TWISTED GEOMETRIC SATAKE EQUIVALENCE

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ABSTRACT. Let k be an algebraically closed field and $\mathbf{O} = \mathsf{k}[[t]] \subset \mathbf{F} = \mathsf{k}((t))$. For an almost simple algebraic group G we classify central extensions $1 \to \mathbb{G}_m \to E \to G(\mathbf{F}) \to 1$; any such extension splits canonically over $G(\mathbf{O})$. Fix a positive integer N and a primitive character $\zeta : \mu_N(\mathsf{k}) \to \overline{\mathbb{Q}}_\ell^*$ (under some assumption on the characteristic of k). Consider the category of $G(\mathbf{O})$-biinvariant perverse sheaves on E with $\mathbb{G}_m$-monodromy ζ. We show that this is a tensor category, which is tensor equivalent to the category of representations of a reductive group $\check{G}_{E,N}$. We compute the root datum of $\check{G}_{E,N}$.

CONTENTS

1. Introduction 1
2. Main result 2
2.1. Notations 2
2.2. Convolution 6
2.3. Fusion 6
2.4. Main result 9
3. Classification of central extensions 10
3.1. Simply-connected case 10
3.2. Proof of Proposition 1 12
4. Proof of Theorem 1 13
4.1. Functors F'_P 13
4.2. Fibre functor 17
4.3. Structure of $\check{G}_N$ 19
4.4. Rank one 20
References 21

1. INTRODUCTION

Let k be an algebraically closed field and $\mathbf{O} = \mathsf{k}[[t]] \subset \mathbf{F} = \mathsf{k}((t))$. For an almost simple algebraic group G we classify central extensions $1 \to \mathbb{G}_m \to E \to G(\mathbf{F}) \to 1$; any such extension splits canonically over $G(\mathbf{O})$. Fix a positive integer N and a primitive character $\zeta : \mu_N(\mathsf{k}) \to \overline{\mathbb{Q}}_\ell^*$ (under some assumption on the characteristic of k). Consider the category of $G(\mathbf{O})$-biinvariant perverse sheaves on E with $\mathbb{G}_m$-monodromy ζ. We show that this is a tensor category, which is tensor equivalent to the category of representations of a reductive group $\check{G}_{E,N}$. We compute the root datum of $\check{G}_{E,N}$ in Theorem 1. A list of examples is given after Theorem 1.

Our result has a natural place in the framework of "Langlands duality for quantum groups" [6]. Namely, if we take $\mathsf{k} = \mathbb{C}$, and $q = \zeta(\exp(\frac{\pi i}{N}))$ in Conjecture 0.4 of *loc. cit.*, then our category of ζ-monodromic perverse sheaves naturally lies inside the twisted Whittaker sheaves $\mathrm{Whit}^c(\mathrm{Gr}_G)$, and corresponds under the equivalence of *loc. cit.* to the category of representations of the quantum Frobenius quotient of $U_q(\check{G})$.

From the physical point of view, our result is a manifestation of electric-magnetic duality for a *rational* parameter Ψ, see [7], Section 11.3. Theorem 1 is a generalization of ([9], Theorem 3) and the classical geometric Satake equivalence [10]. It was probably known to experts for a few years, say it was suggested by an anonymous referee of [9] (compare also to [11]). Also, for G simply connected the root data of $\check{G}_{E,N}$ appeared in Section 7 of [8]. Our result should follow essentially by comparing Lusztig's results on quantum Frobenius homomorphism on the one hand, and Kazhdan-Lusztig-Kashiwara-Tanisaki-Arkhipov-Bezrukavnikov-Ginzburg description of representations of quantum groups at roots of unity in terms of perverse sheaves on affine Grassmanians, on the other. Our goal is to provide a short self-contained proof, following the strategy of [10].

We are obliged to V. Drinfeld who explained to us the classification of central extensions of the loop groups for almost simple groups (not necessarily simply connected), see Proposition 1. We are also indebted to R. Bezrukavnikov and D. Gaitsgory for useful duscussions. M.F. is grateful to University Paris 6 for hospitality and support; he was partially supported by the RFBR grant 09-01-00242 and the Science Foundation of the SU-HSE awards No.09-08-0008 and 09-09-0009.

2. Main result

The general reference for this section is [1] 4.5, 5.3 and, for more details, [10].

2.1. **Notations.** Let k be an algebraically closed field of characteristic $p \geq 0$. Let G be an almost simple algebraic group over k with the simply connected cover G^{sc}. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote the preimage of T in G^{sc} by T^{sc}. The Weyl group of G, T is denoted by W. The weight and coweight lattices of T (resp. T^{sc}) are denoted by $X^*(T)$ and $X_*(T)$ (resp. $X^*(T^{\mathrm{sc}})$ and $X_*(T^{\mathrm{sc}})$). The root system of $T \subset B \subset G$ is denoted by $R^* \subset X^*(T) \subset X^*(T^{\mathrm{sc}})$; the set of simple roots is $\Pi^* = \{\check{\alpha}_1, \dots, \check{\alpha}_r\} \subset R^*$. The sum of all positive roots is denoted $2\check{\rho}$. The coroot system of $T \subset B \subset G$ is denoted by $R_* \subset X_*(T^{\mathrm{sc}}) \subset X_*(T)$; the set of simple coroots is $\Pi_* = \{\alpha_1, \dots, \alpha_r\} \subset R_*$. Denote by $\mathfrak{g}$ the adjoint representation of G.

There is a unique W-invariant bilinear pairing $(,): X_*(T^{\mathrm{sc}}) \times X_*(T^{\mathrm{sc}}) \to \mathbb{Z}$ such that $(\alpha, \alpha) = 2$ for a *short* coroot α. It defines a linear map $\iota: X_*(T^{\mathrm{sc}}) \to X^*(T^{\mathrm{sc}})$ such that $(x, y) = \langle x, \iota(y)\rangle$ for any $x, y \in X_*(T^{\mathrm{sc}})$. The map ι extends uniquely by linearity to the same named map $\iota: X_*(T) \to X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The bilinear form $(,)$ extends uniquely by linearity to the same named bilinear form $(,): X_*(T) \times X_*(T) \to \mathbb{Q}$.

Lemma 1. *For $\lambda \in X_*(T)$ we have $2\check{h}\iota(\lambda) = \sum_{\check{\alpha} \in R^*} \langle \lambda, \check{\alpha}\rangle \check{\alpha}$, where $\check{h}$ is the dual Coxeter number of G.*

Proof Write $\Phi_R(.,.)$ for the canonical bilinear W-invariant linear form on $X^*(T)$ in the sense of ([2], §1, section 12). Formula (17) from *loc.cit.* reads

$$2\Phi_R(\check{\beta}, \check{\beta})^{-1} = \sum_{\check{\alpha} \in R^+} \langle \check{\alpha}, \beta\rangle^2,$$

where $\check{\beta}$ is the root corresponding to a short coroot β. We must check that $\Phi_R(\check{\beta}, \check{\beta}) = \check{h}^{-1}$. This is done case by case for all irreducible reduced root systems using the calculation of Φ_R performed in §4 of *loc.cit.*. $\square$

Set $\mathbf{O} = \mathsf{k}[[t]]$, $\mathbf{F} = \mathsf{k}((t))$. The affine Grassmannian Gr_G is an ind-scheme, the fpqc quotient $G(\mathbf{F})/G(\mathbf{O})$ (cf. [1]). Our conventions about $\mathbb{Z}/2\mathbb{Z}$-gradings are those of [9]. Recall that for free $\mathbf{O}$-modules of finite type V_1, V_2 with an isomorphism $V_1(\mathbf{F}) \xrightarrow{\sim} V_2(\mathbf{F})$ one has the relative determinant $\det(V_1 : V_2)$ ([9], Section 8.1), this is a $\mathbb{Z}/2\mathbb{Z}$-graded line given by

$$\det(V_1 : V_2) = \det(V_1/V) \otimes \det(V_2/V)^{-1}$$

for any $\mathbf{O}$-lattice $V \subset V_1 \cap V_2$.

Let $\mathfrak{L}$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle on Gr_G whose fibre at $gG(\mathbf{O})$ is $\det(\mathfrak{g}(\mathbf{O}) : \mathfrak{g}(\mathbf{O})^g)$. Write Gra_G for the punctured total space (that is, the total space with zero section removed) of the line bundle $\mathfrak{L}$. By abuse of notation, the restriction of $\mathfrak{L}$ under the map $G(\mathbf{F}) \to \mathrm{Gr}_G$, $g \mapsto gG(\mathbf{O})$ is again denoted by $\mathfrak{L}$. Write E^a for the punctured total space of the line bundle $\mathfrak{L}$ on $G(\mathbf{F})$. Since $\mathfrak{L}$ is naturally a multiplicative $\mathbb{G}_m$-torsor on $G(\mathbf{F})$ in the sense of ([3], Section 1.2), E^a gives a central extension

$$1 \to \mathbb{G}_m \to E^a \to G(\mathbf{F}) \to 1, \tag{1}$$

it splits canonically over $G(\mathbf{O})$.

Our first result extends the well-known classification of central extensions of $G^{sc}(\mathbf{F})$ by $\mathbb{G}_m$ to the almost simple case.

Proposition 1. *The isomorphism classes of central extensions of $G(\mathbf{F})$ by $\mathbb{G}_m$ are naturally in bijection with integers $m \in \mathbb{Z}$ such that $m\iota(X_*(T)) \subset X^*(T)$. The corresponding central extension splits canonically over $G(\mathbf{O})$, and its group of automorphisms is* $\mathrm{Hom}(\pi_0(G(\mathbf{F})), \mathbb{G}_m) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(G), \mu_\infty(\mathsf{k}))$.

Let $d > 0$ be the smallest integer such that $d\iota(X_*(T)) \subset X^*(T)$, this is a divisor of $2\check{h}$. We pick and denote by E^c the corresponding central extension of $G(\mathbf{F})$ by $\mathbb{G}_m$. Any central extension of $G(\mathbf{F})$ by $\mathbb{G}_m$ is isomorphic to a multiple of E^c. We also pick an isomorphism of central extensions of $G(\mathbf{F})$ by $\mathbb{G}_m$ identifying E^a with the $(2\check{h}/d)$-th power of E^c. (If G is simply-connected then $d = 1$, and the latter isomorphism is uniquely defined).

Fix a prime ℓ different from p, write $\mathrm{P}(S)$ (resp., $\mathrm{D}(S)$) for the category of étale $\overline{\mathbb{Q}}_\ell$-perverse sheaves (resp., the derived category of étale $\overline{\mathbb{Q}}_\ell$-sheaves) on a k-scheme (or stack) S. Since we are working over an algebraically closed field, we systematically ignore the Tate twists.

Fix a positive integer N and a primitive character $\zeta : \mu_N(\mathsf{k}) \to \overline{\mathbb{Q}}_\ell^*$, we assume that p does not divide $2\check{h}N/d$. For the map $s_N : \mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^N$ let $\mathcal{L}^\zeta$ denote the direct summand of $s_{N!}\overline{\mathbb{Q}}_\ell$ on which $\mu_N(\mathsf{k})$ acts by ζ. If S is a scheme with a $\mathbb{G}_m$-action, we say that a perverse sheaf K on S has $\mathbb{G}_m$-monodromy ζ if it is equipped with a $(\mathbb{G}_m, \mathcal{L}^\zeta)$-equivariant structure.

Let $\mathrm{Perv}_{G,N}$ denote the category of $G(\mathbf{O})$-equivariant $\overline{\mathbb{Q}}_\ell$-perverse sheaves on $E^c/G(\mathbf{O})$ with $\mathbb{G}_m$-monodromy ζ.

Remark 1. *If S is a scheme and L is a line bundle on S, let $\tilde{L}$ be the total space of the punctured line bundle L. Let $f : \tilde{L} \to \widetilde{L^m}$ be the map over S sending l to $l^{\otimes m}$. Assume that p does not divide N, let $\chi : \mu_N(\mathsf{k}) \to \overline{\mathbb{Q}}_\ell^*$ be a character. Then the functor $K \mapsto f^*K$ is an equivalence between the categories of χ-monodromic perverse sheaves on $\widetilde{L^m}$ and χ^m-monodromic perverse sheaves on $\tilde{L}$.*

Pick a primitive character $\zeta_a : \mu_{2\check{h}N/d}(\mathsf{k}) \to \overline{\mathbb{Q}}_\ell^*$ satisfying $\zeta_a^{2\check{h}/d} = \zeta$. By Remark 1, $\mathrm{Perv}_{G,N}$ identifies with the category of $G(\mathbf{O})$-equivariant perverse sheaves on Gra_G with $\mathbb{G}_m$-monodromy ζ_a. Set

$$\mathbb{P}\mathrm{erv}_{G,N} = \mathrm{Perv}_{G,N}[-1] \subset \mathrm{D}(\mathrm{Gra}_G)$$

For the associativity and commutativity constraints we are going to inroduce later to be natural (and avoid some sign problems), one has to work with $\mathbb{P}\mathrm{erv}_{G,N}$ rather than $\mathrm{Perv}_{G,N}$. Let $\widetilde{\mathrm{Gr}}_G$ be the stack quotient of Gra_G by $\mathbb{G}_m$, where $x \in \mathbb{G}_m$ acts as multiplication by $x^{2\check{h}N/d}$. One may think of $\mathbb{P}\mathrm{erv}_{G,N}$ as the category of certain perverse sheaves on $\widetilde{\mathrm{Gr}}_G$.

We have a natural embedding $X_*(T) \subset \mathrm{Gr}_G$ as the set of T-fixed points. Two coweights $\lambda, \mu \in X_*(T) \subset \mathrm{Gr}_G$ lie in the same $G(\mathbf{O})$-orbit iff $\lambda \in W(\mu)$. Thus the set of $G(\mathbf{O})$-orbits in Gr_G coincides with the set of Weyl group orbits $X_*(T)/W$, or equivalently, with the set of dominant coweights $X_*^+(T) \subset X_*(T)$. The orbit corresponding to $\lambda \in X_*^+(T)$ will be denoted by Gr^λ. The G-orbit of λ is isomorphic to a partial flag variety $\mathcal{B}^\lambda = G/P^\lambda$ where P^λ is a parabolic subgroup whose Levi has the Weyl group $W^\lambda \subset W$ coinciding with the stabilizer of λ in W. Write Gra_G^λ for the preimage of Gr_G^λ under $\mathrm{Gra}_G \to \mathrm{Gr}_G$.

Let $\mathrm{Aut}(\mathbf{O})$ denote the group ind-scheme over k such that, for any k-algebra R, $\mathrm{Aut}(\mathbf{O})(R)$ is the automorphism group of the topological R-algebra $R\hat{\otimes}\mathbf{O}$ ([1], 2.6.5). Write $\mathrm{Aut}^0(\mathbf{O})$ for the reduced part of $\mathrm{Aut}(\mathbf{O})$. The group scheme $\mathrm{Aut}^0(\mathbf{O})$ acts naturally on the exact sequence (1) acting trivially on $\mathbb{G}_m$ and preserving $G(\mathbf{O})$.

The action of the loop rotation group $\mathbb{G}_m \subset \mathrm{Aut}^0(\mathbf{O})$ contracts Gr^λ to $\mathcal{B}^\lambda \subset \mathrm{Gr}_\lambda$, and realizes Gr^λ as a composition of affine fibrations over $\mathcal{B}^\lambda$. We denote the projection $\mathrm{Gr}^\lambda \to \mathcal{B}^\lambda$ by ϖ_λ.

If $\check{\nu} \in X^*(T^{\mathrm{sc}})$ is orthogonal to all coroots α satisfying $\langle \lambda, \check{\alpha}\rangle = 0$ then write $\mathcal{O}(\check{\nu})$ for the corresponding G^{sc}-equivariant line bundle on $\mathcal{B}^\lambda$. It is canonically trivialized at $1 \in \mathcal{B}^\lambda$. If $\lambda \in X_*(T)$ then $\iota(\lambda) \in X^*(T^{\mathrm{sc}})$ gives rise to the line bundle $\mathcal{O}(\iota(\lambda))$ on $\mathcal{B}^\lambda$.

For a free $\mathbf{O}$-module $\mathcal{E}$ write $\mathcal{E}_{\bar{c}}$ for its geometric fibre. Write Ω for the completed module of relative differentials of $\mathbf{O}$ over k.

For a root $\check{\alpha}$ write $\mathfrak{g}^{\check{\alpha}} \subset \mathfrak{g}$ for the corresponding root subspace. If $\check{\alpha} = \sum_{i=1}^r a_i\check{\alpha}_i$ we have $\mathfrak{g}^{\check{\alpha}} \widetilde{\to} \otimes_{i=1}^r (\mathfrak{g}^{\check{\alpha}_i})^{a_i}$ canonically. In particular, $\mathfrak{g}^{-\check{\alpha}}$ is identified with the dual of $\mathfrak{g}^{\check{\alpha}}$ via the Killing form.

Pick a trivialization $\phi_i : \mathfrak{g}^{\check{\alpha}_i} \widetilde{\to} \mathsf{k}$ for each simple root $\check{\alpha}_i \in \Pi^*$. Set $\Phi = \{\phi_i\}_{i=1}^r$.

Lemma 2. *Let $\lambda \in X_*(T)$. The family Φ yields a uniquely defined $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism*

$$\mathfrak{L} \mid_{\mathrm{Gr}_G^\lambda} \simeq \Omega_{\bar{c}}^{\check{h}\langle\lambda,\iota(\lambda)\rangle} \otimes \varpi_\lambda^*\mathcal{O}(2\check{h}\iota(\lambda))$$

Set $X^{*+}(\check{T}_N) = \{\lambda \in X_*^+(T) \mid d\iota(\lambda) \in NX^*(T)\}$. By Lemma 2, for $\lambda \in X_*^+(T)$ the scheme Gra_G^λ admits a $G(\mathbf{O})$-equivariant local system with $\mathbb{G}_m$-monodromy ζ_a iff $\lambda \in X^{*+}(\check{T}_N)$.

Remark 2. *In ([1], Section 4.4.9, p. 166, formula (217)) an extension of $G(\mathbf{F})$ by $\mathbb{G}_m$ has been constructed whose square identifies with (1). So, d is a divisor of $\check{h}$. Another way to see this is to note that, by Lemma 1, for $\lambda \in X_*(T)$ we have $\check{h}\iota(\lambda) = \sum_{\check{\alpha}\in R^{*+}} \langle\lambda,\check{\alpha}\rangle\check{\alpha} \in X^*(T)$. Here R^{*+} denotes the set of positive roots.*

Write $\Omega^{\frac{1}{2}}(\mathbf{O})$ for the groupoid of square roots of Ω. For $\mathcal{E} \in \Omega^{\frac{1}{2}}(\mathbf{O})$ and $\lambda \in X^{*+}(\check{T}_N)$ define the line bundle $\mathcal{L}_{\lambda,\mathcal{E}}$ on Gr_G^λ as

$$\mathcal{L}_{\lambda,\mathcal{E}} = \mathcal{E}_{\bar{c}}^{\frac{d}{N}(\lambda,\lambda)} \otimes \varpi_\lambda^*\mathcal{O}(\frac{d}{N}\iota(\lambda))$$

It is equipped with an isomorphism $\mathcal{L}_{\lambda,\mathcal{E}}^{2\check{h}N/d} \widetilde{\to} \mathfrak{L} \mid_{\mathrm{Gr}_G^\lambda}$. Write $\mathring{\mathcal{L}}_{\lambda,\mathcal{E}}$ for the punctured total space of $\mathcal{L}_{\lambda,\mathcal{E}}$. Denote by

$$p_\lambda : \mathring{\mathcal{L}}_{\lambda,\mathcal{E}} \to \mathrm{Gra}_G^\lambda \tag{2}$$

the map over Gr_G^λ sending x to $x^{2\check{h}N/d}$.

For $\lambda \in X^{*+}(\check{T}_N)$ we define $\mathcal{A}^\lambda_\mathcal{E} \in \mathbb{P}\mathrm{erv}_{G,N}$ as the intermediate extension of $E^\lambda_\mathcal{E}[-1 + \dim \mathrm{Gra}^\lambda_G]$. Here $E^\lambda_\mathcal{E}$ is the local system on Gra^λ_G with $\mathbb{G}_m$-monodromy ζ_a equipped with an isomorphism $p^*_\lambda E^\lambda_\mathcal{E} \widetilde{\to} \overline{\mathbb{Q}}_\ell$. Both $E^\lambda_\mathcal{E}$ and $\mathcal{A}^\lambda_\mathcal{E}$ are defined up to a unique isomorphism. The irreducible objects of $\mathbb{P}\mathrm{erv}_{G,N}$ are exactly $\mathcal{A}^\lambda_\mathcal{E}$, $\lambda \in X^{*+}(\check{T}_N)$.

As in ([9], Proposition 11) one shows that each $\mathcal{A}^\lambda_\mathcal{E}$ has nontrivial usual cohomology sheaves only in degrees of the same parity, and derives from this that $\mathbb{P}\mathrm{erv}_{G,N}$ is semi-simple.

2.2. **Convolution.** Consider the automorphism τ of $E^a \times E^a$ sending (g, h) to (g, gh). Let $G(\mathbf{O}) \times G(\mathbf{O}) \times \mathbb{G}_m$ act on $E^a \times E^a$ in such a way that $(\alpha, \beta, b) \in G(\mathbf{O}) \times G(\mathbf{O}) \times \mathbb{G}_m$ send (g, h) to $(g\beta^{-1}b^{-1}, \beta bh\alpha)$. Write $E^a \times_{G(\mathbf{O})\times\mathbb{G}_m} \mathrm{Gra}_G$ for the quotient of $E^a \times E^a$ under this free action. Then τ induces an isomorphism

$$\bar\tau : E^a \times_{G(\mathbf{O})\times\mathbb{G}_m} \mathrm{Gra}_G \widetilde{\to} \mathrm{Gr}_G \times \mathrm{Gra}_G$$

sending $(g, hG(\mathbf{O}))$ to $(\bar g G(\mathbf{O}), ghG(\mathbf{O}))$, where $\bar g$ is the image of $g \in E^a$ in $G(F)$. Set m be the composition of $\bar\tau$ with the projection to Gra_G. Let $p_G : E^a \to \mathrm{Gra}_G$ be the map $h \mapsto hG(\mathbf{O})$. Similarly to ([10], [9]), we get a diagram

$$\mathrm{Gra}_G \times \mathrm{Gra}_G \stackrel{p_G\times\mathrm{id}}{\longleftarrow} E^a \times \mathrm{Gra}_G \stackrel{q_G}{\to} E^a \times_{G(\mathbf{O})\times\mathbb{G}_m} \mathrm{Gra}_G \stackrel{m}{\to} \mathrm{Gra}_G,$$

where q_G is the quotient map under the action of $G(\mathbf{O}) \times \mathbb{G}_m$.

For $K_1, K_2 \in \mathbb{P}\mathrm{erv}_{G,N}$ define the convolution product $K_1 * K_2 \in \mathrm{D}(\mathrm{Gra}_G)$ by $K_1 * K_2 = m_! K$, where $K[1]$ is a perverse sheaf on $E^a \times_{G(\mathbf{O})\times\mathbb{G}_m} \mathrm{Gra}_G$ equipped with an isomorphism $q^*_G K \widetilde{\to} p^*_G K_1 \boxtimes K_2$. Since q_G is a $G(\mathbf{O}) \times \mathbb{G}_m$-torsor and $p^*_G K_1 \boxtimes K_2$ is naturally equivariant under $G(\mathbf{O}) \times \mathbb{G}_m$, K is defined up to a unique isomorphism.

Lemma 3. *If $K_1, K_2 \in \mathbb{P}\mathrm{erv}_{G,N}$ then $K_1 * K_2 \in \mathbb{P}\mathrm{erv}_{G,N}$.*

Proof Following [10], stratify $E^a \times_{G(\mathbf{O})\times\mathbb{G}_m} \mathrm{Gra}_G$ by locally closed subschemes $p^{-1}_G(\mathrm{Gra}^\lambda_G) \times_{G(\mathbf{O})\times\mathbb{G}_m} \mathrm{Gra}^\mu_G$ for $\lambda, \mu \in X^+_*(T)$. Stratify Gra_G by Gra^λ_G, $\lambda \in X^+_*(T)$. By ([10], Lemma 4.4), m is a stratified semi-small map, our assertion follows. $\square$

In a similar way one defines a convolution product $K_1 * K_2 * K_3$ of $K_i \in \mathbb{P}\mathrm{erv}_{G,N}$. Moreover, $(K_1 * K_2) * K_3 \widetilde{\to} K_1 * K_2 * K_3 \widetilde{\to} K_1 * (K_2 * K_3)$ canonically, and $\mathcal{A}^0_\mathcal{E}$ is a unit object in $\mathbb{P}\mathrm{erv}_{G,N}$.

2.3. **Fusion.** As in [10], we will show that the convolution product on $\mathbb{P}\mathrm{erv}_{G,N}$ can be interpreted as a fusion product, thus leading to a commutativity constraint on $\mathbb{P}\mathrm{erv}_{G,N}$.

Fix $\mathcal{E} \in \Omega^{\frac12}(\mathbf{O})$ and consider the group scheme $\mathrm{Aut}_2(\mathbf{O}) := \mathrm{Aut}(\mathbf{O}, \mathcal{E})$ as in ([1], 3.5.2). It fits into an exact sequence $1 \to \mu_2 \to \mathrm{Aut}_2(\mathbf{O}) \to \mathrm{Aut}(\mathbf{O}) \to 1$, and $\mathrm{Aut}_2(\mathbf{O})$ is connected. Write $\mathrm{Aut}^0_2(\mathbf{O})$ for the preimage of $\mathrm{Aut}^0(\mathbf{O})$ in $\mathrm{Aut}_2(\mathbf{O})$.

The map (2) is $\mathrm{Aut}_2^0(\mathbf{O})$-equivariant, so the action of $\mathrm{Aut}^0(\mathbf{O})$ on Gra_G lifts to a $\mathrm{Aut}_2^0(\mathbf{O})$-equivariant structure on each $\mathcal{A}_{\mathcal{E}}^\lambda \in \mathbb{P}\mathrm{erv}_{G,N}$. The corresponding $\mathrm{Aut}_2^0(\mathbf{O})$-equivariant structure on each $\mathcal{A}_{\mathcal{E}}^\lambda$ is unique, as the action of $\mathrm{Aut}_2^0(\mathbf{O})$ on $\overline{\mathrm{Gra}}_G^\lambda$ factors through a smooth connected quotient group of finite type. Here $\overline{\mathrm{Gra}}_G^\lambda$ is the preimage of $\overline{\mathrm{Gr}}_G^\lambda$ under the projection $\mathrm{Gra}_G \to \mathrm{Gr}_G$.

Let X be a smooth connected projective curve over k. For a closed $x \in X$ let $\mathbf{O}_x$ be the completed local ring of X at x, and $\mathbf{F}_x$ its fraction field. Write $\mathcal{F}_G^0$ for the trivial G-torsor on a scheme (or stack). Write $\mathrm{Gr}_{G,x} = G(\mathbf{F}_x)/G(\mathbf{O}_x)$ for the corresponding affine grassmanian. Then $\mathrm{Gr}_{G,x}$ identifies canonically with the ind-scheme classifying a G-torsor $\mathcal{F}_G$ on X together with a trivialization $\nu : \mathcal{F}_G \widetilde{\to} \mathcal{F}_G^0 \mid_{X-x}$.

For $m \geq 1$ write Gr_{G,X^m} for the ind-scheme classifying $(x_1, \dots, x_m) \in X^m$, a G-torsor $\mathcal{F}_G$ on X, and a trivialization $\mathcal{F}_G \widetilde{\to} \mathcal{F}_G^0 \mid_{X-\cup x_i}$.

Let G_{X^m} be the group scheme over X^m classifying $\{(x_1, \dots, x_m) \in X^m, \mu\}$, where μ is an automorphism of $\mathcal{F}_G^0$ restricted to the formal neighborhood of $D = x_1 \cup \dots \cup x_m$ in X. The fibre of G_{X^m} over $(x_1, \dots, x_m) \in X^m$ is $\prod_i G(\mathbf{O}_{y_i})$ with $\{y_1, \dots, y_s\} = \{x_1, \dots, x_m\}$ and y_i pairwise distinct.

Let $\mathfrak{L}_{X^m}$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle on Gr_{G,X^m} whose fibre is $\det \mathrm{R}\Gamma(X, \mathfrak{g}_{\mathcal{F}_G^0}) \otimes \det \mathrm{R}\Gamma(X, \mathfrak{g}_{\mathcal{F}_G})^{-1}$. Here for a G-module V and a G-torsor $\mathcal{F}_G$ on a base S we write $V_{\mathcal{F}_G}$ for the induced vector bundle on S.

Lemma 4. *For a* k*-point* $(x_1, \dots, x_m, \mathcal{F}_G)$ *of* Gr_{G,X^m} *let* $\{y_1, \dots, y_s\} = \{x_1, \dots, x_m\}$ *with* y_i *pairwise distinct. The fibre of* $\mathfrak{L}_{X^m}$ *at this* k*-point is canonically isomorphic as* $\mathbb{Z}/2\mathbb{Z}$*-graded to*

$$\otimes_{i=1}^s \det(\mathfrak{g}(\mathbf{O}_{y_i}) : \mathfrak{g}_{\mathcal{F}_G}(\mathbf{O}_{y_i}))$$

Write Gra_{G,X^m} for the punctured total space of $\mathfrak{L}_{X^m}$. The group scheme G_{X^m} acts naturally on Gra_{G,X^m} and Gr_{G,X^m}, and the projection $\mathrm{Gra}_{G,X^m} \to \mathrm{Gr}_{G,X^m}$ is G_{X^m}-equivariant. Let Perv_{G,N,X^m} be the category of G_{X^m}-equivariant perverse sheaves on Gra_{G,X^m} with $\mathbb{G}_m$-monodromy ζ_a. Set

$$\mathbb{P}\mathrm{erv}_{G,N,X^m} = \mathrm{Perv}_{G,N,X^m}[-m-1] \subset \mathrm{D}(\mathrm{Gra}_{G,X^m})$$

For $x \in X$ write $D_x = \operatorname{Spec} \mathbf{O}_x$, $D_x^* = \operatorname{Spec} \mathbf{F}_x$. Consider the diagram, where the left and right square is cartesian

$$\begin{array}{ccccccc}
\mathrm{Gra}_{G,X} \times \mathrm{Gra}_{G,X} & \overset{\tilde{p}_{G,X}}{\leftarrow} & \tilde{C}_{G,X} & \overset{\tilde{q}_{G,X}}{\to} & \widetilde{\mathrm{Conv}}_{G,X} & \overset{\tilde{m}_X}{\to} & \mathrm{Gra}_{G,X^2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}_{G,X} \times \mathrm{Gr}_{G,X} & \overset{p_{G,X}}{\leftarrow} & C_{G,X} & \overset{q_{G,X}}{\to} & \mathrm{Conv}_{G,X} & \overset{m_X}{\to} & \mathrm{Gr}_{G,X^2}
\end{array}$$

Here the low row is the convolution diagram from [10]. Namely, $C_{G,X}$ is the ind-scheme classifying collections:

$$\left\{ \begin{array}{l} x_1, x_2 \in X,\ G-\text{torsors } \mathcal{F}_G^1, \mathcal{F}_G^2 \text{ on } X \text{ with } \nu_i : \mathcal{F}_G^i \widetilde{\to} \mathcal{F}_G^0 \mid_{X-x_i}, \\ \mu_1 : \mathcal{F}_G^1 \widetilde{\to} \mathcal{F}_G^0 \mid_{D_{x_2}} \end{array} \right. \tag{3}$$

The map $p_{G,X}$ forgets μ_1. The ind-scheme $\mathrm{Conv}_{G,X}$ classifies collections:

$$\left\{\begin{array}{l} x_1, x_2\in X,\ G-\text{torsors } \mathcal{F}^1_G, \mathcal{F}_G \text{ on } X,\\ \text{isomorphisms } \nu_1 : \mathcal{F}^1_G \widetilde{\to} \mathcal{F}^0_G\mid_{X-x_1} \text{ and } \eta : \mathcal{F}^1_G \widetilde{\to} \mathcal{F}_G\mid_{X-x_2}\end{array}\right. \tag{4}$$

The map m_X sends this collection to $(x_1, x_2, \mathcal{F}_G)$ together with the trivialization $\eta\circ\nu_1^{-1} : \mathcal{F}^0_G \widetilde{\to} \mathcal{F}_G\mid_{X-x_1-x_2}$.

The map $q_{G,X}$ sends (3) to (4), where $\mathcal{F}_G$ is obtained by gluing $\mathcal{F}^1_G$ on $X - x_2$ and $\mathcal{F}^2_G$ on D_{x_2} using their identification over $D^*_{x_2}$ via $\nu_2^{-1}\circ\mu_1$.

The canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$q^*_{G,X} m^*_X \mathfrak{L}_{X^2} \widetilde{\to} p^*_{G,X}(\mathfrak{L}_X\boxtimes\mathfrak{L}_X)$$

allows to define $\tilde{q}_{G,X}$, it sends (3) together with $v_i\in\det(\mathfrak{g}(\mathbf{O}_{x_i}) : \mathfrak{g}_{\mathcal{F}^i_G}(\mathbf{O}_{x_i}))$ for $i = 1, 2$ to the image of (3) under $q_{G,X}$ together with $v_1\otimes v_2$. Here we used the isomorphism

$$\begin{aligned}\det(\mathfrak{g}(\mathbf{O}_{x_1}) : \mathfrak{g}_{\mathcal{F}^1_G}(\mathbf{O}_{x_1}))\otimes\det(\mathfrak{g}(\mathbf{O}_{x_2}) : \mathfrak{g}_{\mathcal{F}^2_G}(\mathbf{O}_{x_2})) \widetilde{\to}\\ \det(\mathfrak{g}(\mathbf{O}_{x_1}) : \mathfrak{g}_{\mathcal{F}^1_G}(\mathbf{O}_{x_1})\otimes\det(\mathfrak{g}_{\mathcal{F}^1_G}(\mathbf{O}_{x_2}) : \mathfrak{g}_{\mathcal{F}_G}(\mathbf{O}_{x_2}))\end{aligned} \tag{5}$$

given by μ_1 and $\mathfrak{g}_{\mathcal{F}_G}(\mathbf{O}_{x_2}) \widetilde{\to} \mathfrak{g}_{\mathcal{F}^2_G}(\mathbf{O}_{x_2})$, so (5) is the fibre of $\mathfrak{L}_{X^2}$ over $\mathcal{F}_G$.

For $K_1, K_2\in\mathbb{P}\mathrm{erv}_{G,N,X}$ there is a (defined up to a unique isomorphism) perverse sheaf $K_{12}[3]$ on $\widetilde{\mathrm{Conv}}_{G,X}$ with $\tilde{q}^*_{G,X}K_{12}\widetilde{\to}\tilde{p}^*_{G,X}(K_1\boxtimes K_2)$. Moreover, K_{12} has $\mathbb{G}_m$-monodromy ζ_a. We then let

$$K_1 *_X K_2 = \tilde{m}_{X!}K_{12}$$

Let $U\subset X^2$ be the complement to the diagonal. Let $j : \mathrm{Gra}_{G,X^2}(U)\hookrightarrow\mathrm{Gra}_{G,X^2}$ be the preimage of U. Recall that m_X is stratified small, an isomorphism over the preimage of U ([10]), so the same holds for $\tilde{m}_X$. Thus, $(K_1 *_X K_2)[3]$ is a perverse sheaf, the intermediate extension from $\mathrm{Gra}_{G,X^2}(U)$. Clearly, $K_1 *_X K_2$ is G_{X^2}-equivariant over $\mathrm{Gra}_{G,X^2}(U)$, and this property is preserved under the intermediate extension. So, $K_1 *_X K_2\in\mathrm{Perv}_{G,N,X^2}$.

Let Ω_X be the canonical line bundle on X. Write $\Omega^{\frac12}_X(X)$ for the groupoid of square roots of Ω_X. For $\mathcal{E}_X\in\Omega^{\frac12}_X(X)$ let $\hat{X}_2\to X$ be the $\mathrm{Aut}^0_2(\mathbf{O})$-torsor whose fibre over x is the scheme of isomorphisms between $(\mathcal{E}_x, \mathbf{O}_x)$ and $(\mathcal{E}, \mathbf{O})$. Then $\mathrm{Gr}_{G,X}\widetilde{\to}\hat{X}_2\times_{\mathrm{Aut}^0_2(\mathbf{O})}\mathrm{Gr}_G$ (cf. [1], 5.3.11), and similarly $\mathrm{Gra}_{G,X}\widetilde{\to}\hat{X}_2\times_{\mathrm{Aut}^0_2(\mathbf{O})}\mathrm{Gra}_G$. Since any $K\in\mathrm{Perv}_{G,N}$ is $\mathrm{Aut}^0_2(\mathbf{O})$-equivariant (in a unique way), we get a fully faithful functor

$$\tau^0 : \mathbb{P}\mathrm{erv}_{G,N}\to\mathbb{P}\mathrm{erv}_{G,N,X} \tag{6}$$

sending K to the descent of $\overline{\mathbb{Q}}_\ell\boxtimes K$ under $\hat{X}_2\times\mathrm{Gra}_G\to\mathrm{Gra}_{G,X}$.

Let $i : \mathrm{Gra}_{G,X} \to \mathrm{Gra}_{G,X^2}$ be the preimage of the diagonal in X^2. For $F_i \in \mathbb{P}\mathrm{erv}_{G,N}$ letting $K_i = \tau^0 F_i$ define

$$K_{12} \mid_U := K_{12} \mid_{\mathrm{Gra}_{G,X^2}(U)}$$

as above (it is placed in perverse degree 3). We get $K_1 *_X K_2 \widetilde{\to} j_{!*}(K_{12} \mid_U)$ and $\tau^0(F_1 * F_2) \widetilde{\to} i^*(K_1 *_X K_2)$. So, the involution σ of Gra_{G,X^2} interchanging x_i yields

$$\tau^0(F_1 * F_2) \widetilde{\to} i^* j_{!*}(K_{12} \mid_U) \widetilde{\to} i^* j_{!*}(K_{21} \mid_U) \widetilde{\to} \tau^0(F_2 * F_1),$$

because $\sigma^*(K_{12} \mid_U) \widetilde{\to} K_{21} \mid_U$ canonically. As in ([1], 5.3.13-5.3.17) one shows that the associativity and commutativity constraints are compatible. Thus, $\mathbb{P}\mathrm{erv}_{G,N}$ is a symmetric monoidal category.

The idea to use τ^0 instead of $\tau^0[1]$ in the above definition of the commutativity constraint goes back to ([1], 5.3.17), this is a way to avoid sign problems.

Remark 3. *Write* $\mathrm{P}_{G(\mathbf{O})}(\mathrm{Gra}_G)$ *for the category of* $G(\mathbf{O})$*-equivariant perverse sheaves on* Gra_G*. Let* $\star$ *be the covariant self-functor on* $\mathrm{P}_{G(\mathbf{O})}(\mathrm{Gra}_G)$ *induced by the map* $E^a \to E^a$*,* $e \mapsto e^{-1}$*. Then* $K \mapsto K^\vee := \mathbb{D}(\star K)[-2]$ *is a contravariant functor* $\mathbb{P}\mathrm{erv}_{G,N} \to \mathbb{P}\mathrm{erv}_{G,N}$*. As in ([9], Remark 6), one checks that for* $K_i \in \mathbb{P}\mathrm{erv}_{G,N}$ *we have canonically* $\mathrm{RHom}(K_1 * K_2, K_3) \widetilde{\to} \mathrm{RHom}(K_1, K_3 * K_2^\vee)$*. So,* $K_3 * K_2^\vee$ *represents the internal* $\mathcal{H}om(K_2, K_3)$ *in the sense of the tensor structure on* $\mathbb{P}\mathrm{erv}_{G,N}$*. Besides,* $\star(K_1 * K_2) \widetilde{\to} (\star K_2) * (\star K_1)$ *canonically.*

2.4. **Main result.** In Section 4.2 below we introduce a tensor category $\mathbb{P}\mathrm{erv}^\natural_{G,N}$ obtained from $\mathbb{P}\mathrm{erv}_{G,N}$ by some modification of the commutativity constraint. Set

$$X^*(\check{T}_N) = \{\nu \in X_*(T) \mid d\iota(\nu) \in NX^*(T)\}$$

Let $\check{T}_N = \operatorname{Spec} k[X^*(\check{T}_N)]$ be the torus whose weight lattice is $X^*(\check{T}_N)$. The natural inclusion $X^*(T) \subset X_*(\check{T}_N)$ allows to see each root $\check{\alpha} \in R^*$ as a coweight of $\check{T}_N$. For $a \in \mathbb{Q}, a > 0$ written as $a = a_1/a_2$ with $a_i \in \mathbb{N}$ prime to each other say that a_2 is *the denominator* of a. Recall that p does not divide $2\check{h}N/d$.

Theorem 1. *There is a connected semi-simple group* $\check{G}_N$ *and a canonical equivalence of tensor categories*

$$\mathbb{P}\mathrm{erv}^\natural_{G,N} \widetilde{\to} \mathrm{Rep}(\check{G}_N)$$

There is a canonical inclusion $\check{T}_N \subset \check{G}_N$ *whose image is a maximal torus in* $\check{G}_N$*. The Weyl groups of* G *and* $\check{G}_N$ *viewed as subgroups of* $\mathrm{Aut}(X^*(\check{T}_N))$ *are the same. Our choice of a Borel subgroup* $T \subset B \subset G$ *yields a Borel subgroup* $\check{T}_N \subset \check{B}_N \subset \check{G}_N$*. The corresponding simple roots (resp., coroots) of* $(\check{G}_N, \check{T}_N)$ *are* $\delta_i\alpha_i$ *(resp.,* $\frac{\check{\alpha}_i}{\delta_i}$*), where* δ_i *is the denominator of* $\frac{d(\alpha_i,\alpha_i)}{2N}$*.*

Examples. (If G is simply-connected then $d = 1$.)

- $G = \mathrm{SL}_2$ then $\check{G}_N \widetilde{\to} \mathrm{SL}_2$ for N even, and $\check{G}_N \widetilde{\to} \mathrm{PSL}_2$ for N odd.

- $G = \mathrm{PSL}_2$ then $d = 2$, and $\check{G}_N \widetilde{\to} \mathrm{SL}_2$ for N odd, $\check{G}_N \widetilde{\to} \mathrm{PSL}_2$ for N even.
- $G = \mathbb{S}\mathrm{p}_{2n}$ then $\check{G}_N \widetilde{\to} \mathbb{SO}_{2n+1}$ for N odd, and $\check{G}_N \widetilde{\to} \mathbb{S}\mathrm{p}_{2n}$ for N even. For $N = 2$ this has been also proved in [9].
- $G = \mathbb{S}\mathrm{pin}_{2n+1}$ with $n \geq 2$ then

$$\check{G}_N \widetilde{\to} \begin{cases} \mathbb{S}\mathrm{p}_{2n}/\{\pm 1\}, & N \text{ odd} \\ \mathbb{S}\mathrm{pin}_{2n+1}, & N \text{ even and } nN/2 \text{ even} \\ \mathbb{SO}_{2n+1}, & N \text{ even and } nN/2 \text{ odd} \end{cases}$$

- $G = G_2$ has trivial center, and $\check{G}_N \widetilde{\to} G_2$ for any N
- $G = F_4$ has trivial center, and $\check{G}_N \widetilde{\to} F_4$ for any N
- $G = E_8$ has trivial center, and $\check{G}_N \widetilde{\to} E_8$ for any N
- G simply-connected of type E_6, its center identifies with $\mathbb{Z}/3\mathbb{Z}$ and

$$\check{G}_N \widetilde{\to} \begin{cases} \text{adjoint of type } E_6, & 3 \nmid N \\ \text{simply-connected of type } E_6, & 3 \mid N \end{cases}$$

- G is simply-connected of type E_7, its center identifies with $\mathbb{Z}/2\mathbb{Z}$ and

$$\check{G}_N \widetilde{\to} \begin{cases} \text{simply-connected of type } E_7, & N \text{ even} \\ \text{adjoint of type } E_7, & N \text{ odd} \end{cases}$$

Remark 4. *The case when p divides $2\check{h}/d$, but p does not divide N can also be treated. In this case one can pick a character $\zeta_a : \mu_{2\check{h}N/d}(\mathsf{k}) \to \overline{\mathbb{Q}}_\ell^*$ satisfying $\zeta_a^{2\check{h}/d} = \zeta$ and define $\mathbb{P}\mathrm{erv}^\natural_{G,N}$ in the same way. But the category $\mathbb{P}\mathrm{erv}_{T,G,N}$ will have more objects, we excluded this case to simplify the proof.*

3. Classification of central extensions

3.1. **Simply-connected case.** In this subsection we remind the classification of central extensions of $G^{sc}(\mathbf{F})$ by $\mathbb{G}_m$ in relation with [3].

By [3], the central extensions of G^{sc} by (the sheaf version of) K_2 are classified by integer-valued W-invariant quadratic forms on $X_*(T^{sc})$ (and have no automorphisms). Let Q be the unique $\mathbb{Z}$-valued quadratic form on $X_*(T^{sc})$ satisfying $Q(\alpha) = 1$ for a short coroot α. So, $(\lambda_1, \lambda_2) = Q(\lambda_1 + \lambda_2) - Q(\lambda_1) - Q(\lambda_2)$ for $\lambda_i \in X_*(T^{sc})$. Let

$$1 \to K_2 \to E_Q \to G^{sc} \to 1 \tag{7}$$

denote the central extension corresponding to Q.

Write $v(f)$ for the valuation of $f \in \mathbf{F}^*$. Write $(.,.)_{st}$ for the tame symbol given by

$$(f, g)_{st} = (-1)^{v(f)v(g)}(g^{v(f)} f^{-v(g)})(0)$$

for $f, g \in F^*$. We may view it as a map $K_2(\mathbf{F}) \to \mathsf{k}^*$. Taking the $\mathbf{F}$-valued points of (7) and further the push-forward by the tame symbol, one gets a central extension

$$1 \to \mathsf{k}^* \to \bar{G} \to G^{sc}(\mathbf{F}) \to 1 \tag{8}$$

For $\theta \in \pi_1(G)$ write Gr_G^θ for the connected component of Gr_G that contains $t^\lambda G(\mathbf{O})$ for $\lambda \in X_*(T)$ whose image in $\pi_1(G)$ equals θ. The natural map $\mathrm{Gr}_{G^{sc}} \to \mathrm{Gr}_G^0$ is an isomorphism.

From [5] one knows that there is a line bundle $\mathcal{L}$ on Gr_G generating the Picard group $\mathrm{Pic}(\mathrm{Gr}_G^\theta) \widetilde{\to} \mathbb{Z}$ of each connected component Gr_G^θ of Gr_G, and an isomorphism $\mathcal{L}^{2\check{h}} \widetilde{\to} \mathfrak{L}$. Write $\bar{G}_Q$ for the punctured total space of the inverse image of $\mathcal{L}$ under $G^{sc}(F) \to \mathrm{Gr}_G$, $x \mapsto xG(\mathbf{O})$. It can be seen as the Mumford extension

$$1 \to \mathbb{G}_m \to \bar{G}_Q \to G^{sc}(\mathbf{F}) \to 1, \tag{9}$$

that is, the ind-scheme classifying pairs $(g \in G^{sc}(\mathbf{F}), \sigma)$, where $\sigma : g^*\mathcal{L} \widetilde{\to} \mathcal{L}$ is an isomorphism over Gr_G^0. The central extension (9) splits canonically over $G^{sc}(\mathbf{O})$. Any central extension of $G^{sc}(\mathbf{F})$ by $\mathbb{G}_m$ is a multiple of (9) and has no automorphisms. In the rest of this section we prove the following.

Lemma 5. *Passing to* k*-points in (9) one gets a central extension isomorphic to (8).*

For a central extension $1 \to A \to E \to H \to 1$ we write $(.,.)_c : H \times H \to A$ for the corresponding commutator given by

$$(h_1, h_2)_c = \tilde{h}_1 \tilde{h}_2 \tilde{h}_1^{-1} \tilde{h}_2^{-1},$$

where $\tilde{h}_i$ is any lifting of h_i to E. If H is abelian then the commutator $(h_1, h_2)_c$ depends only on the isomorphism class of the central extension.

Note that $T(\mathbf{F}) = X_*(T) \otimes_{\mathbb{Z}} \mathbf{F}^*$. For $f_i \in \mathbf{F}^*$ and $\lambda_i \in X_*(T)$ the commutator for the central extension (1) is given by

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{2\check{h}(\lambda_1,\lambda_2)}$$

Indeed, for $\lambda \in X_*(T)$, $f \in \mathbf{F}^*$ the fibre of $\mathfrak{L}$ at $\lambda \otimes f$ identifies as $\mathbb{Z}/2\mathbb{Z}$-graded line with

$$\otimes_{\check{\alpha} \in R^*} \det(\mathfrak{g}^{\check{\alpha}}(\mathbf{O}) : f^{\langle \lambda, \check{\alpha} \rangle} \mathfrak{g}^{\check{\alpha}}(\mathbf{O}))$$

and, using Lemma 1, it suffices to calculate the commutator of the canonical central extension of $\mathbb{G}_m(\mathbf{F})$ by $\mathbb{G}_m$ given by the relative determinant. But the latter commutator is given by the tame symbol (cf. [3], 12.13, p. 82). We learn that the commutator for the central extension (9) is given on the torus by

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{(\lambda_1,\lambda_2)}$$

for $\lambda_i \in X_*(T^{sc})$, $f_i \in \mathbf{F}^*$.

We will check that the commutators corresponding to (9) and to (8) are the same on $T^{sc}(\mathbf{F})$. The commutator for (8) can by calculated using, for example, ([3], Proposition 11.11, p. 77). Namely, consider first the case of $G^{sc}=\mathrm{SL}_2$. In this case identify T^{sc} with $\mathbb{G}_m$ via the positive coroot $\alpha:\mathbb{G}_m\widetilde{\to}T^{sc}$ then the commutator for (8) becomes

$$(f_1,f_2)_c=(f_1,f_2)_{st}^2$$

Indeed, for $h_i\in T^{sc}$ consider in the notation of ([3], formula (11.1.4) on p. 73) Steinberg's cocycle $c(h_1,h_2)\in K_2$. The image of $c(h_1,h_2)$ under the tame symbol $K_2(\mathbf{F})\to \mathsf{k}^*$ equals $(h_1,h_2)_{st}$. So, the commutator $(f_1,f_2)_c$ is the image of

$$\frac{c(f_1,f_2)}{c(f_2,f_1)}\in K_2(\mathbf{F})$$

under the tame symbol $K_2(\mathbf{F})\to\mathsf{k}^*$. For $G^{sc}=\mathrm{SL}_2$ our assertion follows.

The general case can be reduced to the case $G^{sc}=\mathrm{SL}_2$ by restricting to the SL_2-subgroups $S_{\check\alpha}\subset G$ corresponding to the roots $\check\alpha$ as in ([3], Section 11.2, p. 74). We are done.

Remark 5. *For $\lambda\in X_*(T^{sc})$ we have $(\lambda,\lambda)\in 2\mathbb{Z}$. Indeed, $(\lambda,\lambda)=Q(2\lambda)-2Q(\lambda)=2Q(\lambda)$.*

3.2. **Proof of Proposition 1.** The idea of the argument below was communicated to us by Drinfeld.

For $m\in\mathbb{Z}$ write $\bar G_{mQ}$ for the m-th multiple of the central extension $\bar G_Q$. There is a canonical action δ_0 of $G(\mathbf{F})$ on the exact sequence

$$1\to\mathbb{G}_m\to\bar G_{mQ}\to G^{sc}(\mathbf{F})\to 1 \tag{10}$$

such that $G(\mathbf{F})$ acts trivially on $\mathbb{G}_m$ and by conjugation on $G^{sc}(\mathbf{F})$. Indeed, we know that the extension (9) comes from the canonical extension (7), so that the automorphisms of G^{sc} act on it. Write $\delta_0: G(\mathbf{F})\times\bar G_{mQ}\to\bar G_{mQ}$ for the action map.

If $\lambda\in X_*(T^{sc}),\mu\in X_*(T),\ f,g\in\mathbf{F}^*$ then $\mu\otimes g\in T(\mathbf{F})\subset G(\mathbf{F})$ acts on the fibre of $\bar G_{mQ}$ over $\lambda\otimes f\in T^{sc}(\mathbf{F})$ via δ_0 as a multiplication by

$$(g,f)_{st}^{m(\mu,\lambda)}$$

This is a kind of 'explanation' of the fact that the form $(.,.)$ initially defined on $X_*(T^{sc})$ extends by linearity to a form $(.,.):X_*(T)\times X_*(T^{sc})\to\mathbb{Z}$ taking values in $\mathbb{Z}$ and not just in $\mathbb{Q}$.

3.2.1. The isomorphism classes of central extensions

$$1\to\mathbb{G}_m\to\mathcal{E}_b\to T(\mathbf{F})\to 1 \tag{11}$$

are classified by symmetric bilinear forms $(.,.)_b : X_*(T) \times X_*(T) \to \mathbb{Z}$, namely for the corresponding extension (11) we have

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{(\lambda_1,\lambda_2)_b}$$

for $f_i \in \mathbf{F}^*$, $\lambda_i \in X_*(T)$.

The group of automorphisms of the central extension (11) is $\mathrm{Hom}(T(\mathbf{F}), \mathbb{G}_m)$. Since $T(\mathbf{F})$ is abelian, the commutator $(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c$ is invariant under these automorphisms. The extension (11) admits a (non unique) splitting over $T(\mathbf{O})$.

3.2.2. The group T acts on G^{sc} by conjugation, let $\tilde{G} = G^{sc} \ltimes T$ denote the corresponding semi-direct product. The map $G^{sc} \ltimes T \to G$, $(g,t) \mapsto \bar{g}t$, where $\bar{g}$ is the image of $g \in G^{sc}$ in G, yields an exact sequence $1 \to T^{sc} \to \tilde{G} \to G \to 1$. Hence, an exact sequence

$$1 \to T^{sc}(\mathbf{F}) \to \tilde{G}(\mathbf{F}) \to G(\mathbf{F}) \to 1 \tag{12}$$

The category of central extensions of $G(\mathbf{F})$ by $\mathbb{G}_m$ is equivalent to the category of pairs: a central extension

$$1 \to \mathbb{G}_m \to ? \to \tilde{G}(\mathbf{F}) \to 1 \tag{13}$$

together with a splitting of its pull-back under $T^{sc}(\mathbf{F}) \to \tilde{G}(\mathbf{F})$.

By a slight (we have to drop off the assumption of being of finite type) generalization of ([3], Construction 1.7), the category of central extensions (13) is equivalent to the category of triples: central extensions (10) and (11) together with an action δ of $T(\mathbf{F})$ on $\bar{G}_{mQ}$ extending the action of $T(\mathbf{F})$ on $G^{sc}(F)$ by conjugation. Write $\delta : T(\mathbf{F}) \times \bar{G}_{mQ} \to \bar{G}_{mQ}$ for the action map.

Since (10) has no automorphisms, δ coincides with the restriction of δ_0 to $T(\mathbf{F}) \times \bar{G}_{mQ}$. Write

$$1 \to \mathbb{G}_m \to \bar{G}^T_{mQ} \to T^{sc}(\mathbf{F}) \to 1 \tag{14}$$

for the restriction of (10) to $T^{sc}(\mathbf{F})$. We conclude that the category of central extensions of $G(\mathbf{F})$ by $\mathbb{G}_m$ is equivalent to the category of pairs: a central extension (11) together with an isomorphism of its restriction to $T^{sc}(\mathbf{F})$ with (14). Clearly, the corresponding form $(.,.)_b : X_*(T) \times X_*(T) \to \mathbb{Z}$ is given by $(\lambda_1, \lambda_2)_b = m(\lambda_1, \lambda_2)$. Proposition 1 follows.

4. Proof of Theorem 1

4.1. **Functors F'_P.** Let P be a parabolic subgroup of G containing B. Let $M \subset P$ be a Levi subgroup containing T. Write

$$1 \to \mathbb{G}_m \to E^a_M \to M(\mathbf{F}) \to 1$$

for the restriction of (1) to $M(\mathbf{F})$, it is equipped with an action of $\mathrm{Aut}^0(\mathbf{O})$ and a section over $M(\mathbf{O})$ coming from the corresponding objects for (1). Let $\mathrm{Perv}_{M,G,N}$

denote the category of $M(\mathbf{O})$-equivariant $\overline{\mathbb{Q}}_\ell$-perverse sheaves on $E^a_M/M(\mathbf{O})$ with $\mathbb{G}_m$-monodromy ζ_a. Set

$$\mathbb{P}\mathrm{erv}_{M,G,N} = \mathrm{Perv}_{M,G,N}[-1] \subset \mathrm{D}(E^a_M/M(\mathbf{O}))$$

Write $\mathfrak{L}_{M,G}$ for the restriction of $\mathfrak{L}$ under $\mathrm{Gr}_M \to \mathrm{Gr}_G$, we equip it with the action of $\mathrm{Aut}^0(\mathbf{O})$ coming from that on $\mathfrak{L}$.

Write Gr_M for the affine grassmanian for M. The connected components of Gr_M are indexed by $\pi_1(M)$. For $\theta \in \pi_1(M)$ write Gr^θ_M for the connected component of Gr_M containing $t^\lambda M(\mathbf{O})$ for any coweight λ whose image in $\pi_1(M)$ is θ. The diagram $M \leftarrow P \hookrightarrow G$ yields the following diagram of affine grassmanians

$$\mathrm{Gr}_M \stackrel{\mathfrak{t}_P}{\leftarrow} \mathrm{Gr}_P \stackrel{\mathfrak{s}_P}{\to} \mathrm{Gr}_G$$

The map $\mathfrak{t}_P$ yields a bijection between the connected components of Gr_P and those of Gr_M. Let Gr^θ_P be the connected component of Gr_P such that $\mathfrak{t}_P$ restricts to a map $\mathfrak{t}^\theta_P : \mathrm{Gr}^\theta_P \to \mathrm{Gr}^\theta_M$. Write $\mathfrak{s}^\theta_P : \mathrm{Gr}^\theta_P \to \mathrm{Gr}_G$ for the restriction of $\mathfrak{s}_P$. The restriction of $\mathfrak{s}^\theta_P$ to $(\mathrm{Gr}^\theta_P)_{red}$ is a closed immersion.

The section $M \hookrightarrow P$ yields a section $\mathfrak{r}_P : \mathrm{Gr}_M \to \mathrm{Gr}_P$ of $\mathfrak{t}_P$. By abuse of notation, we write

$$\mathrm{Gra}_M \stackrel{\mathfrak{r}_P}{\to} \mathrm{Gra}_P \stackrel{\mathfrak{s}_P}{\to} \mathrm{Gra}_G$$

for the diagram obtained from $\mathrm{Gr}_M \stackrel{\mathfrak{r}_P}{\to} \mathrm{Gr}_P \stackrel{\mathfrak{s}_P}{\to} \mathrm{Gr}_G$ by the base change $\mathrm{Gra}_G \to \mathrm{Gr}_G$. Clearly, $\mathfrak{t}_P$ lifts naturally to a map $\mathfrak{t}_P : \mathrm{Gra}_P \to \mathrm{Gra}_M$. Define the functor

$$F'_P : \mathbb{P}\mathrm{erv}_{G,N} \to \mathrm{D}(\mathrm{Gra}_M)$$

by $F'_P(K) = \mathfrak{t}_{P!}\mathfrak{s}^*_P K$. Write Gra^θ_M (resp., Gra^θ_P) for the connected component of Gra_M (resp., Gra_P) over Gr^θ_M (resp., Gr^θ_P). Write

$$\mathbb{P}\mathrm{erv}^\theta_{M,G,N} \subset \mathbb{P}\mathrm{erv}_{M,G,N}$$

for the full subcategory of objects that vanish off Gra^θ_M. Set

$$\mathbb{P}\mathrm{erv}'_{M,G,N} = \bigoplus_{\theta\in\pi_1(M)} \mathbb{P}\mathrm{erv}^\theta_{M,G,N}[\langle\theta, 2\check{\rho}_M - 2\check{\rho}\rangle]$$

As in ([1], 5.3.29) one shows that F'_P sends $\mathbb{P}\mathrm{erv}_{G,N}$ to $\mathbb{P}\mathrm{erv}'_{M,G,N}$ (cf. also [9], appendix A.4).

The above construction applied to the Borel subgroup yields a functor $F'_B : \mathbb{P}\mathrm{erv}_{G,N} \to \mathbb{P}\mathrm{erv}'_{T,G,N}$.

Let $B(M) \subset M$ be a Borel subgroup containing T such that the preimage of $B(M)$ under $P \to M$ equals B. The functor $F'_{B(M)} : \mathbb{P}\mathrm{erv}'_{M,G,N} \to \mathrm{D}(\mathrm{Gra}_T)$ is defined as follows. As above, the inclusions $T \subset B(M) \subset M$ yield a diagram

$$\mathrm{Gr}_T \stackrel{\mathfrak{r}_{B(M)}}{\to} \mathrm{Gr}_{B(M)} \stackrel{\mathfrak{s}_{B(M)}}{\to} \mathrm{Gr}_M \tag{15}$$

Write

$$\mathrm{Gra}_T \stackrel{\mathfrak{r}_{B(M)}}{\to} \mathrm{Gra}_{B(M)} \stackrel{\mathfrak{s}_{B(M)}}{\to} \mathrm{Gra}_M$$

for the diagram obtained from (15) by the base change $\mathrm{Gra}_M \to \mathrm{Gr}_M$. The projection $B(M) \to T$ yields $\mathfrak{t}_{B(M)} : \mathrm{Gr}_{B(M)} \to \mathrm{Gr}_T$, which lifts naturally to $\mathfrak{t}_{B(M)} : \mathrm{Gra}_{B(M)} \to \mathrm{Gra}_T$. For $K \in \mathbb{P}\mathrm{erv}'_{M,G,N}$ set

$$F'_{B(M)}(K) = (\mathfrak{t}_{B(M)})_! \mathfrak{s}^*_{B(M)} K$$

As in ([1], 5.3.29), one shows that $F'_{B(M)}$ is a functor

$$F'_{B(M)} : \mathbb{P}\mathrm{erv}'_{M,G,N} \to \mathbb{P}\mathrm{erv}'_{T,G,N}$$

By base change, we have canonically

$$F'_{B(M)} \circ F'_P \widetilde{\to} F'_B \tag{16}$$

4.1.1. Write $X_*^{+M}(T) \subset X_*(T)$ for the coweights of T dominant for M. For $\lambda \in X_*^{+M}(T)$ denote by Gr_M^λ the $M(\mathbf{O})$-orbit through $t^\lambda M(\mathbf{O})$. Let Gra_M^λ be the preimage of Gr_M^λ under $\mathrm{Gra}_M \to \mathrm{Gr}_M$. The M-orbit on Gr_M through $t^\lambda M(\mathbf{O})$ is isomorphic to a partial flag variety $\mathcal{B}_M^\lambda = M/P_M^\lambda$, where the Levi subgroup of P_M^λ has the Weyl group coinciding with the stabilizer of λ in W_M. Write $\tilde{\omega}_{M,\lambda} : \mathrm{Gr}_M^\lambda \to \mathcal{B}_M^\lambda$ for the projection. As in Lemma 2, one gets a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\mathfrak{L}_{M,G}\mid_{\mathrm{Gr}_M^\lambda} \widetilde{\to} \Omega_{\bar{c}}^{\check{h}(\lambda,\lambda)} \otimes \tilde{\omega}^*_{M,\lambda} \mathcal{O}(2\check{h}\iota(\lambda)),$$

here the line bundles $\mathcal{O}(\check{\nu})$ on $\mathcal{B}_M^\lambda$ are defined as in Section 2.1. Set

$$X_M^{*+}(\check{T}_N) = \{\lambda \in X_*^{+M}(T) \mid d\iota(\lambda) \in NX^*(T)\}$$

As for G itself, for $\lambda \in X_*^{+M}(T)$ the scheme Gra_M^λ admits a $M(\mathbf{O})$-equivariant local system with $\mathbb{G}_m$-monodromy ζ_a iff $\lambda \in X_M^{*+}(\check{T}_N)$. For $\lambda \in X_M^{*+}(\check{T}_N)$ denote by $\mathcal{A}^\lambda_{M,\mathcal{E}}$ the irreducible object of $\mathbb{P}\mathrm{erv}_{M,G,N}$ defined as in Section 2.1.

4.1.2. *More tensor structures.* One equips $\mathbb{P}\mathrm{erv}_{M,G,N}$ and $\mathbb{P}\mathrm{erv}'_{M,G,N}$ with a convolution product as in Section 2.2. Let us define the commutativity constraint on these categories via fusion.

Recall the line bundles $\mathfrak{L}_{X^m}$ on Gr_{G,X^m} from Section 2.3. For the convenience of the reader we remind the *factorization structure* on these line bundles, which allowed to do fusion for $\mathbb{P}\mathrm{erv}_{G,N}$.

For a surjective map of finite sets $\alpha : J \to I$ one has a cartesian square

$$\begin{array}{ccc} \mathrm{Gr}_{G,X^I} & \stackrel{\tilde{\triangle}^\alpha}{\to} & \mathrm{Gr}_{G,X^J} \\ \downarrow & & \downarrow \\ X^I & \stackrel{\triangle^\alpha}{\to} & X^J, \end{array}$$

where $\triangle^\alpha$ is the corresponding diagonal. We have canonically $(\tilde{\triangle}^\alpha)^* \mathfrak{L}_{X^J} \widetilde{\to} \mathfrak{L}_{X^I}$.

Write $\nu^\alpha : U^\alpha \hookrightarrow X^J$ for the open subscheme given by the condition that the divisors D_i do not intersect pairwise, where $D_i = \sum_{j\in J,\, \alpha(j)=i} x_j$ for $(x_j)\in X^J$. We have a cartesian square

$$\begin{array}{ccc} (\prod_{i\in I} \mathrm{Gr}_{G,X^{\alpha^{-1}(i)}}) \mid_{U^\alpha} & \stackrel{\tilde\nu^\alpha}{\hookrightarrow} & \mathrm{Gr}_{G,X^J} \\ \downarrow & & \downarrow \\ U^\alpha & \stackrel{\nu^\alpha}{\to} & X^J \end{array}$$

We have canonically

$$(\tilde\nu^\alpha)^*\mathfrak{L}_{X^J} \widetilde{\to} (\boxtimes_{i\in I}\mathfrak{L}_{X^{\alpha^{-1}(i)}}) \mid_{U^\alpha}$$

Let $\mathfrak{L}_{M,G,X^m}$ be the restriction of $\mathfrak{L}_{X^m}$ under the map $\mathrm{Gr}_{M,X^m}\to \mathrm{Gr}_{G,X^m}$ induced by $M\hookrightarrow G$. The collection $\{\mathcal{L}_{M,G,X^m}\}$ is endowed with the induced factorization structure. Write Gra_{M,G,X^m} for the punctured line bundle of $\mathfrak{L}_{M,G,X^m}$.

Let M_{X^m} be the group scheme over X^m classifying $\{(x_1,\ldots,x_m)\in X^m, \mu\}$, where μ is an automorphism of $\mathcal{F}^0_M$ restricted to the formal neighborhood of $x_1\cup\ldots\cup x_m$ in X. The group scheme M_{X^m} acts naturally on Gra_{M,G,X^m}. Write $\mathrm{Perv}_{M,G,N,X^m}$ for the category of M_{X^m}-equivariant perverse sheaves on Gra_{M,G,X^m} with $\mathbb{G}_m$-monodromy ζ_a. Set

$$\mathbb{P}\mathrm{erv}_{M,G,N,X^m} = \mathrm{Perv}_{M,G,N,X^m}[-m-1]$$

Let $\mathrm{Aut}^0_2(\mathbf{O})$ act on Gra_M via its quotient $\mathrm{Aut}^0(\mathbf{O})$. Then every object of $\mathrm{Perv}_{M,G,N}$ admits a unique $\mathrm{Aut}^0_2(\mathbf{O})$-equivariant structure. Note that $\mathrm{Gra}_{M,X} \widetilde{\to} \hat{X}_2\times_{\mathrm{Aut}^0_2(\mathbf{O})} \mathrm{Gra}_M$. As in Section 2.3, we get a fully faithful functor

$$\tau^0 : \mathbb{P}\mathrm{erv}_{M,G,N}\to \mathbb{P}\mathrm{erv}_{M,G,N,X}$$

Now we define the commutativity constraint on $\mathbb{P}\mathrm{erv}_{M,G,N}$ and $\mathbb{P}\mathrm{erv}'_{M,G,N}$ using the above factorization structure as in Section 2.3. As in ([1], 5.3.16) one checks that $\mathbb{P}\mathrm{erv}_{M,G,N}$ and $\mathbb{P}\mathrm{erv}'_{M,G,N}$ are symmetric monoidal categories.

Lemma 6. *The functors F'_P, $F'_{B(M)}$, F'_B are tensor functors, and (16) is an isomorphism of tensor functors.*

Proof We will only check that F'_P is a tensor functor, the rest is similar.
1) Let us show that F'_P is compatible with the convolution. Let Gr_{P,X^m} be the ind-scheme classifying $(x_1,\ldots,x_m)\in X^m$, a P-torsor $\mathcal{F}_P$ on X, and a trivialization $\mathcal{F}_P \widetilde{\to} \mathcal{F}^0_P \mid_{X-\cup x_i}$. Write Gra_{P,X^m} for the ind-scheme obtained from Gra_{G,X^m} by the base change $\mathrm{Gr}_{P,X^m}\to \mathrm{Gr}_{G,X^m}$. As in Section 4.1, we get a diagram

$$\mathrm{Gra}_{M,X^m} \stackrel{\mathfrak{t}_{P,X^m}}{\leftarrow} \mathrm{Gra}_{P,X^m} \stackrel{\mathfrak{s}_{P,X^m}}{\to} \mathrm{Gra}_{G,X^m}$$

and a functor

$$F'_{P,X^m} : \mathrm{D}(\mathrm{Gra}_{G,X^m}) \to \mathrm{D}(\mathrm{Gra}_{M,G,X^m})$$

given by $F'_{P,X^m}(K)=(\mathfrak{t}_{P,X^m})_!\mathfrak{s}^*_{P,X^m}$. For $i=1,2$ let $F_i\in\mathbb{P}\mathrm{erv}_{G,N}$ and $K_i=\tau^0 F_i$. Recall that $U\subset X^2$ is the complement to the diagonal. We have a natural diagram, where both squares are cartesian

$$\begin{array}{ccc}
(\mathrm{Gra}_{G,X}\times\mathrm{Gra}_{G,X})\mid_U & \stackrel{\nu_{G,U}}{\to} & \mathrm{Gra}_{G,X^2}\\
\uparrow & & \uparrow \mathfrak{s}_{P,X^2}\\
(\mathrm{Gra}_{P,X}\times\mathrm{Gra}_{P,X})\mid_U & \to & \mathrm{Gra}_{P,X^2}\\
\downarrow & & \downarrow \mathfrak{t}_{P,X^2}\\
(\mathrm{Gra}_{M,X}\times\mathrm{Gra}_{M,X})\mid_U & \stackrel{\nu_{M,U}}{\to} & \mathrm{Gra}_{M,X^2},
\end{array}$$

and the maps $\nu_{G,U}$ and $\nu_{M,U}$ come from the above factorization structures. As in ([9], Proposition 14), one shows that $F'_{P,X^2}(K_1*_X K_2)$ is the Goresky-MacPherson extension from $\mathrm{Gra}_{M,X^2}\mid_U$. Now the isomorphism

$$\nu^*_{M,U}F'_{P,X^2}(K_1*_X K_2)\widetilde{\to}\tau^0(F'_P(F_1))\boxtimes\tau^0(F'_P(F_2))$$

yields an isomorphism

$$\epsilon_{12}:F'_{P,X^2}(K_1*_X K_2)\widetilde{\to}\tau^0(F'_P(F_1))*_X\tau^0(F'_P(F_2))$$

Restricting it to the diagonal in X one gets

$$\tau^0(F'_P(F_1*F_2))\widetilde{\to}\tau^0(F'_P(F_1)*F'_P(F_2))$$

2) Let us check that F'_P is compatible with the commutativity constraints. Recall that σ is the involution of X^2 permuting the two coordinates. One has a commutative diagram, where the vertical arrows are canonical isomorphisms

$$\begin{array}{ccc}
\sigma^*F'_{P,X^2}(K_1*_X K_2) & \stackrel{\sigma^*\epsilon_{12}}{\to} & \sigma^*(\tau^0(F'_P(F_1))*_X\tau^0(F'_P(F_2)))\\
\downarrow & & \downarrow\\
F'_{P,X^2}(K_2*_X K_1) & \stackrel{\epsilon_{21}}{\to} & \tau^0(F'_P(F_2))*_X\tau^0(F'_P(F_1))
\end{array}$$

Restricting it to the diagonal, one gets the desired compatibility isomorphism. □

4.2. **Fibre functor.** In Section 4.1 we introduced the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle $\mathfrak{L}_{T,G}$ on Gr_T. The action of $T(\mathbf{O})$ on this line bundle comes from the action of $T(\mathbf{O})$ on E^a_T by left multpilication. The fibre of $\mathfrak{L}_{T,G}$ at $\nu\in X_*(T)$ is $T(\mathbf{O})$-equivariantly isomorphic to $\Omega_{\bar c}^{\check h\langle\nu,\iota(\nu)\rangle}$, where $T(\mathbf{O})$ acts on the latter space via $T(\mathbf{O})\to T\stackrel{2\check h\iota(\nu)}{\to}\mathbb{G}_m$. Recall the torus $\check T_N$ introduced in Section 2.4. For $\nu\in X_*(T)$ the orbit Gra^ν_T supports a nonzero object of $\mathbb{P}\mathrm{erv}_{T,G,N}$ iff $\nu\in X^*(\check T_N)$.

For $\nu\in X^*(\check T_N)$ consider the map $a_\nu:\mathcal{E}_{\bar c}^{d(\nu,\nu)/N}-\{0\}\to\Omega_{\bar c}^{\check h(\nu,\nu)}-\{0\}$ sending x to $x^{2\check hN/d}$. For $K\in\mathbb{P}\mathrm{erv}_{T,G,N}$ the complex $a^*_\nu K$ is a constant sheaf placed in degree zero, so we view it as a vector space denoted $F^\nu_T(K)$. Then

$$F_T=\bigoplus_{\nu\in X^*(\check T_N)}F^\nu_T$$

is a fibre functor $\mathbb{P}\mathrm{erv}_{T,G,N} \to \mathrm{Vect}$. Let $\check{T}_N = \mathrm{Spec}\, k[X^*(\check{T}_N)]$ be the torus whose weight lattice is $X^*(\check{T}_N)$. By ([4], Theorem 2.11), we get

$$\mathbb{P}\mathrm{erv}_{T,G,N} \widetilde{\to} \mathrm{Rep}(\check{T}_N)$$

For $\nu \in X^*(\check{T}_N)$ write $F'^{\nu}_{B(M)}$ for the functor $F'_{B(M)}$ followed by restriction to Gra^{ν}_T. Write $F^{\nu}_M : \mathbb{P}\mathrm{erv}_{M,G,N} \to \mathrm{Vect}$ for the functor

$$F^{\nu}_T F'^{\nu}_{B(M)}[\langle \nu, 2\check{\rho}_M \rangle]$$

For $\nu \in X^*(\check{T}_N)$ any $x \in \mathcal{E}_{\bar{c}}^{d(\nu,\nu)/N}$ yields a section $a_{B(M),\nu} : \mathrm{Gr}^{\nu}_{B(M)} \to \mathrm{Gra}^{\nu}_{B(M)}$ of the projection $\mathrm{Gra}^{\nu}_{B(M)} \to \mathrm{Gr}^{\nu}_{B(M)}$ sending x to $x^{2\check{h}N/d}$.

Lemma 7. *If $\nu \in X^*(\check{T}_N)$, $\lambda \in X^{*+}_M(\check{T}_N)$ then $F^{\nu}_M(\mathcal{A}^{\lambda}_{M,\mathcal{E}})$ has a canonical base consisting of those connected components of*

$$\mathrm{Gr}^{\nu}_{B(M)} \cap \mathrm{Gr}^{\lambda}_M$$

over which the (shifted) local system $a^*_{B(M),\nu}\mathcal{A}^{\lambda}_{M,\mathcal{E}}$ is constant. In particular, $F^{\lambda}_M(\mathcal{A}^{\lambda}_{\mathcal{E}}) \widetilde{\to} \overline{\mathbb{Q}}_\ell$ canonically, and $F^{w(\lambda)}(\mathcal{A}^{\lambda}_{\mathcal{E}}) \widetilde{\to} \overline{\mathbb{Q}}_\ell$ for $w \in W_M$.

Proof The first claim is proved as in ([10], Proposition 3.10). If $\lambda \in X^{*+}_M(\check{T}_N)$ then $\mathrm{Gr}^{w(\lambda)}_{B(M)} \cap \mathrm{Gr}^{\lambda}_M$ is an affine space ([10], proof of Theorem 3.2), and the local system $a^*_{B(M),\lambda}\mathcal{A}^{\lambda}_{M,\mathcal{E}}$ is constant. □

Consider the following $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{P}\mathrm{erv}'_{M,G,N}$. For $\theta \in \pi_1(M)$ call an object of $\mathbb{P}\mathrm{erv}^{\theta}_{M,G,N}[\langle \theta, 2\check{\rho}_M - 2\check{\rho}\rangle]$ *of parity* $\langle \theta, 2\check{\rho}\rangle \bmod 2$. Write $E^{a,\theta}_M$ for the connected component of E^a_M such that

$$E^{a,\theta}_M / M(\mathbf{O}) = \mathrm{Gra}^{\theta}_M$$

The product in E^a_M is compatible with this $\mathbb{Z}/2\mathbb{Z}$-grading of $\pi_1(M)$, so the $\mathbb{Z}/2\mathbb{Z}$-grading we get on $\mathbb{P}\mathrm{erv}'_{M,G,N}$ is compatible with the tensor structure. In particular, for $M = G$ we get a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{P}\mathrm{erv}_{G,N}$. If $(\mathrm{Gra}^{\theta}_P)_{red}$ is contained in the connected component $\mathrm{Gra}^{\bar{\theta}}_G$ of Gra_G then $\bar{\theta}$ is the image of θ in $\pi_1(G)$. So, the functors F'_P and $F'_{B(M)}$ are compatible with these gradings.

Write Vect^{ϵ} for the tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Let $\mathbb{P}\mathrm{erv}^{\natural}_{M,G,N}$ be the category of even objects in $\mathbb{P}\mathrm{erv}'_{M,G,N} \otimes \mathrm{Vect}^{\epsilon}$. Let $\mathbb{P}\mathrm{erv}^{\natural}_{G,N}$ be the category of even objects in $\mathbb{P}\mathrm{erv}_{G,N} \otimes \mathrm{Vect}^{\epsilon}$. We get a canonical equivalence of tensor categories $sh : \mathbb{P}\mathrm{erv}^{\natural}_{T,G,N} \widetilde{\to} \mathbb{P}\mathrm{erv}_{T,G,N}$ The functors $F'_{B(M)}, F'_P, F'_B$ yield tensor functors

$$\mathbb{P}\mathrm{erv}^{\natural}_{G,N} \stackrel{F^{\natural}_P}{\to} \mathbb{P}\mathrm{erv}^{\natural}_{M,G,N} \stackrel{F^{\natural}_{B(M)}}{\to} \mathbb{P}\mathrm{erv}^{\natural}_{T,G,N} \tag{17}$$

whose composition is $F^{\natural}_B$. Write $F^{\natural} : \mathbb{P}\mathrm{erv}^{\natural}_{G,N} \to \mathrm{Vect}$ for $F_T \circ sh \circ F^{\natural}_B$. By Lemma 7, $F^{\natural}$ does not annihilate a nonzero object, so it is faithful. By Remark 3,

$\mathbb{P}\mathrm{erv}^{\natural}_{G,N}$ is a rigid abelian tensor category, so $F^{\natural}$ is a fibre functor. By ([4], Theorem 2.11), $\mathrm{Aut}^{\otimes}(F^{\natural})$ is represented by an affine group scheme $\check{G}_N$, and we have an equivalence of tensor categories

$$\mathbb{P}\mathrm{erv}^{\natural}_{G,N} \widetilde{\to} \mathrm{Rep}(\check{G}_N) \tag{18}$$

An analog of Remark 3 holds also for M, so $F_T \circ sh \circ F^{\natural}_{B(M)} : \mathbb{P}\mathrm{erv}^{\natural}_{M,G,N} \to \mathrm{Vect}$ is a fibre functor that yields an affine group scheme $\check{M}_N$ and an equivalence of tensor categories $\mathbb{P}\mathrm{erv}^{\natural}_{M,G,N} \widetilde{\to} \mathrm{Rep}(\check{M}_N)$. The diagram (17) yields homomorphisms $\check{T}_N \to \check{M}_N \to \check{G}_N$. Since $X^{*+}(\check{T}_N)$ does not contain a nontrivial subgroup, $\check{G}_N$ is semisimple of rank equal to the rank of G.

4.3. **Structure of $\check{G}_N$.**

Lemma 8. *If $\lambda, \mu \in X^{*+}(\check{T}_N)$ then $\mathcal{A}^{\lambda+\mu}_{\mathcal{E}}$ appears in $\mathcal{A}^{\lambda}_{\mathcal{E}} * \mathcal{A}^{\mu}_{\mathcal{E}}$ with multiplicity one.*

Proof Write $\bar{E}^{a,\lambda}$ (resp., $E^{a,\lambda}$) for the preimage of $\overline{\mathrm{Gra}}^{\lambda}_G$ (resp., Gra^{λ}_G) under $E^a \to \mathrm{Gra}_G$, $x \mapsto xG(\mathbf{O})$. Write $m^{\lambda,\mu} : \bar{E}^{a,\lambda} \times_{G(\mathbf{O})\times\mathbb{G}_m} \overline{\mathrm{Gra}}^{\mu}_G \to \overline{\mathrm{Gra}}^{\lambda+\mu}_G$ for the convolution diagram as in Section 2.2. If W is the preimage of $\mathrm{Gra}^{\lambda+\mu}_G$ under $m^{\lambda,\mu}$ then $m^{\lambda,\mu} : W \to \mathrm{Gra}^{\lambda+\mu}_G$ is an isomorphism, and W is open in $E^{a,\lambda} \times_{G(\mathbf{O})\times\mathbb{G}_m} \mathrm{Gra}^{\mu}_G$. $\square$

Clearly, if $X^{*+}(\check{T}_N)$ is a $\mathbb{Z}_+$-span of $\lambda_1, \dots, \lambda_r$ then $\oplus_i \mathcal{A}^{\lambda_i}_{\mathcal{E}}$ is a tensor generator for $\mathbb{P}\mathrm{erv}_{G,N}$. So, $\check{G}_N$ is algebraic by ([4], Proposition 2.20). By Lemma 7, for $\mu \in X^{*+}(\check{T}_N)$ and $w \in W$ the weight $w(\mu)$ of $\check{T}_N$ appears in $F^{\natural}(\mathcal{A}^{\mu}_{\mathcal{E}})$. So, $\check{T}_N$ is closed in $\check{G}_N$ by ([4], Proposition 2.21). By Lemma 8, there is no tensor subcategory of $\mathbb{P}\mathrm{erv}_{G,N}$ whose objects are direct sums of finitely many fixed irreducible objects, so $\check{G}_N$ is connected by ([4], Corollary 2.22). Since $\mathbb{P}\mathrm{erv}_{G,N}$ is semisimple, $\check{G}_N$ is reductive by ([4], Proposition 2.23). We will use the following.

Lemma 9. *Let $\mathbb{G}$ be a connected reductive group with a maximal torus $\mathbb{T} \subset \mathbb{G}$. Let $\check{\Lambda}^+$ be a subsemigroup in the group $\check{\Lambda}$ of weights of $\mathbb{T}$. Assume that we are given a bijection $\nu \mapsto V^{\nu}$ between $\check{\Lambda}^+$ and the set of irreducible representations of $\mathbb{G}$ such that the following holds:*

- *if $\nu \in \check{\Lambda}^+$ then the ν-weight space L^{ν} of $\mathbb{T}$ in V^{ν} is of dimension one;*
- *if $\nu_1, \nu_2 \in \check{\Lambda}^+$ then $V^{\nu_1+\nu_2}$ appears with multiplicity one in $V^{\nu_1} \otimes V^{\nu_2}$, and the subspace $L^{\nu_1} \otimes L^{\nu_2} \subset V^{\nu_1} \otimes V^{\nu_2}$ coincides with the image of $L^{\nu_1+\nu_2} \hookrightarrow V^{\nu_1+\nu_2} \hookrightarrow V^{\nu_1} \otimes V^{\nu_2}$.*

Then there is a unique Borel subgroup $\mathbb{T} \subset \mathbb{B} \subset \mathbb{G}$ such that $\check{\Lambda}^+$ is the set of dominant weights for $\mathbb{B}$. $\square$

Write V^{ν} for the irreducible representation of $\check{G}_N$ corresponding to $\mathcal{A}^{\nu}_{\mathcal{E}}$ via (18).

Lemma 10. *The torus $\check{T}_N$ is maximal in $\check{G}_N$. There is a unique Borel subgroup $\check{T}_N \subset \check{B}_N \subset \check{G}_N$ whose set of dominant weights coincides with $X^{*+}(\check{T}_N)$.*

Proof Let $T' \subset \check{G}_N$ be a maximal torus containing $\check{T}_N$. By Lemma 7, for each $\nu \in X^{*+}(\check{T}_N)$ there is a unique character ν' of T' such that the composition $\check{T}_N \to T' \xrightarrow{\nu'} \mathbb{G}_m$ is ν, and the T'-weight ν' appears in V^ν. Clearly, $\nu \mapsto \nu'$ is a homomorphism of semigroups, and we can apply Lemma 9. Since $\nu \mapsto \nu'$ is a bijection between $X^{*+}(\check{T}_N)$ and the dominant weights of $\check{B}_N$, $\check{T}_N$ is maximal. $\square$

Applying similar arguments for M, one checks that $\check{M}_N$ is reductive, and $\check{T}_N \to \check{M}_N \to \check{G}_N$ are closed immersions, so $\check{M}_N$ is a Levi subgroup of $\check{G}_N$.

4.4. **Rank one.** Let M be the subminimal Levi subgroup of G corresponding to the simple root $\check{\alpha}_i$. As in Lemma 10, there is a unique Borel subgroup $\check{T}_N \subset \check{B}(M)_N \subset \check{M}_N$ whose set of dominant weights is $X^{*+}_M(\check{T}_N)$. View $\check{\alpha}_i$ as a coweight of $\check{T}_N$. Then

$$\{\check{\nu} \in X_*(\check{T}_N) \mid \langle \lambda, \check{\nu}\rangle \geq 0 \text{ for all } \lambda \in X^{*+}_M(\check{T}_N)\}$$

is a $\mathbb{Z}_+$-span of a multiple of $\check{\alpha}_i$. So, $\check{M}_N$ is of semisimple rank one, and its unique simple coroot is of the form $\check{\alpha}_i/\kappa_i$ for some $\kappa_i \in \mathbb{Q}$, $\kappa_i > 0$.

Take any $\lambda \in X^{*+}_M(\check{T}_N)$ with $\langle \lambda, \check{\alpha}_i\rangle > 0$. Write $s_i \in W$ for the simple reflection corresponding to $\check{\alpha}_i$. By Lemma 7, $F^\lambda_M(\mathcal{A}^\lambda_{M,\mathcal{E}})$ and $F^{s_i(\lambda)}_M(\mathcal{A}^\lambda_{M,\mathcal{E}})$ do not vanish, so $\lambda - s_i(\lambda)$ is a multiple of the positive root of $\check{M}_N$. So, this positive root is $\kappa_i \alpha_i$. We learn that the simple reflection for $(\check{M}_N, \check{T}_N)$ acts on $X^*(\check{T}_N)$ as s_i. So, the Weyl groups of G and of $\check{G}_N$, viewed as subgroups of $\mathrm{Aut}(X^*(\check{T}_N))$ are the same. We must show that $\kappa_i = \delta_i$.

Recall that the scheme $\mathrm{Gr}^\nu_{B(M)} \cap \mathrm{Gr}^\lambda_M$ is non empty iff

$$\nu = \lambda, \lambda - \alpha_i, \lambda - 2\alpha_i, \ldots, \lambda - \langle \lambda, \check{\alpha}_i\rangle \alpha_i$$

For $0 < k < \langle \lambda, \check{\alpha}_i\rangle$ and $\nu = \lambda - k\alpha_i$ one has

$$\mathrm{Gr}^\nu_{B(M)} \cap \mathrm{Gr}^\lambda_M \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{A}^{\langle \lambda, \check{\alpha}_i\rangle - k - 1}$$

Write M_0 for the derived group of M, let $T_0 \subset M_0$ be the maximal torus such that $T_0 \subset T$. Consider the central extension

$$1 \to \mathbb{G}_m \to E^a_{M_0} \to M_0(\mathbf{F}) \to 1$$

obtained by pulling back of $1 \to \mathbb{G}_m \to E^a_M \to M(\mathbf{F}) \to 1$ via $M_0(\mathbf{F}) \to M(\mathbf{F})$. It corresponds to the restriction of the bilinear form $2\check{h}\iota$ under

$$X_*(T_0) \times X_*(T_0) \subset X_*(T) \times X_*(T)$$

So, $E^a_{M_0}/M_0(\mathbf{O}) \to \mathrm{Gr}_{M_0}$ is isomorphic to the punctured total space of $\mathcal{L}^{\check{h}(\alpha_i,\alpha_i)}_{M_0}$, where $\mathcal{L}_{M_0}$ is an ample generator of the Picard group of (each connected component of) Gr_{M_0}.

Assume that $\lambda = a\alpha_i$ with $a > 0, a \in \mathbb{Z}$ such that $\lambda \in X^*(\check{T}_N)$. Let $\nu = b\alpha_i$ with $b \in \mathbb{Z}$ satisfy $-\lambda < \nu < \lambda$.

Write $U \subset M(\mathbf{F})$ for the 1-parameter unipotent subgroup corresponding to the affine root space $t^{-a+b}\mathfrak{g}_{\check{\alpha}_i}$. Let Y be the closure of the U-orbit through $t^\nu M(\mathbf{O})$ in Gr_M. It is a T-stable subscheme $Y \widetilde{\to} \mathbb{P}^1$, the T-fixed points in Y are $t^\nu M(\mathbf{O})$ and $t^{-\lambda}M(\mathbf{O})$. The restriction of $\mathcal{L}_{M_0}$ to Y identifies with $\mathcal{O}_{\mathbb{P}^1}(a+b)$. The section

$$a_{B(M),\nu} : \mathrm{Gr}^\nu_{B(M)} \to \mathrm{Gra}^\nu_{B(M)}$$

viewed as a section of the line bundle $\mathcal{L}_{M_0}^{\check{h}(\alpha_i,\alpha_i)}$ over Y will vanish only at $t^{-\lambda}M(\mathbf{O})$ with multiplicity $(a+b)\check{h}(\alpha_i,\alpha_i)$. So, the local system $a^*_{B(M),\nu}\mathcal{A}^\lambda_{M,\mathcal{E}}$ will have the $\mathbb{G}_m$-monodromy $\zeta_a^{(a+b)\check{h}(\alpha_i,\alpha_i)}$. This local system is trivial iff

$$(a+b)\check{h}(\alpha_i,\alpha_i) \in \frac{2\check{h}N}{d}\mathbb{Z}$$

We may assume that $\frac{da}{2N}(\alpha_i,\alpha_i) \in \mathbb{Z}$. Then the above condition is equivalent to $b \in \frac{2N}{d(\alpha_i,\alpha_i)}\mathbb{Z}$. The smallest positive integer b satisfying this condition is δ_i. So, $\kappa_i = \delta_i$. Theorem 1 is proved.

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