C_k)$, $x_k \neq x_{k-2}$ and x_k is adjacent to x_1.

An accordion is *of the first type* if $\deg(y_1) = \deg(y_p) = 2$; otherwise call it an *accordion of the second type*. In an accordion of the second type, y_1 is adjacent to x_{k-1} or y_p is adjacent to x_{p+2}.

Prove that Problem IS for the simplest graphs is reduced to Problem IS for the simplest graphs containing only accordions of the first type. To this end, suppose that a set of vertices

$$\{x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p\}$$

induces an accordion of the second type in G. Assume for definiteness that y_1 is adjacent to x_{k-1} (it is possible that also y_p is adjacent to x_{p+2}). Clearly, if x_k is adjacent to $y \notin \{x_1, x_2, \dots, x_k\}$ then either $(y, x_{k-2}) \in E(G)$ or $(y, x_{k-3}) \in E(G)$ (otherwise G would contain the subgraph $T_{2,2,2}$ induced by x_{k-1}, y_1, y_2, x_k, y, x_{k-2}, and x_{k-3}. From G, construct a graph G' by the following rules:

(P1) if $\deg(x_k) = 2$ or x_k is adjacent to a vertex $y \notin \{x_1, x_2, \dots, x_k\}$ and $(y, x_{k-3}) \in E(G)$ then G' is obtained from G by removing the edge (x_{k-1}, y_1);

(P2) if x_k is adjacent to a vertex y not lying in $\{x_1, x_2, \dots, x_k\}$ and the edges $(y, x_{k-2}) \in E(G)$ and $(y, x_{k-3}) \notin E(G)$ then G' is obtained from G by removing the vertices x_{k-1}, x_k, x_1, y_1 and adding the edges (x_{k-2}, y_2) and (y, x_2).

Lemma 7. *If G' is obtained from G by* (P1) *then these graphs have the same independence numbers; moreover, $G' \in \mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$.*

Proof. Observe that G' is s skeleton subgraph in G. Therefore, $\alpha(G) \leq \alpha(G')$. Prove the reverse inequality. Let IS be a m.i.s. for G'. If the vertices y_1 and x_{k-1} do not belong to IS simultaneously then IS is a m.i.s. for G. If $y_1 \in IS$ and $x_{k-1} \in IS$ then consider the degree of the vertex x_k. If $\deg(x_k) = 2$ then $IS \cup \{x_k\} \setminus \{x_{k-1}\}$ is a m.i.s. for G. If $\deg(x_k) = 3$ (i.e., $(y, x_k) \in E(G)$) and there exists a vertex z adjacent to x_{k-2} and different from x_{k-1} and x_{k-3} then either $z = y$ or z is adjacent to y. Otherwise, the vertices x_{k-1}, x_{k-2}, z, x_k, y, y_1, and y_2 would induce a subgraph $T_{2,2,2}$ in G. Then $IS \cup \{x_k\} \setminus \{x_{k-1}\}$ (if $y \notin IS$) or $IS \cup \{x_k, x_{k-2}\} \setminus \{x_{k-1}, y\}$ (if $y \in IS$) is a m.i.s. for G. Thus, in both cases, there is an independent set in G with the number of vertices equal to $\alpha(G')$. Therefore, $\alpha(G) \geq \alpha(G')$. Hence, the independence numbers of G' and G are equal.

Show that $G' \in \mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$. For this, it suffices to check that G' has no induced subgraphs $T_{2,2,2}$ containing x_{k-1} and y_1. Suppose that such a subgraph exists. Let z^* denote its vertex of degree 3. Obviously, z^* belongs to the intersection of the sets

$$N_1(y_1) \cup N_2(y_1), \qquad N_1(x_{k-1}) \cup N_2(x_{k-1}).$$

Furthermore,

$$N_1(y_1) \cup N_2(y_1) \subseteq \{x_1, x_2, x_k, y_2, y_3\},$$

$$N_1(x_{k-1}) \cup N_2(x_{k-1}) \subseteq \{x_{k-3}, x_{k-2}, x_k, x_1, x, y\},$$

where $x \notin V(C_k)$ is a vertex adjacent to x_{k-2}. Since

$$\{x_1, x_2, x_k, y_2, y_3\} \cap \{x_{k-3}, x_{k-2}, x_k, x_1, x, y\} = \{x_1, x_k\},$$

this implies that $z^* \in \{x_1, x_k\}$. If $z^* = x_1$ then the formation of an induced subgraph $T_{2,2,2}$ is impossible because it is prevented by the edge (x_2, y_2). If $z^* = x_k$ then a subgraph $T_{2,2,2}$ can be induced by vertices x_k, x_1, y_1, x_{k-1}, x_{k-2}, y, and z', where $(z', y) \in E(G')$ and $z' \notin \{x_{k-3}, x_k\}$, which is impossible since for $z' \neq x_3$ the vertices x_k, x_1, x_2, x_{k-1}, x_{k-2}, y, and z' induce a subgraph $T_{2,2,2}$ in G, and for $z' = x_3$ the vertices x_3, x_2, x_1, y, x_{k-3}, x_4, and x_5 induce such a subgraph in G. Therefore, also in the second case an induced subgraph $T_{2,2,2}$ does not appear.

Lema 7 is proved. □

Lemma 8. *If G' is obtained from G by* (P2) *then*

$$\alpha(G) = \alpha(G') + 2, \qquad G' \in \mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\}).$$

Proof. Prove there exists a m.i.s. for G containing at least one of the vertices x_1 and y_1. Obviously, if some m.i.s. IS of this graph contains none of these vertices then, by the maximality of IS, at least one of the vertices in $\{x_{k-1}, y_2\}$ and at least one of the vertices in $\{x_k, x_2\}$ belong to this set. At the same time, in each of these sets, exactly one vertex possesses this property. Therefore, if $x_{k-1} \in IS$ then $IS \cup \{y_1\} \setminus \{x_{k-1}\}$ is a m.i.s. for G and if $y_2 \in IS$ then $IS \cup \{y_1\} \setminus \{y_2\}$ is also m.i.s. for G. Hence, there is always a m.i.s. for G containing either x_1 or y_1.

Prove now that there exists a m.i.s. for G containing the vertices x_1, x_{k-1}, y, y_2 or the vertices x_{k-2}, x_k, y_1, x_2. Suppose for definiteness that y_1 belongs to some m.i.s. IS' in G. We may assume that $x_k \in IS'$ (it is clear that, by the maximality of this independent set, either x_k or y belongs to it; if $y \in IS'$ the it can be replaced with x_k). A similar argument leads to the conclusion that x_{k-2} and x_2 belong to IS'. It is proved similarly that if $x_1 \in IS'$ then $\{x_1, x_{k-1}, y, y_2\} \subseteq IS'$.

Note that $IS' \setminus \{x_k, y_1\}$ is an i.s. for G'. If IS' contains x_1, x_{k-1}, y, y_2 then $IS' \setminus \{x_{k-1}, x_1\}$ is an i.s. for G'. Hence, $\alpha(G') \geq \alpha(G) - 2$. Prove the reverse inequality. By analogy with the previous arguments, we can prove that there exists a m.i.s. IS'' for G' containing either the vertices x_{k-2} and x_2 or y and y_2. In the first case $IS'' \cup \{x_k, y_1\}$ is an i.s. for G, and in the second, so is $IS'' \cup \{x_{k-1}, x_1\}$. Therefore, $\alpha(G) \geq \alpha(G') + 2$. These inequalities imply that

$$\alpha(G) = \alpha(G') + 2.$$

Check that the graph G' belongs to the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$. Clearly, $G \in \mathcal{P}(3)$. Suppose that G contains an induced subgraph H isomorphic to $T_{2,2,2}$ with a vertex z of degree 3. Clearly, H contains exactly one of the edges (y, x_2), (x_{k-2}, y_2). It is also clear that $z \notin \{x_2, y_2\}$. Therefore, only the following cases are possible:

Case 1: $z = y$ and H contains the simple path $P = (y, x_2, x_3)$.

Case 2: $z = x_{k-2}$ and H contains the simple path $P = (x_{k-2}, y_2, y_3)$.

Case 3: $z = x_{k-3}$ and H contains the simple path $P = (x_{k-3}, x_{k-2}, y_2)$.

Case 4: $z = x_3$ and H contains the simple path $P = (x_3, x_2, y)$.

Case 5: $z = y_3$ and H contains the simple path $P = (y_3, y_2, x_{k-2})$.

Case 6: H contains the simple path $P = (z, y, x_2)$.

In each of the cases, there is an induced subgraph $T_{2,2,2}$ in G obtained from H as follows: in Cases 1–2, by replacing the path P with the path (y, x_k, x_1) or with (x_{k-2}, x_{k-1}, y_1); and in Cases 3–6, by replacing the edges (x_{k-2}, y_2), (x_2, y), (y_2, x_{k-2}), and (y, x_2) with the edges (x_{k-2}, x_{k-1}), (x_2, x_1), (y_2, y_1), and (y, x_k); a contradiction. Hence, $G' \in \mathcal{F}ree(\{T_{2,2,2}\})$.

Lemma 8 is proved. □

Lemma 9. *Let G be a simplest graph containing an accordion of the first type induced by the vertices $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_p$. Then*

$$\alpha(G) = \begin{cases} \alpha(G \setminus \{x_1\}) & \text{if } p \text{ odd,} \\ \alpha(G \setminus \{y_1, y_2, \ldots, y_p\}) + p/2 & \text{if } p \text{ even.} \end{cases}$$

Proof. Consider the case of p odd. If there is a m.i.s. for G not containing x_1 then $\alpha(G) = \alpha(G \setminus \{x_1\})$. If there is a m.i.s. IS in G containing x_1 then

$$IS \cup \{y_1\} \setminus \{x_1\}, \quad \text{or} \quad IS \cup \{y_1, x_2\} \setminus \{x_1, y_2\},$$
$$\text{or} \quad IS \cup \{y_1, x_2, y_3\} \setminus \{x_1, y_2, x_3\}, \ \ldots, \quad \text{or} \quad IS \cup \{y_1, x_2, y_3, x_4, \ldots, y_p\} \setminus \{x_1, y_2, x_3, y_4, \ldots, x_p\}$$

is a m.i.s. for G. Therefore, in all possible cases,

$$\alpha(G) = \alpha(G \setminus \{x_1\}).$$

Let p be even. Denote the graph $G \setminus \{y_1, y_2, \ldots, y_p\}$ by G'. Obviously, there exists a m.i.s. IS for G' containing exactly one half of the vertices $x_1, x_2, \ldots, x_p$. This implies that there is a m.i.s. for G containing exactly $\alpha(G') + p/2$ vertices (it is obtained by adding $p/2$ vertices from the set $\{y_1, y_2, \ldots, y_p\}$ nonadjacent to the vertices in $\{x_1, x_2, \ldots, x_p\} \cap IS$). Therefore, $\alpha(G) \geq \alpha(G') + p/2$. It is also not hard to validate the reverse inequality, which follows from the fact that any m.i.s. for G contains at most one half of the vertices $y_1, y_2, \ldots, y_p$. Comparing both inequalities, we infer that

$$\alpha(G) = \alpha(G \setminus \{y_1, y_2, \ldots, y_p\}) + p/2.$$

Lemma 9 is proved. □

Theorem 2. *The* IS *problem for the graphs of the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ is polynomially reduced to the* IS *problem the for graphs of the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{A_{10}, A_{11}, \ldots\})$.*

Proof. In proving Theorem 1, we have showed how to check in polynomial time whether a vertex x of degree 3 in an arbitrary C-block in $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ is a vertex of degree 3 of some of its induced apples A_k, $k \geq 10$ (and find this apple if it exists). If there is no cycle adjoint to this apple and the graph itself is simplest then there exists an accordion (of the first or second type) containing x. Clearly, this accordion can be determined in polynomial time.

The reduction rules formulated in this and previous sections imply that Problem IS for the graphs in $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$ is reduced to the same problem for simplest graphs in

$$\mathcal{P}(3) \cap \mathcal{F}ree(\{A_{10}, A_{11}, \ldots\}).$$

At each reduction iteration, one of the following actions is implemented for one of the graphs:

(1) distinguishing C-blocks;

(2) removal of a vertex;

(3) check if the graph is isomorphic to a tire or a punctured tire;

(4) destruction of an accordion.

Every action is executed in polynomial time. At each iteration, we consider the quantity $7n_3 + 3n_2 + n_1 + n_0$, where n_i is the total number of vertices of degree i in the current graphs. Show that the application of operations (1)–(4) decreases this quantity. For (2), (3), and for the destruction of an accordion of the first type in (4), this is obvious. It is easy to see that the destruction of an accordion by (P1) decreases this quantity by 8, and its destruction by (P2), by 28. Note that if $G \in \mathcal{P}(3)$ and a vertex v belongs to more than one C-block in G then:

(a) either the degree of v is equal to 3 and v belongs to two C-blocks in G in each of which it has degree at most 2;

(b) or the degree of v is equal to 3 and v belongs to three C-blocks in G in each of which it has degree 1;

(c) or the degree of v is equal to 2 and v belongs to two C-blocks in G in each of which it has degree 1.

With these cases taken into account, it is easy to verify that then sum under consideration decreases after distinguishing C-blocks. Thus, at any moment of the process, the weighted sum of the degrees of the vertices does not exceed $7|V(G)|$, which gives the upper bound $7|V(G)|$ for the number of iterations. Therefore, the reduction is also polynomial.

Theorem 2 is proved. □

4. THE MAIN RESULTS

Theorem 3. *For each $i > 2$, Problem IS for the graphs of the class $\mathcal{D}(3) \cap \mathcal{F}ree(\{T_{2,2,i}\})$ is polynomially reduced to the same problem for graphs in $\mathcal{D}(3) \cap \mathcal{F}ree(\{T_{2,2,2}\})$.*

Proof. Let us first demonstrate that, for every connected graph $G \in \mathcal{D}(3) \cap \mathcal{F}ree(\{T_{2,2,i}\})$, either $G \in \mathcal{F}ree(\{T_{2,2,2}\})$ or G has at most $3 \cdot 2^{i+2} - 2$ vertices.

Suppose that G contains an induced subgraph $T_{2,2,2}$ such that $\deg(x) = 3$ and x is adjacent to y_k and y_k is adjacent to z_k for each $k \in \overline{1,3}$. Prove that G does not contain a vertex y situated at distance $i + 3$ from x. Suppose the contrary and consider the shortest path from x to y:

$$P = (x_1 = x, x_2, x_3, \dots, x_{i+4} = y).$$

Clearly, $x_2 \in \{y_1, y_2, y_3\}$. Without loss of generality, we may assume that $x_2 = y_1$. By the choice of P, each of the vertices y_2 and y_3 can be adjacent to none of the vertices x_j with $j > 3$. For the same reasons, none of the vertices z_1, z_2, z_3 can be adjacent to x_j for which $j > 4$. The two cases are possible: $x_3 \neq z_1$ or $x_3 = z_1$.

Case 1: $x_3 \neq z_1$. If z_1 and x_4 are adjacent then the vertices x, y_1, z_1, y_2, z_2, y_3, z_3, x_4,$x_5, \dots, x_{i+1}$ generate a subgraph $T_{2,2,i}$ in G. Therefore, we may assume that $(z_1, x_4) \notin E(G)$. If none of the vertices y_2, y_3, z_2, and z_3 is adjacent to x_3 or x_4 then G contains the subgraph $T_{2,2,i}$ induced by the vertices x_1, y_2, y_3, z_2, z_3, x_2, $x_3, \dots, x_{i+1}$. If the vertices y_2, y_3, z_2, and z_3 include a vertex y_k or z_k adjacent to x_3 ($k \in \{2, 3\}$, such a vertex is unique) and none of them is adjacent to x_4 then G contains the subgraph $T_{2,2,i}$ induced by the vertices y_k, z_k, x_2, x_3, z_1, $x_4, \dots, x_{i+3}$. If the vertices y_2, y_3, z_2, and z_3 include a vertex z_k adjacent to x_4 (by the choice of P, such a vertex can belong only to the set $\{z_2, z_3\}$) then x, y_1, z_1, y_2, z_2, y_3, z_3, x_4, $x_5, \dots, x_{i+1}$ induce a subgraph $T_{2,2,i}$.

Case 2: $x_3 = z_1$. If none of the vertices z_2 and z_3 is adjacent to x_4 then G contains the subgraph $T_{2,2,i}$ induced by $x_1, y_2, z_2, y_3, z_3, x_2, x_3, \dots, x_{i+1}$. If there is a vertex z_k ($k \in \{2, 3\}$) adjacent to x_4 (such a vertex is unique) then $y_k, z_k, x_2, x_3, x_4, \dots, x_{i+4}$ induce a subgraph $T_{2,2,i}$ in G.

Thus, there is no vertex y. This means that if G contains an induced subgraph $T_{2,2,2}$ than each of its vertices is at distance at most $i + 2$ from x. Hence,

$$|V(G)| \leqslant 1 + 3 + 3 \cdot 2 + \dots + 3 \cdot 2^{i+1} = 3 \cdot 2^{i+2} - 2.$$

Therefore, there are only finitely many connected graphs in $\mathcal{D}(3) \cap \mathcal{F}ree(\{T_{2,2,i}\}) \setminus \mathcal{F}ree(\{T_{2,2,2}\})$ for each fixed i. This yields the reduction mentioned in the statement of the theorem.

Theorem 3 is proved. □

The main result of the article is as follows:

Theorem 4. *For each fixed i, the* IS *problem for graphs of the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,i}\})$ is polynomially solvable.*

Proof. It follows from Theorems 2 and 3 that Problem IS for the graphs in $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,i}\})$ is polynomially reduced to the same problem for graphs in $\mathcal{P}(3) \cap \mathcal{F}ree(\{A_{10}, A_{11}, \ldots\})$. In [4], it was shown that, for each fixed k, the class $\mathcal{P} \cap \mathcal{F}ree(\{A_k, A_{k+1}, \ldots\})$ is IS-easy. Therefore, for each fixed i, the class $\mathcal{P}(3) \cap \mathcal{F}ree(\{T_{2,2,i}\})$ is IS-easy, too.

Theorem 4 is proved. □

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