Derivational modal logics with the difference modality

Andrey Kudinov
kudinov [at here] iitp [dot] ru
Institute for Information Transmission Problems, Russian Academy of Sciences
National Research University Higher School of Economics, Moscow, Russia
Moscow Institute of Physics and Technology

Valentin Shehtman
shehtman [at here] netscape [dot] net
Institute for Information Transmission Problems, Russian Academy of Sciences
Moscow State University
National Research University Higher School of Economics, Moscow, Russia

May 27, 2014

Abstract

In this chapter we study modal logics of topological spaces in the combined language with the derivational modality and the difference modality. We give axiomatizations and prove completeness for the following classes: all spaces, T_1-spaces, dense-in-themselves spaces, a zero-dimensional dense-in-itself separable metric space, $\mathbf{R}^n$ ($n \geq 2$). We also discuss the correlation between languages with different combinations of the topological, the derivational, the universal and the difference modality in terms of definability.

1 Introduction

Topological modal logic was initiated by the works of A. Tarski and J.C.C. McKinsey in the 1940s. They were first to consider both topological interpretations of the diamond modality: one as closure, and another as derivative.

Their studies of closure modal logics were rather detailed and profound. In particular, in the fundamental paper [32] they have shown that the logic of any metric separable dense-in-itself space is **S4**. This remarkable result also demonstrates a relative weakness of the closure operator to distinguish between interesting topological properties.

The derivational interpretation gives more expressive power. For example, the real line can be distinguished from the real plane (the observation made by K. Kuratowski as early as in 1920s, cf. [27]); the real line can be distinguished from the rational line [37]; T_0 and T_D separation axioms become expressible [5], [14]. However, in [32] McKinsey and Tarski only gave basic definitions for derivational modal logics and put several problems that were solved much later.

The derivational semantics also has its limitations (for example, it is still impossible to distinguish $\mathbf{R}^2$ from $\mathbf{R}^3$). Further increase of expressive power can be provided by the well-known methods of adding universal or difference modalities [18], [17]. In the context of topological semantics this approach also has proved fruitful — for example, connectedness is expressible in modal logic with the closure and the universal modality [38], and the T_1 separation axiom in modal logic with the closure and the difference modality [22].

Until the early 1990s, when the connections between topological modal logic and Computer Science were established, the interest in that subject was moderate. Leo Esakia was one of the enthusiasts of modal logical approach to topology, and he was probably the first to appreciate the role of the derivational modality, in particular, in modal logics of provability [13]. Another strong motivation for further studies of derivational modal logics ('d-logics') were the axiomatization problems left open in [32].[1] In recent years d-logics have been studied rather intensively, a brief summary of results can be found in section 3 below.

In this chapter the first thorough investigation is provided for logics in the most expressive language in this context[2], namely the derivational modal logics with the difference modality ('dd-logics'). It unifies earlier studies by the first author in closure modal logics with the difference modality ('cd-logics') and by the second author in d-logics.

The diagram in section 12 compares the expressive power of different kinds of topomodal logics. Our conjecture is that dd-logics are strictly more expressive than the others, but it is still an open question if the dd-language is stronger than the cd-language. Speaking informally, it is more convenient — for example, the Kuratowski's axiom for $\mathbf{R}^2$ (Definition 9.1) is expressible in cd-logic as well, but in a more complicated form [23].

We show that still in many cases properties of dd-logics are similar to those of d-logics: finite axiomatizability, decidability and the finite model property (fmp). Besides specific results characterizing logics of some particular spaces, our goal was to propose some general methods. In fact, nowadays in topomodal logic there are many technical proofs, but few general methods. In this chapter we propose only two simplifying novelties — dd-morphisms (section 6) and the Glueing lemma 6.9, but we hope that much more can be done in this direction, cf. the recent paper [20].

In more detail, the plan of the chapter is as follows. Preliminary sections 2–4 include standard definitions and basic facts about modal logics and their semantics. Some general completeness results for dd-logics can be found in sections 5, 7. In section 5 we show that every extension of

[1] The early works of the second author in this field were greatly influenced by Leo Esakia.

[2] Some other kinds of topomodal logics arise when we deal with topological spaces with additional structures, e.g. spaces with two topologies, spaces with a homeomorphism etc. (cf. [19]).

the minimal logic $\mathbf{K4}^\circ\mathbf{D}^+$ by variable-free axioms is topologically complete. In section 8 we prove the same for extensions of $\mathbf{DT_1}$ (the logic of dense-in-themselves T_1-spaces); the proof is based on a construction of d-morphisms from the recent paper [8].

In section 6 we consider validity-preserving maps from topological to Kripke frames (d-morphisms and dd-morphisms) and prove a modified version of McKinsey–Tarski's lemma on dissectable spaces. In section 7 we prove that $\mathbf{DT_1}$ is complete w.r.t. an arbitrary zero-dimensional dense-in-itself separable metric space by the method from [37], [39].

Sections 8–10 study the axiom of connectedness AC and Kuratowski's axiom Ku related to local 1-componency. In particular we prove that the logic $\mathbf{DT_1CK}$ with both these axioms has the fmp. This is a refinement of an earlier result [37], [39] on the fmp of the d-logic $\mathbf{D4} + Ku$ (the new proof uses a simpler construction).

Section 11 contains our central result: $\mathbf{DT_1CK}$ is the dd-logic of $\mathbf{R}^n$ for $n > 1$. The proof uses an inductive construction of dd-morphisms onto finite frames of the corresponding logic, and it combines methods from [37], [39], [23], with an essential improvement motivated by [31] and based on the Glueing lemma.

The final section discusses some further directions and open questions. The Appendix contains technical details of some proofs.

2 Basic notions

The material of this section is quite standard, and most of it can be found in [11]. We consider *n-modal (propositional) formulas* constructed from a countable set of propositional variables PV and the connectives $\bot$, $\to$, $\Box_1, \dots, \Box_n$. The derived connectives are $\wedge$, $\vee$, $\neg$, $\top$, $\leftrightarrow$, $\Diamond_1, \dots, \Diamond_n$. A formula without occurrences of propositional variables is called *closed*.

A *(normal) n-modal logic* is a set of modal formulas containing the classical tautologies, the axioms $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$ and closed under the standard inference rules: Modus Ponens $(A,\ A \to B/B)$, Necessitation $(A/\Box_i A)$, and Substitution $(A(p_j)/A(B))$.

To be more specific, we use the terms '$(\Box_1, \dots, \Box_n)$-modal formula' and '$(\Box_1, \dots, \Box_n)$-modal logic'.

$\mathbf{K}_n$ denotes the minimal n-modal logic (and $\mathbf{K} = \mathbf{K}_1$). An n-modal logic containing a certain n-modal logic Λ is called an *extension* of Λ, or a Λ*-logic*. The minimal Λ-logic containing a set of n-modal formulas Γ is denoted by $\Lambda + \Gamma$. In particular,

$$\mathbf{K4} := \mathbf{K} + \Box p \to \Box\Box p,\ \mathbf{S4} := \mathbf{K4} + \Box p \to p,\ \mathbf{D4} := \mathbf{K4} + \Diamond\top,$$

$$\mathbf{K4}^\circ := \mathbf{wK4} := \mathbf{K} + p \wedge \Box p \to \Box\Box p.$$

The *fusion* $L_1 * L_2$ of modal logics $L_1,\ L_2$ with distinct modalities is the smallest modal logic in the joined language containing $L_1 \cup L_2$.

A *(normal) n-modal algebra* is a Boolean algebra with extra n unary operations preserving $\mathbf{1}$ (the unit) and distributing over $\cap$; they are often denoted by $\Box_1, \dots, \Box_n$, in the same way as the modal connectives. A *valuation* in a modal algebra $\mathfrak{A}$ is a set-theoretic map $\theta : PV \longrightarrow \mathfrak{A}$. It extends to all n-modal formulas by induction:

$$\theta(\bot) = \varnothing,\ \theta(A \to B) = -\theta(A) \cup \theta(B),\ \theta(\Box_i A) = \Box_i \theta(A).$$

A formula A is *true in* $\mathfrak{A}$ (in symbols: $\mathfrak{A} \vDash A$) if $\theta(A) = \mathbf{1}$ for any valuation θ. The set $\mathbf{L}(\mathfrak{A})$ of all n-modal formulas true in an n-modal algebra $\mathfrak{A}$ is an n-modal logic called the *logic of* $\mathfrak{A}$.

An n-modal *Kripke frame* is a tuple $F = (W, R_1, \ldots, R_n)$, where W is a nonempty set (of worlds), R_i are binary relations on W. We often write $x \in F$ instead of $x \in W$. In this chapter (except for Section 2) all 1-modal frames are assumed to be transitive. The associated n-modal algebra $MA(F)$ is 2^W (the Boolean algebra of all subsets of W) with the operations $\Box_1, \ldots, \Box_n$ such that $\Box_i V = \{x \mid R_i(x) \subseteq V\}$ for any $V \subseteq W$.

A *valuation* in F is the same as in $MA(F)$, i.e., this is a map from PV to $\mathcal{P}(W)$ (the power set of W). A *(Kripke) model* over F is a pair $M = (F, \theta)$, where θ is a valuation in F. The notation $M, x \vDash A$ means $x \in \theta(A)$, which is also read as 'A is *true in M at x*'. A (modal) formula A is *true in* M (in symbols: $M \vDash A$) if A is true in M at all worlds. A formula A is called *valid in* a Kripke frame F (in symbols: $F \vDash A$) if A is true in all Kripke models over F; this is obviously equivalent to $MA(F) \vDash A$.

The *modal logic* $\mathbf{L}(F)$ of a Kripke frame F is the set of all modal formulas valid in F, i.e., $\mathbf{L}(MA(F))$. For a class of n-modal frames $\mathcal{C}$, the *modal logic* of $\mathcal{C}$ (or *the modal logic determined* by $\mathcal{C}$) is $\mathbf{L}(\mathcal{C}) := \bigcap\{\mathbf{L}(F) \mid F \in \mathcal{C}\}$. Logics determined by classes of Kripke frames are called *Kripke complete*. An n-modal frame validating an n-modal logic Λ is called a Λ-*frame*. A modal logic has the *finite model property (fmp)* if it is determined by some class of finite frames.

It is well known that $(W, R) \vDash \mathbf{K4}$ iff R is transitive; $(W, R) \vDash \mathbf{S4}$ iff R is reflexive transitive (a *quasi-order*).

A *cluster* in a transitive frame (W, R) is an equivalence class under the relation $\sim_R := (R \cap R^{-1}) \cup I_W$, where I_W is the equality relation on W. A *degenerate cluster* is an irreflexive singleton. A cluster that is a reflexive singleton, is called *trivial*, or *simple*. A *chain* is a frame (W, R) with R transitive, antisymmetric and linear, i.e., it satisfies $\forall x \forall y\ (xRy \lor yRx \lor x = y)$. A point $x \in W$ is *strictly (R-)minimal* if $R^{-1}(x) = \varnothing$.

A *subframe* of a frame $F = (W, R_1, \ldots, R_n)$ obtained by restriction to $V \subseteq W$, is $F|V := (V, R_1|V, \ldots, R_n|V)$. Then for any Kripke model $M = (F, \theta)$ we have a *submodel* $M|V := (F|V, \theta|V)$, where $(\theta|V)(q) := \theta(q) \cap V$ for each $q \in PV$. If $R_i(V) \subseteq V$ for any i, the subframe $F|V$ and the submodel $M|V$ are called *generated*.

The *union* of subframes $F_j = F|W_j,\ j \in J$ is the subframe $\bigcup\limits_{j \in J} F_j := F|\bigcup\limits_{j \in J} W_j$.

A *generated subframe (cone) with the root* x is $F^x := F|R^*(x)$, where R^* is the reflexive transitive closure of $R_1 \cup \ldots \cup R_n$; so for a transitive frame (W, R), $R^* = R \cup I_W$ is the reflexive closure of R (which is also denoted by $\overline{R}$). A frame F is called *rooted* with the root u if $F = F^u$. Similarly we define a cone M^x of a Kripke model M.

Every finite rooted transitive frame $F = (W, R)$ can be presented as the union $(F|C) \cup F^{x_1} \cup \ldots \cup F^{x_m}$ $(m \geq 0)$, where C is the root cluster, x_i are its successors (i.e., $x_i \notin C$, $\overline{R}^{-1}(x_i) = \sim_R (x_i) \cup C$). If C is non-degenerate, the frame $F|C$ is (C, C^2), which we usually denote just by C. If $C = \{a\}$ is degenerate, $F|C$ is $(\{a\}, \varnothing)$, which we denote by $\breve{a}$.

Let us fix the propositional language (and the number n) until the end of this section.

Lemma 2.1. (Generation Lemma)

(1) $\mathbf{L}(F) = \bigcap\{\mathbf{L}(F^x) \mid x \in F\}$.

(2) If F is a generated subframe of G, then $\mathbf{L}(G) \subseteq \mathbf{L}(F)$.

(3) If M is a generated submodel of N, then for any formula A for any x in M

$$N, x \vDash A \text{ iff } M, x \vDash A.$$

Lemma 2.2. *For any Kripke complete modal logic Λ,*

$$\Lambda = \mathbf{L}(\textit{all } \Lambda\textit{-frames}) = \mathbf{L}(\textit{all rooted } \Lambda\textit{-frames}).$$

A *p-morphism* from a frame $(W, R_1, \ldots, R_n)$ onto a frame $(W', R'_1, \ldots, R'_n)$ is a surjective map $f : W \longrightarrow W'$ satisfying the following conditions (for any i):

(1) $\forall x \forall y\ (xR_iy \Rightarrow f(x)R'_if(y))$ (monotonicity);

(2) $\forall x \forall z\ (f(x)R'_iz \Rightarrow \exists y(f(y) = z \ \&\ xR_iy))$ (the lift property).

If xR_iy and $f(x)R'_if(y)$, we say that xR_iy *lifts* $f(x)R'_if(y)$.

Note that (1) & (2) is equivalent to

$$\forall x\ f(R_i(x)) = R'_i(f(x)).$$

$f:\ F \twoheadrightarrow F'$ denotes that f is a p-morphism from F onto F'.

Every set-theoretic map $f : W \longrightarrow W'$ gives rise to the dual morphism of Boolean algebras $2^f : 2^{W'} \longrightarrow 2^W$ sending every subset $V \subseteq W'$ to its inverse image $f^{-1}(V) \subseteq W$.

Lemma 2.3. (P-morphism Lemma)

(1) $f:\ F \twoheadrightarrow F'$ iff 2^f is an embedding of $MA(F')$ in $MA(F)$.

(2) $f:\ F \twoheadrightarrow F'$ implies $\mathbf{L}(F) \subseteq \mathbf{L}(F')$.

(3) If $f:\ F \twoheadrightarrow F'$, then $F \vDash A \Leftrightarrow F' \vDash A$ for any closed formula A.

In proofs of the fmp in this chapter we will use the well-known filtration method [11]. Let us recall the construction we need.

Let Ψ be a set of modal formulas closed under subformulas. For a Kripke model $M = (F, \varphi)$ over a frame $F = (W, R_1, \ldots, R_n)$, there is the equivalence relation on W

$$x \equiv_\Psi y \Longleftrightarrow \forall A \in \Psi(M, x \vDash A \Leftrightarrow M, y \vDash A).$$

Put $W' := W/\equiv_\Psi$; $\ x^\sim := \equiv_\Psi (x)$ (the equivalence class of x), $\varphi'(q) := \{x^\sim \mid x \in \varphi(q)\}$ for $q \in PV \cap \Psi$ (and let $\varphi'(q)$ be arbitrary for $q \in PV - \Psi$).

Lemma 2.4. (Filtration Lemma) *Under the above assumptions, consider the relations $\underline{R}_i, R'_i$ on W' such that*

$$a\underline{R}_i b \text{ iff } \exists x \in a \ \exists y \in b \ xR_i y,$$

$$R'_i = \begin{cases} \text{the transitive closure of } \underline{R}_i & \text{if } R_i \text{ is transitive,} \\ \underline{R}_i & \text{otherwise.} \end{cases}$$

Put $M' := (W', R'_1, \ldots, R'_n, \varphi')$. Then for any $x \in W$, $A \in \Psi$:

$$M, x \vDash A \text{ iff } M', x^{\sim} \vDash A.$$

Definition 2.5. *An m-*formula *is a modal formula in propositional variables $\{p_1, \ldots, p_m\}$. For a modal logic Λ we define the m-*weak *(or m-*restricted) canonical frame $F_{\Lambda\lceil m} := (W, R_1, \ldots, R_n)$ and canonical model $M_{\Lambda\lceil m} := (F_{\Lambda\lceil m}, \varphi)$*, where W is the set of all maximal Λ-consistent sets of m-formulas, xR_iy iff for any m-formula A*
$(\Box_i A \in x \Rightarrow A \in y)$,

$$\varphi(p_i) := \begin{cases} \{x \mid p_i \in x\} & \text{if } i \le m, \\ \varnothing & \text{if } i > m. \end{cases}$$

Λ *is called* weakly canonical *if $F_{\Lambda\lceil m} \vDash \Lambda$ for any finite m.*

Proposition 2.6. *For any m-formula A and a modal logic Λ*

(1) $M_{\Lambda\lceil m}, x \vDash A$ iff $A \in x$;

(2) $M_{\Lambda\lceil m} \vDash A$ iff $A \in \Lambda$;

(3) if Λ is weakly canonical, then it is Kripke complete.

Corollary 2.7. *If for any m-formula A, $M_{\Lambda\lceil m}, x \vDash A \Leftrightarrow M_{\Lambda\lceil m}, y \vDash A$, then $x = y$.*

Definition 2.8. *A cluster C in a transitive frame (W, R) is called* maximal *if $\overline{R}(C) = C$.*

Lemma 2.9. *Let $F_{\Lambda\lceil m} = (W, R_1, \ldots, R_n)$ and suppose $\Lambda \vdash \Box_1 p \to \Box_1\Box_1 p$ (i.e., R_1 is transitive). Then every generated subframe of (W, R_1) contains a maximal cluster.*

The proof is based on the fact that the general Kripke frame corresponding to a canonical model is descriptive; cf. [11], [15] for further details[3].

3 Derivational modal logics

We denote topological spaces by $\mathfrak{X}, \mathfrak{Y}, \ldots$ and the corresponding sets by $X, Y, \ldots$.[4] The interior operation in a space $\mathfrak{X}$ is denoted by $\mathbf{I}_X$ and the closure operation by $\mathbf{C}_X$, but we often omit the subscript X. A set S is a *neighbourhood* of a point x if $x \in \mathbf{I}S$; then $S - \{x\}$ is called a *punctured neighbourhood* of x.

[3] For the 1-modal case this lemma has been known as folklore since the 1970s; the second author learned it from Leo Esakia in 1975.

[4] Sometimes we neglect this difference.

Definition 3.1. *Let $\mathfrak{X}$ be a topological space, $V \subseteq X$. A point $x \in X$ is said to be* limit *for V if $x \in \mathbf{C}(V - \{x\})$; a non-limit point of V is called* isolated.

The derived set of V *(denoted by $\mathbf{d}V$, or by $\mathbf{d}_X V$) is the set of all limit points of V. The unary operation $V \mapsto \mathbf{d}V$ on $\mathcal{P}(X)$ is called* the derivation *(in $\mathfrak{X}$).*

A set without isolated points is called dense-in-itself.

Lemma 3.2. *[28] For a subspace $\mathcal{Y} \subseteq \mathfrak{X}$ and $V \subseteq X$ $\mathbf{d}_Y(V \cap Y) = \mathbf{d}_X(V \cap Y) \cap Y$; if Y is open, then $\mathbf{d}_Y(V \cap Y) = \mathbf{d}_X V \cap Y$.*

Definition 3.3. *The* derivational algebra of a topological space $\mathfrak{X}$ *is $DA(X) := (2^X, \tilde{\mathbf{d}})$, where 2^X is the Boolean algebra of all subsets of X, $\tilde{\mathbf{d}}V := -\mathbf{d}(-V)$*[5]*. The* closure algebra of a space $\mathfrak{X}$ *is $CA(\mathfrak{X}) := (2^X, \mathbf{I})$.*

Remark 3.4. In [32] the derivational algebra of $\mathfrak{X}$ is defined as $(2^X, \mathbf{d})$, and the closure algebra as $(2^X, \mathbf{C})$, but here we adopt equivalent dual definitions.

It is well known that $CA(\mathfrak{X})$, $DA(\mathfrak{X})$ are modal algebras, $CA(\mathfrak{X}) \vDash \mathbf{S4}$ and $DA(\mathfrak{X}) \vDash \mathbf{K4}^\circ$ (the latter is due to Esakia).

Every Kripke **S4**-frame $F = (W, R)$ is associated with a topological space $N(F)$ on W, with the *Alexandrov* (or *right*) *topology* $\{V \subseteq W \mid R(V) \subseteq V\}$. In $N(F)$ we have $\mathbf{C}V = R^{-1}(V)$, $\mathbf{I}V = \{x \mid R(x) \subseteq V\}$; thus $MA(F) = CA(N(F))$.

Definition 3.5. *A modal formula A is called* d-valid in a topological space $\mathfrak{X}$ *(in symbols, $\mathfrak{X} \vDash^d A$) if it is true in the algebra $DA(\mathfrak{X})$. The logic $\mathbf{L}(DA(\mathfrak{X}))$ is called the* derivational modal logic *(or the* d-logic*) of $\mathfrak{X}$ and denoted by $\mathbf{Ld}(\mathfrak{X})$.*

A formula A is called c-valid *in $\mathfrak{X}$ (in symbols, $\mathfrak{X} \vDash^c A$) if it is true in $CA(\mathfrak{X})$. $\mathbf{Lc}(\mathfrak{X}) := \mathbf{L}(CA(\mathfrak{X}))$ is called the* closure modal logic, *or the* c-logic *of $\mathfrak{X}$.*

Definition 3.6. *For a class of topological spaces $\mathcal{C}$ we also define the* d-logic $\mathbf{Ld}(\mathcal{C}) := \bigcap\{\mathbf{Ld}(\mathfrak{X}) \mid \mathfrak{X} \in \mathcal{C}\}$ *and the* c-logic $\mathbf{Lc}(\mathcal{C}) := \bigcap\{\mathbf{Lc}(\mathfrak{X}) \mid \mathfrak{X} \in \mathcal{C}\}$*. Logics of this form are called* d-complete *(respectively,* c-complete *).*

Definition 3.7. *A* valuation *in a topological space $\mathfrak{X}$ is a map $\varphi : PV \longrightarrow \mathcal{P}(\mathfrak{X})$. Then $(\mathfrak{X}, \varphi)$ is called a* topological model *over $\mathfrak{X}$.*

So valuations in $\mathfrak{X}$, $CA(\mathfrak{X})$, and $DA(\mathfrak{X})$ are the same. Every valuation φ can be prolonged to all formulas in two ways, according either to $CA(\mathfrak{X})$ or $DA(\mathfrak{X})$. The corresponding maps are denoted respectively by φ_c or φ_d. Thus

$$\varphi_d(\Box A) = \tilde{\mathbf{d}}\varphi_d(A), \qquad \varphi_d(\Diamond A) = \mathbf{d}\varphi_d(A),$$
$$\varphi_c(\Box A) = \mathbf{I}\varphi_c(A), \qquad \varphi_c(\Diamond A) = \mathbf{C}\varphi_c(A).$$

A formula A is called *d-true* (respectively, *c-true*) in $(\mathfrak{X}, \varphi)$ if $\varphi_d(A) = X$ (respectively, $\varphi_c(A) = X$). So A is d-valid in $\mathfrak{X}$ iff A is d-true in every topological model over $\mathfrak{X}$, similarly for c-validity.

[5] There is no common notation for this operation; some authors use τ.

Definition 3.8. *A modal formula A is called* d-true at a point x *in a topological model* $(\mathfrak{X}, \varphi)$ *if* $x \in \varphi_d(A)$.

Instead of $x \in \varphi_d(A)$, we write $x \vDash^d A$ if the model is clear from the context. Similarly we define the c-truth at a point and use the corresponding notation.

From the definitions we obtain

Lemma 3.9. *For a topological model over a space* $\mathfrak{X}$

- $x \vDash^d \Box A$ *iff* $\exists U \ni x$ (U *is open in* $\mathfrak{X}$ & $\forall y \in U - \{x\}$ $y \vDash^d A$)*;*
- $x \vDash^d \Diamond A$ *iff* $\forall U \ni x$ (U *is open in* $\mathfrak{X}$ $\Rightarrow$ $\exists y \in U - \{x\}$ $y \vDash^d A$)*.*

Definition 3.10. *A* local T_1-space *(or a* T_D-space *[4]) is a topological space, in which every point is* locally closed*, i.e, closed in some neighbourhood.*

Note that a point x in an Alexandrov space $N(W, R)$ is closed iff it is minimal (i.e., $R^{-1}(x) = \{x\}$); x is locally closed iff $R(x) \cap R^{-1}(x) = \{x\}$. Thus $N(F)$ is local T_1 iff F is a poset.

Lemma 3.11. *[14] For a topological space* $\mathfrak{X}$

(1) $\mathfrak{X} \vDash^d \mathbf{K4}$ *iff* $\mathfrak{X}$ *is local* T_1;

(2) $\mathfrak{X} \vDash^d \Diamond\top$ *iff* $\mathfrak{X}$ *is dense-in-itself.*

Definition 3.12. *A Kripke frame (W,R) is called* weakly transitive *if* $R \circ R \subseteq \overline{R}$.

It is obvious that the weak transitivity of R is equivalent to the transitivity of $\overline{R}$.

Proposition 3.13. *[14] (1)* $(W, R) \vDash \mathbf{K4}^\circ$ *iff* (W, R) *is weakly transitive*;
(2) $\mathbf{K4}^\circ$ *is Kripke-complete.*

Lemma 3.14. *[14] (1) Let* $F = (W, R)$ *be a Kripke* $\mathbf{S4}$*-frame, and let* $R^\circ := R - I_W$, $F^\circ := (W, R^\circ)$*. Then* $\mathbf{Ld}(N(F)) = \mathbf{L}(F^\circ)$*.*
(2) Let $F = (W, R)$ *be a weakly transitive irreflexive Kripke frame, and let* $\overline{F} =: (W, \overline{R})$ *be its reflexive closure. Then* $\mathbf{Ld}(N(\overline{F})) = \mathbf{L}(F)$*.*
(3) If $\Lambda = \mathbf{L}(\mathcal{C})$*, for some class* $\mathcal{C}$ *of weakly transitive irreflexive Kripke frames, then* Λ *is d-complete.*

Proof. (1) Note that $R^\circ(x)$ is the smallest punctured neighbourhood of x in the space $N(F)$. So the inductive d-truth definition in a topological model $(N(F), \varphi)$ coincides with the inductive truthdefinition in the Kripke model (W, R°, φ).

(2) Readily follows from (1), since $\overline{R}$ is transitive and $(\overline{R})^\circ = R$.

(3) Follows from (2). ■

Definition 3.15. *For a 1-modal formula* A *we define* $A^\sharp$ *as the formula obtained by replacing every occurrence of every subformula* $\Box B$ *with* $\overline{\Box} B := \Box B \wedge B$*. For a 1-modal logic* Λ *its* reflexive fragment *is* ${}^\sharp\Lambda := \{A \mid \Lambda \vdash A^\sharp\}$.

Proposition 3.16. *[5] (1) If* Λ *is a* $\mathbf{K4}^\circ$*-logic, then* ${}^\sharp\Lambda$ *is an* $\mathbf{S4}$*-logic.*
(2) For any topological space X, $\mathbf{Lc}(\mathfrak{X}) = {}^\sharp\mathbf{Ld}(\mathfrak{X})$*,*
(3) For any weakly transitive Kripke frame F, $\mathbf{L}(\overline{F}) = {}^\sharp\mathbf{L}(F)$*.*

Proof. (1) It is clear that for a weakly transitive Λ, $\overline{\Box}$ satisfies the axioms of **S4**, so ${}^\sharp\Lambda$ contains these axioms. Since $^\sharp$ distributes over implication, it follows that ${}^\sharp\Lambda$ is closed under Modus Ponens. For the substitution closedness, note that for any variable p and formulas A, B $([B/p]A)^\sharp = [B^\sharp/p]A^\sharp$; thus $A \in {}^\sharp\Lambda$ implies $[B/p]A \in {}^\sharp\Lambda$.Finally, since $(\Box A)^\sharp = \overline{\Box}\, A^\sharp$, it is clear that $A \in {}^\sharp\Lambda$ only if $\Box A \in {}^\sharp\Lambda$.

(2) By definitions,

$$\mathbf{Lc}(X) \vdash A \text{ iff } CA(X) \vDash A,$$

$${}^\sharp\mathbf{Ld}(X) \vdash A \text{ iff } \mathbf{Ld}(X) \vdash A^\sharp \text{ iff } DA(X) \vDash A^\sharp.$$

Let us show that that $CA(X) \nvDash A$ iff $DA(X) \nvDash A^\sharp$. In fact, consider a topological model (X, φ). We claim that

$$\varphi_c(B) = \varphi_d(B^\sharp) \tag{$*$}$$

for any formula B. This is easily checked by induction, the crucial case is when $B = \Box B_1$; then by definitions and the induction hypothesis we have:

$$\varphi_c(B) = \mathbf{I}\varphi_c(B_1) = \mathbf{I}\varphi_d(B_1^\sharp) = \boxdot\varphi_d(B_1^\sharp) \cap \varphi_d(B_1^\sharp) = \varphi_d(\overline{\Box}\, B_1^\sharp) = \varphi_d(B^\sharp).$$

The claim (*) implies that $\varphi_c(A) \neq X$ iff $\varphi_d(A^\sharp) \neq X$ as required.

(3) On the one hand,

$$\mathbf{L}(\overline{F}) = \mathbf{L}(MA(\overline{F})) = \mathbf{L}(CA(N(\overline{F})) = \mathbf{Lc}(N(\overline{F})).$$

On the other hand, by Lemma 3.14(2),

$$\mathbf{L}(F) = \mathbf{Ld}(N(\overline{F})),$$

and we can apply (2) to $N(F)$. ■

Let us give some examples of d-complete logics.

(1) $\mathbf{Ld}$(all topological spaces) $= \mathbf{K4}^\circ$. This was proved by L. Esakia in the 1970s and published in [14].

(2) $\mathbf{Ld}$(all local T_1-spaces) $= \mathbf{K4}$. This is also a result from [14].

(3) $\mathbf{Ld}$(all T_0-spaces) $= \mathbf{K4}^\circ + p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q)$. This result is from [7].

(4) L. Esakia [13] also proved that Gödel - Löb logic $\mathbf{GL} := \mathbf{K} + \Box(\Box p \to p) \to \Box p$ is the derivational logic of the class of all topological scattered spaces (a space is *scattered* if each its nonempty subset has an isolated point).

(5) The papers [1], [2], [9] give a complete description of d-logics of ordinals with the interval topology: $\mathbf{Ld}(\alpha)$ is either $\mathbf{GL}$ (if $\alpha \geq \omega^\omega$), or $\mathbf{GL} + \Box^n\bot$ (if $\omega^{n-1} \leq \alpha < \omega^n$). In particular, $\mathbf{Ver} := \mathbf{K} + \Box\bot$ is the d-logic of any finite ordinal (and of any discrete space).

(6) The well-known "difference logic" [36], [12] $\mathbf{DL} := \mathbf{K4}^\circ + \Diamond\Box p \to p$, is determined by Kripke frames with the difference relation: $\mathbf{DL} = \mathbf{L}(\{(W, \neq_W) \mid W \neq \varnothing\})$, where $\neq_W := W^2 - I_W$; hence by 3.16, **DL** is the d-logic of the class of all trivial topological spaces. However, for any particular trivial space $\mathfrak{X}$, $\mathbf{Ld}(\mathfrak{X}) \neq \mathbf{DL}$. Moreover, $\mathbf{Ld}(\mathfrak{X})$ is not finitely axiomatizable for any infinite trivial $\mathfrak{X}$[26]; this surprising result is easily proved by a standard technique using Jankov formulas (cf. [24]).

(7) In [39] it was proved that **Ld**(all 0-dimensional separable metric spaces) = **K4**. All these spaces are embeddable in **R** [28].

(8) In [39] it was also proved that for any dense-in itself separable metric space $\mathfrak{X}$, $\mathbf{Ld}(\mathfrak{X}) = \mathbf{D4}$; this was a generalization of an earlier proof [37] for $\mathfrak{X} = \mathbf{Q}$. A more elegant proof for **Q** is in [30].

(9) Every extension of **K4** by a set of closed axioms is a d-logic of some subspace of **Q** [8]. This gives us a continuum of d-logics of countable metric spaces.

(10) In [37] $\mathbf{Ld}(\mathbf{R}^2)$ was axiomatized and it was also proved that the d-logics of $\mathbf{R}^n$ for $n \geq 2$ coincide. We will simplify and extend that proof in the present chapter.

(11) **Ld**(**R**) was described in [39]; for a simpler completeness proof cf. [31].

(12) **Ld**(all Stone spaces) = **K4** and **Ld**(all weakly scattered Stone spaces) = $\mathbf{K4} + \Diamond\top \to \Diamond\Box\bot$, cf. [6].

(13) d-logics of special types of spaces were studied in [5], [30]. They include submaximal, perfectly disconnected, maximal, weakly scattered and some others.

However, not all extensions of $\mathbf{K4}^\circ$ are d-complete. In fact, the formula $p \to \Diamond p$ never can be d-valid, because $\mathbf{d}Y = \varnothing$ for a singleton Y. So every extension of **S4** is d-incomplete, and thus Kripke completeness does not imply d-completeness.

Proposition 3.17. *Let $F = (\omega^*, \prec)$ be the "standard irreflexive transitive tree", where ω^* is the set of all finite sequences in ω; $\alpha \prec \beta$ iff α is a proper initial segment of β. Then*

$$\mathbf{D4} = \mathbf{L}(F) = \mathbf{Ld}(N(\overline{F})) = \mathbf{Ld}(\mathcal{D}),$$

where $\mathcal{D}$ denotes the class of all dense-in-themselves local T_1-spaces.

Proof. The first equality is well known [41]; the second one holds by 3.14. By 3.11, **D4** is d-valid exactly in spaces from $\mathcal{D}$. So $N(\overline{F}) \in \mathcal{D}$, $\mathbf{D4} \subseteq \mathbf{Ld}(\mathcal{D})$, and the third equality follows. ■

4 Adding the universal modality and the difference modality

Recall that the *universal modality* [∀] and the *difference modality* [≠] correspond to Kripke frames with the universal and the difference relation.

So (under a valuation in a set W) these modalities are interpreted in the standard way:

$$x \vDash [\forall]A \text{ iff } \forall y \in W\ y \vDash A; \qquad x \vDash [\neq]A \text{ iff } \forall y \in W\ (y \neq x \Rightarrow y \vDash A).$$

The corresponding dual modalities are denoted by $\langle\exists\rangle$ and $\langle\neq\rangle$.

Definition 4.1. *For a* $[\forall]$*-modal formula* A *we define the* $[\neq]$*-modal formula* A^u *by induction:*

$$A^u := A \textit{ for } A \textit{ atomic}, \quad (A \to B)^u := A^u \to B^u, \quad ([\forall]B)^u := [\neq]B^u \wedge B^u.$$

We can consider 2-modal topological logics obtained from $\mathbf{Lc}(\mathfrak{X})$ or $\mathbf{Ld}(\mathfrak{X})$ by adding the universal or the difference modality[6]. Thus for a topological space $\mathfrak{X}$ we obtain four 2-modal logics : $\mathbf{Lc}_\forall(\mathfrak{X})$ (the *closure universal (cu-) logic*), $\mathbf{Ld}_\forall(\mathfrak{X})$ (the *derivational universal (du-) logic*), $\mathbf{Lc}_{\neq}(\mathfrak{X})$ (the *closure difference (cd-) logic*), $\mathbf{Ld}_{\neq}(\mathfrak{X})$ (the *derivational difference (dd-) logic*). Similar notations ($\mathbf{Lc}_\forall(\mathcal{C})$ etc.) are used for logics of a class of spaces $\mathcal{C}$, and respectively we can define four kinds of topological completeness (cu-, du-, cd-, dd-) for 2-modal logics.

cd-logics were first studied in [16], cu-logics in [38], du-logics in [31], but dd-logics have never been addressed so far.

For a $\Box$-modal logic $\mathbf{L}$ we define the 2-modal logics

$$\mathbf{LD} := \mathbf{L} * \mathbf{DL} + [\neq]p \wedge p \to \Box p, \qquad \mathbf{LD}^+ := \mathbf{L} * \mathbf{DL} + [\neq]p \to \Box p,$$
$$\mathbf{LU} := \mathbf{L} * \mathbf{S5} + [\forall]p \to \Box p.$$

Here we suppose that $\mathbf{S5}$ is formulated in the language with $[\forall]$ and $\mathbf{DL}$ in the language with $[\neq]$. The following is checked easily:

Lemma 4.2. *For any topological space* $\mathfrak{X}$*,*

$$\mathbf{Lc}_\forall(\mathfrak{X}) \supseteq \mathbf{S4U}, \quad \mathbf{Ld}_\forall(\mathfrak{X}) \supseteq \mathbf{K4^\circ U}, \quad \mathbf{Lc}_{\neq}(\mathfrak{X}) \supseteq \mathbf{S4D}, \quad \mathbf{Ld}_{\neq}(\mathfrak{X}) \supseteq \mathbf{K4^\circ D^+}.$$

Definition 4.3. *For a 1-modal Kripke frame* $F = (W, R)$ *we define 2-modal frames* $F_\forall := (F, W^2)$*,* $F_{\neq} := (F, \neq_W)$ *and modal logics* $\mathbf{L}_\forall(F) := \mathbf{L}(F_\forall)$*,* $\mathbf{L}_{\neq}(F) := \mathbf{L}(F_{\neq})$*.*

Sahlqvist theorem [11] implies

Proposition 4.4. *The logics* $\mathbf{S4U}$*,* $\mathbf{K4^\circ U}$*,* $\mathbf{S4D}$*,* $\mathbf{K4^\circ D^+}$ *are Kripke complete.*

Using the first-order equivalents of the modal axioms for these logics (in particular, Proposition 3.13) we obtain

Lemma 4.5. *For a rooted Kripke frame* $G = (W, R, S)$

(1) $G \vDash \mathbf{S4U}$ *iff* R *is a quasi-order &* $S = W^2$*,*

(2) $G \vDash \mathbf{K4^\circ U}$ *iff* R *is weakly transitive &* $S = W^2$*,*

(3) $G \vDash \mathbf{S4D}$ *iff* R *is a quasi-order &* $\overline{S} = W^2$*,*

(4) $G \vDash \mathbf{K4^\circ D^+}$ *iff* R *is weakly transitive &* $\overline{S} = W^2$*&* $R \subseteq S$*.*

[6] So we extend the definitions of the d-truth or the c-truth by adding the item for $[\forall]$ or $[\neq]$.

Also note that $\overline{S} = W^2$ *iff* $\neq_W \subseteq S$.

Definition 4.6. *A rooted Kripke* $\mathbf{K4^\circ D^+}$*-frame described by Lemma 4.5 (4) is called* basic. *The class of these frames is denoted by* $\mathfrak{F}_0$.

Next, we easily obtain the 2-modal analogue to Lemma 3.14.

Lemma 4.7. *(1) Let* F *be an* $\mathbf{S4}$*-frame. Then*

$$\mathbf{Ld}_{\neq}(N(F)) = \mathbf{L}_{\neq}(F^\circ),\ \mathbf{Ld}_{\forall}(N(F)) = \mathbf{L}_{\forall}(F^\circ).$$

(2) Let F *be a weakly transitive irreflexive Kripke frame. Then*

$$\mathbf{Ld}_{\neq}(N(\overline{F})) = \mathbf{L}_{\neq}(F),\ \mathbf{Ld}_{\forall}(N(\overline{F})) = \mathbf{L}_{\forall}(F).$$

(3) Let $\mathcal{C}$ *be a class of weakly transitive irreflexive Kripke 1-frames. Then* $\mathbf{L}_{\neq}(\mathcal{C})$ *is dd-complete,* $\mathbf{L}_{\forall}(\mathcal{C})$ *is du-complete.*

Let us extend the translations $(-)^\sharp$, $(-)^u$ to 2-modal formulas.

Definition 4.8. $(-)^u$ *translates* $(\Box, [\forall])$*-modal formulas to* $(\Box, [\neq])$*-modal formulas so that* $([\forall]B)^u = [\neq]B^u \wedge B^u$ *and* $(-)^u$ *distributes over the other connectives.*

Similarly, $(-)^\sharp$ *translates* $(\Box, [\neq])$*-modal formulas and* $(\Box, [\forall])$*-modal formulas to formulas of the same kind, so that* $(\Box B)^\sharp = \Box B^\sharp \wedge B^\sharp$ *and* $(-)^\sharp$ *distributes over the other connectives.*

$^u\Lambda := \{A \mid A^u \in \Lambda\}$ *for a* $(\Box, [\forall])$*-modal logic* Λ *(the* universal fragment*),*

$^\sharp\Lambda := \{A \mid A^\sharp \in \Lambda\}$ *for a* $(\Box, [\neq])$*- or a* $(\Box, [\forall])$*-modal* Λ *(the* reflexive fragment*),*

$^{\sharp u}\Lambda := {}^\sharp({}^u\Lambda)$ *for a* $(\Box, [\neq])$*-modal* Λ *(the* reflexive universal fragment*).*

Proposition 4.9. *(1) The map* $\Lambda \mapsto {}^\sharp\Lambda$ *sends* $\mathbf{K4^\circ D^+}$*-logics to* $\mathbf{S4U}$*-logics.*
(2) The map $\Lambda \mapsto {}^u\Lambda$ *sends* $\mathbf{K4^\circ D^+}$*-logics to* $\mathbf{K4^\circ U}$*-logics and* $\mathbf{S4D}$*-logics to* $\mathbf{S4U}$*-logics.*
(3) The map $\Lambda \mapsto {}^{\sharp u}\Lambda$ *sends* $\mathbf{K4^\circ D^+}$*-logics to* $\mathbf{S4U}$*-logics.*
(4) For a topological space $\mathfrak{X}$

$$\mathbf{Lc}_{\neq}(\mathfrak{X}) = {}^\sharp\mathbf{Ld}_{\neq}(\mathfrak{X}),\ \mathbf{Ld}_{\forall}(\mathfrak{X}) = {}^u\mathbf{Ld}_{\neq}(\mathfrak{X}),\ \mathbf{Lc}_{\forall}(\mathfrak{X}) = {}^u\mathbf{Lc}_{\neq}(\mathfrak{X}) = {}^\sharp\mathbf{Ld}_{\forall}(\mathfrak{X}).$$

(5) For a weakly transitive Kripke frame F

$$\mathbf{L}_{\neq}(\overline{F}) = {}^\sharp\mathbf{L}_{\neq}(F),\quad \mathbf{L}_{\forall}(F) = {}^u\mathbf{L}_{\neq}(F),\ \mathbf{L}_{\forall}(\overline{F}) = {}^\sharp\mathbf{L}_{\forall}(F) = {}^{\sharp\, u}\mathbf{L}_{\neq}(F).$$

Proposition 4.9 (4) implies that dd-logics are the most expressive of all kinds of the logics we consider.

Corollary 4.10. *If* $\mathbf{Ld}_{\neq}(\mathfrak{X}) = \mathbf{Ld}_{\neq}(\mathfrak{Y})$ *for spaces* $\mathfrak{X}, \mathfrak{Y}$*, then all the other logics (du-, cu-, cd-, d-, c-) of these spaces coincide.*

Let

$$AT_1 := [\neq]p \to [\neq]\Box p, \quad AC := [\forall]\,(\Box p \vee \Box\neg p) \to [\forall]\,p \vee [\forall]\,\neg p.$$

Proposition 4.11. *For a topological space* $\mathfrak{X}$

(1) $\mathfrak{X} \models^d \Diamond\top$ *iff* $\mathfrak{X}$ *is dense-in-itself;*

(2) $\mathfrak{X} \models^d AT_1$ *iff* $\mathfrak{X} \models^c AT_1$ *iff* $\mathfrak{X}$ *is a* T_1*-space;*

(3) $\mathfrak{X} \vDash^d AC^\sharp$ *iff* $\mathfrak{X} \models^c AC$ *iff* $\mathfrak{X}$ *is connected.*

Proof. (1) and the first equivalence in (2) are trivial. The first equivalence in (3) follows from 4.9(4). The remaining ones are checked easily, cf. [23], [38]. ■

For a $\Box$-modal logic **L** put

$$\mathbf{LD^+T_1} := \mathbf{LD^+} + AT_1, \qquad \mathbf{LD^+T_1C} := \mathbf{LD^+} + AT_1 + AC^{\sharp u}.$$

Also put

$$\mathbf{KT_1} := \mathbf{K4D^+T_1},\ \mathbf{DT_1} := \mathbf{D4D^+T_1},\ \mathbf{DT_1C} := \mathbf{D4D^+T_1C}.$$

Proposition 4.12. *[23] If* $F = (W, R, R_D)$ *is basic, then* $F \models AT_1$ *iff all* R_D*-irreflexive points are strictly* R*-minimal iff* $R_D \circ R \subseteq R_D$*.*

Remark 4.13. Density-in-itself is expressible in cd-logic and dd-logic by the formula $DS := [\neq]p \supset \Diamond p$, So for any space $\mathfrak{X}$, $\mathfrak{X} \vDash^c DS$ iff $\mathfrak{X} \vDash^d DS$ iff $\mathfrak{X} \vDash^d \Diamond\top$. It is known that DS axiomatizes dense-in-themselves spaces in cd-logic [23]. However, in dd-logic this axiom is insufficient: $\mathbf{Ld}_{\neq}$(all dense-in-themselves spaces) $= \mathbf{D4^\circ D^+} = \mathbf{K4^\circ D^+} + \Diamond\top$, and it is *stronger* than $\mathbf{K4^\circ D^+} + DS$. (To see the latter, consider a singleton Kripke frame, which is R_D-reflexive, but R-irreflexive.) Therefore $\mathbf{K4^\circ D^+} + DS$ is dd-incomplete.

Remark 4.14. Every T_1-space is a local T_1-space, so the dd-logic of all T_1-spaces contains $\Box p \to \Box\Box p$. However, $\mathbf{K4^\circ D^+T_1} \not\vdash \Box p \to \Box\Box p$. In fact, consider a 2-point frame $F := (W, \neq_W, W^2)$. It is clear that $F \vDash \mathbf{K4^\circ D^+}$. Also $F \vDash AT_1$, by Proposition 4.12, but $F \not\vDash \Box p \to \Box\Box p$, since $\neq_W$ is not transitive.

It follows that $\mathbf{K4^\circ D^+T_1}$ is dd-incomplete; T_1-spaces are actually axiomatized by $\mathbf{KT_1}$ (Corollary 7.13).

Let us give some examples of du-, cu- and cd-complete logics.

(1) $\mathbf{Lc}_\forall$(all spaces) $= \mathbf{S4U}$.

(2) $\mathbf{Lc}_\forall$(all connected spaces) $= \mathbf{Lc}_\forall(\mathbf{R}^n) = \mathbf{S4U} + AC$ for any $n \geq 1$ [38][7]

(3) $\mathbf{Ld}_\forall$(all spaces) $= \mathbf{S4D}$ [12].

(4) $\mathbf{Lc}_{\neq}(\mathfrak{X}) = \mathbf{S4DT_1} + \mathbf{DS}$, where $\mathfrak{X}$ is a 0-dimensional separable metric space [23].

(5) $\mathbf{Lc}_{\neq}(\mathbf{R}^n)$ for any $n \geq 2$ is finitely axiomatized in [22]; all these logics coincide.

(6) $\mathbf{Ld}_\forall(\mathbf{R})$ is finitely axiomatized in [31].

[7] The paper [38] contains a stronger claim: $\mathbf{Lc}_\forall(\mathfrak{X}) = \mathbf{S4U} + AC$ for any connected dense-in-itself separable metric $\mathfrak{X}$. However, recently we found a gap in the proof of Lemma 17 from that paper. Now we state the main result only for the case $\mathfrak{X} = \mathbf{R}^n$; a proof can be obtained by applying the methods of the present Chapter, but we are planning to publish it separately.

5 dd-completeness of $\mathbf{K4^\circ D^+}$ and some of its extensions

This section contains some simple arguments showing that there are many dd-complete bimodal logics.

All formulas and logics in this section are $(\Box, [\neq])$-modal. An arbitrary Kripke frame for $(\Box, [\neq])$-formulas is often denoted by (W, R, R_D).

Lemma 5.1. *(1) Every weakly transitive Kripke 1-frame is a p-morphic image of some irreflexive weakly transitive Kripke 1-frame.*
(2) Every rooted $\mathbf{K4^\circ D^+}$*-frame is a p-morphic image of some* R*- and* R_D*-irreflexive rooted* $\mathbf{K4^\circ D^+}$*-frame.*

Proof. (1) Cf. [14].

(2) Similar to the proof of (1). For $F = (W, R, R_D) \in \mathfrak{F}_0$ put

$$W_r := \{a \mid aR_Da\},\ W_i = W - W_r, \quad \tilde{W} := W_i \cup (W_r \times \{0,1\}).$$

Then we define the relation $\tilde{R}$ on $\tilde{W}$ such that

$$(b,j)\tilde{R}a \quad \text{iff} \quad bRa, \qquad\qquad a\tilde{R}(b,j) \quad \text{iff} \quad aRb,$$

$$(b,j)\tilde{R}(b',k) \quad \text{iff} \quad bRb' \ \&\ b \neq b' \vee b = b' \ \&\ j \neq k, \qquad a\tilde{R}a' \quad \text{iff} \quad aRa'.$$

Here $a, a' \in W_i$; $b, b' \in W_r$; $j, k \in \{0,1\}$. So we duplicate all R_D-reflexive points making them irreflexive (under both relations). It follows that $\tilde{F} := (\tilde{W}, \tilde{R}, \neq_{\tilde{W}}) \in \mathfrak{F}_0$ and $\tilde{R}$ is irreflexive; the map $f : \tilde{W} \to W$ sending (b,j) to b and a to itself (for $b \in W_r,\ a \in W_i$) is a p-morphism $\tilde{F} \twoheadrightarrow F$. ■

Proposition 5.2. *Let* Γ *be a set of closed 2-modal formulas,* $\Lambda := \mathbf{K4^\circ D^+} + \Gamma$*. Then*

(1) Λ *is Kripke complete.*

(2) Λ *is dd-complete.*

Proof. (1) $\mathbf{K4^\circ D^+}$ is axiomatized by Sahlqvist formulas. One can easily check that (in the minimal modal logic) every closed formula is equivalent to a positive formula, so we can apply Sahlqvist theorem.

(2) Suppose $A \notin \Lambda$. By (1) and the Generation lemma there exists a rooted Kripke 2-frame F such that $F \vDash L$ and $F \nvDash A$. Then by Lemma 5.1, for some irreflexive weakly transitive 1-frame $G = (W, R)$ there is a p-morphism $(G, \neq_W) \twoheadrightarrow F$. By the p-morphism lemma $(G, \neq_W) \nvDash A$ and $(G, \neq_W) \vDash \Lambda$ (since Γ consists of closed formulas). Hence by Lemma 4.7, $\Lambda \subseteq \mathbf{Ld}_{\neq}(N(\overline{G}))$, $A \notin \mathbf{Ld}_{\neq}(N(\overline{G}))$. ■

Remark 5.3. Using Proposition 5.2 and the construction from [8] one can prove that there is a continuum of dd-complete logics. Such a claim is rather weak, because Proposition 5.2 deals only with Alexandrov spaces. In section 7 we will show how to construct many dd-complete logics of metric spaces.

6 d-morphisms and dd-morphisms; extended McKinsey - Tarski's Lemma

In this section we recall the notion of a d-morphism (a validity-preserving map for d-logics) and introduce dd-morphisms, the analogues of d-morphisms for dd-logics. This is the main technical tool in the present chapter. Two basic lemmas are proved here, an analogue of McKinsey–Tarski's lemma on dissectability for d-morphisms and the Glueing lemma.

The original McKinsey–Tarski's lemma [32] states the existence of a c-morphism (cf. Remark 6.4) from an arbitrary separable dense-in-itself metric space onto a certain quasi-tree of depth 2. The separability condition is actually redundant [33, Ch. 3] (note that the latter proof is quite different from [32][8]). But c-morphisms preserve validity only for c-logics, and unfortunately, the constructions by McKinsey–Tarski and Rasiowa–Sikorski cannot be used for d-morphisms. So we need another construction to prove a stronger form of McKinsey–Tarski's lemma.

Definition 6.1. *Let $\mathfrak{X}$ be a topological space, $F = (W, R)$ a transitive Kripke frame. A map $f : X \longrightarrow W$ is called a* d-morphism *from $\mathfrak{X}$ to F if f is open and continuous as a map $\mathfrak{X} \longrightarrow N(\overline{F})$ and also satisfies*

$$\textit{r-density}: \ \forall w \in W (wRw \Rightarrow f^{-1}(w) \subseteq \mathbf{d} f^{-1}(w)),$$
$$\textit{i-discreteness}: \ \forall w \in W (\neg wRw \Rightarrow f^{-1}(w) \cap \mathbf{d} f^{-1}(w) = \varnothing).$$

If f is surjective, we write $f : \mathfrak{X} \twoheadrightarrow^d F$.

Proposition 6.2. *[5] (1) f is a d-morphism from $\mathfrak{X}$ to F iff 2^f is a homomorphism from $MA(F)$ to $DA(\mathfrak{X})$.*
(2) If $f : \mathfrak{X} \twoheadrightarrow^d F$, then $\mathbf{Ld}(\mathfrak{X}) \subseteq \mathbf{L}(F)$.

Corollary 6.3. *[37] A map f from a topological space $\mathfrak{X}$ to a finite transitive Kripke frame F is a d-morphism iff*

$$\forall w \in W \ \mathbf{d} f^{-1}(w) = f^{-1}(R^{-1}(w)).$$

Proof. 2^f preserves Boolean operations. It is a homomorphism of modal algebras iff it preserves diamonds, i.e., iff for any $V \subseteq W$,

$$f^{-1}(R^{-1}(V)) = \mathbf{d} f^{-1}(V).$$

Inverse images and $\mathbf{d}$ distribute over finite unions, so the above equality holds for any (finite) V iff it holds for singletons, i.e.,

$$f^{-1}(R^{-1}(w)) = \mathbf{d} f^{-1}(w). \blacksquare$$

Remark 6.4. For a space $\mathfrak{X}$ and a Kripke **S4**-frame $F = (W, R)$ one can also define a *c-morphism* $\mathfrak{X} \longrightarrow F$ just as an open and continuous map $f : \mathfrak{X} \longrightarrow N(F)$. So every d-morphism to an **S4**-frame is a c-morphism. It is well known [33] that $f : X \longrightarrow W$ is a c-morphism iff 2^f is a homomorphism $MA(F) \longrightarrow CA(\mathfrak{X})$. Again for a finite F this is equivalent to

$$\forall w \in W \ \mathbf{C} f^{-1}(w) = f^{-1}(R^{-1}(w)).$$

[8]Recently P. Kremer [21] has showed that **S4** is *strongly complete* w.r.t. any dense-in-itself metric space. His proof uses much of the construction from [33].

Lemma 6.5. *If $f : \mathfrak{X} \twoheadrightarrow^d F$ for a finite frame F and $\mathcal{Y} \subseteq \mathfrak{X}$ is an open subspace, then $f|Y$ is a d-morphism.*

Proof. We apply Proposition 6.2. Note that $f|Y$ is the composition $f \cdot j$, where $j : Y \hookrightarrow X$ is the inclusion map. Then $2^{f|Y} = 2^j \cdot 2^f$. Since 2^f is a homomorphism $MA(F) \longrightarrow DA(\mathfrak{X})$, it remains to show that 2^j is a homomorphism $DA(\mathfrak{X}) \longrightarrow DA(\mathcal{Y})$, i.e., it preserves the derivation: $j^{-1}(\mathbf{d}V) = \mathbf{d}_Y j^{-1}(V)$, or $\mathbf{d}V \cap Y = \mathbf{d}_Y(V \cap Y)$, which follows from 3.2. ■

Definition 6.6. *A set γ of subsets of a topological space $\mathfrak{X}$ is called* dense *at $x \in X$ if every neighbourhood of x contains a member of γ.*

Proposition 6.7. *For $m > 0,\ l > 0$ let Φ_{ml} be a "quasi-tree" of height 2, with singleton maximal clusters and An m-element root cluster (Fig. 2). For $l = 0,\ m > 0,\ \Phi_{ml}$ denotes an m-element cluster.*

Let $\mathfrak{X}$ be a dense-in-itself separable metric space, $B \subset X$ a closed nowhere dense set. Then there exists a d-morphism $g : \mathfrak{X} \twoheadrightarrow^d \Phi_{ml}$ with the following properties:

(1) $B \subseteq g^{-1}(b_1)$;

(2) every $g^{-1}(a_i)$ (for $i \leq l$) is a union of a set α_i of disjoint open balls, which is dense at any point of $g^{-1}(\{b_1, ..., b_m\})$.

Proof. Let $X_1, \dots, X_n, \dots$ be a countable base of $\mathfrak{X}$ consisting of open balls. We construct sets $A_{ik},\ B_{jk}$ for $1 \leq i \leq l,\ 1 \leq j \leq m,\ k \in \omega$, with the following properties:

(1) A_{ik} is the union of a finite set α_{ik} of nonempty open balls whose closures are disjoint;

(2) $\mathbf{C}A_{ik} \cap \mathbf{C}A_{i'k} = \varnothing$ for $i \neq i'$;

(3) $\alpha_{ik} \subseteq \alpha_{i,k+1};\ A_{ik} \subseteq A_{i,k+1}$;

(4) B_{jk} is finite;

(5) $B_{jk} \subseteq B_{j,k+1}$;

(6) $A_{ik} \cap B_{jk} = \varnothing$;

(7) $X_{k+1} \subseteq \bigcup_{i=1}^{l} A_{ik} \Rightarrow \alpha_{i,k+1} = \alpha_{ik},\ B_{j,k+1} = B_{jk}$;

(8) if $X_{k+1} \not\subseteq \bigcup_{i=1}^{l} A_{ik}$, there are closed nontrivial balls $P_1, \dots, P_l$ such that for any i, j

$$P_i \subseteq X_{k+1} - A_{ik},\ \alpha_{i,k+1} = \alpha_{ik} \cup \{\mathbf{I}P_i\},\ (B_{j,k+1} - B_{jk}) \cap X_{k+1} \neq \varnothing;$$

(9) $A_{ik} \subseteq X - B$;

(10) $B_{jk} \subseteq X - B$;

(11) $j \neq j' \Rightarrow B_{j'k} \cap B_{jk} = \varnothing$.

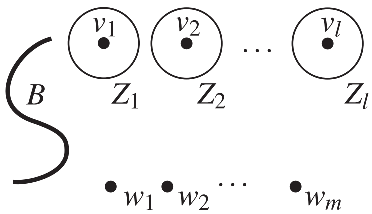


Figure 1: Case k = 0

We carry out both the construction and the proof by induction on k.

Let $k = 0$. $(X - B)$ is infinite, since it is nonempty and open in a dense-in-itself $\mathfrak{X}$. Take distinct points $v_1, \dots, v_l \notin B$ and disjoint closed nontrivial balls $Z_1, \dots, Z_l \subset X - B$ with centres at $v_1, \dots, v_l$ respectively (see Fig.1).

Put

$$\alpha_{i0} := \{\mathbf{I}Z_i\};\ A_{i0} := \mathbf{I}Z_i;$$

then $Z_i = \mathbf{C}A_{i0}$. As above, since $(X - B) - \bigcup_{i=1}^{l} Z_i$ is nonempty and open, it is infinite. Pick distinct $w_1, \dots, w_m \in X - B$ and put $B_{j0} := \{w_j\}$. Then the required properties hold for $k = 0$.

At the induction step we construct $A_{i,k+1}, B_{j,k+1}$. Put $Y_k := \bigcup_{i=1}^{l} A_{ik}$ and consider two cases.

(a) $X_{k+1} \subseteq Y_k$. Then put:

$$\alpha_{i,k+1} := \alpha_{ik};\ A_{i,k+1} := A_{ik};\ B_{j,k+1} := B_{jk}.$$

(b) $X_{k+1} \not\subseteq Y_k$. Then $X_{k+1} \not\subseteq \mathbf{C}Y_k$. In fact, $X_{k+1} \subseteq \mathbf{C}Y_k$ implies $X_{k+1} \subseteq \mathbf{IC}Y_k = Y_k$, since X_{k+1} is open and by (1) and (2). So we put

$$W_0 := X_{k+1} - \mathbf{C}Y_k - \bigcup_{j=1}^{m} B_{jk},\ W := W_0 - B.$$

Since $(X_{k+1} - \mathbf{C}Y_k)$ is nonempty and open and every B_{jk} is finite by (4), W_0 is also open and nonempty (by the density of $\mathfrak{X}$). By the assumption of 6.7, B is closed, and thus W is open.

W is also nonempty. In fact, otherwise $W_0 \subseteq B$, and then $W_0 \subseteq \mathbf{I}B = \varnothing$ (since B is nowhere dense by the assumption of 6.7).

Now we argue similarly to the case $k = 0$. Take disjoint closed nontrivial balls $P_1, \dots, P_l \subset W$. Then $W - \bigcup_{i=1}^{l} P_i$ is infinite, so we pick distinct $b_{1,k+1}, \dots, b_{m,k+1}$ in this set and put

$$B_{j,k+1} := B_{jk} \cup \{b_{j,k+1}\},\ \alpha_{i,k+1} := \alpha_{ik} \cup \{\mathbf{I}P_i\},\ A_{i,k+1} := A_{ik} \cup \mathbf{I}P_i.$$

In the case (a) all the required properties hold for $(k+1)$ by the construction.

In the case (b) we have to check only (1), (2), (6), (8)–(11).

(8) holds, since by construction we have

$$P_i \subset W \subset X_{k+1} - \mathbf{C}Y_k \subset X_{k+1} - A_{ik};$$
$$b_{j,k+1} \in W \subseteq X_{k+1},\ b_{j,k+1} \in (B_{j,k+1} - B_{jk}).$$

(1): From IH it is clear that $\alpha_{i,k+1}$ is a finite set of open balls and their closures are disjoint; note that $P_i \cap \mathbf{C}A_{ik} = \varnothing$, since $P_i \subseteq W \subseteq -\mathbf{C}A_{ik}$.

(2): We have

$$\mathbf{C}A_{i,k+1} \cap \mathbf{C}A_{i',k+1} = (\mathbf{C}A_{ik} \cup P_i) \cap (\mathbf{C}A_{i'k} \cup P_{i'}) =$$
$$=(\mathbf{C}A_{ik} \cap \mathbf{C}A_{i'k}) \cup (\mathbf{C}A_{ik} \cap P_{i'}) \cup (\mathbf{C}A_{i'k} \cap P_i) \cup (P_i \cap P_{i'}) = \mathbf{C}A_{ik} \cap \mathbf{C}A_{i'k} = \varnothing$$

by IH and by the construction; note that $P_i, P_i' \subseteq W \subseteq -\mathbf{C}Y_k$.

(6): We have

$$A_{i,k+1} \cap B_{j,k+1} = (A_{ik} \cap B_{jk}) \cup (\mathbf{I}P_i \cap \{b_{j,k+1}\}) \cup (A_{ik} \cap \{b_{j,k+1}\}) \cup (\mathbf{I}P_i \cap B_{jk}) = \varnothing$$

by IH and since $b_{j,k+1} \notin P_i,\ b_{j,k+1} \in W \subseteq X - Y_k,\ P_i \subset W \subseteq X - B_{jk}$.

(9): We have $A_{i,k+1} = A_{ik} \cup \mathbf{I}P_i \subseteq -B$, since $A_{ik} \subseteq -B$ by IH, and $P_i \subset W \subseteq -B$ by the construction.

Likewise, (10) follows from $B_{jk} \subseteq -B$ and $b_{j,k+1} \in W \subseteq -B$.

To check (11), assume $j \neq j'$. We have $B_{j',k+1} \cap B_{j,k+1} = B_{j'k} \cap B_{jk}$, since $b_{j',k+1} \neq b_{j,k+1},\ b_{j,k+1} \in W \subseteq -B_{j'k}$ and $b_{j',k+1} \in W \subseteq -B_{jk}$. Then apply IH.

Therefore the required sets A_{ik}, B_{jk} are constructed. Now put

$$\alpha_i := \bigcup_k \alpha_{ik},\ A_i := \bigcup \alpha_i = \bigcup_k A_{ik},\ B_j := \bigcup_k B_{jk},$$

$$B_1' := X - (\bigcup_i A_i \cup \bigcup_j B_j),$$

and define a map $g : X \longrightarrow \Phi_{ml}$ as follows:

$$g(x) := \begin{cases} a_i & \text{if } x \in A_i, \\ b_j & \text{if } x \in B_j,\ j \neq 1, \\ b_1 & \text{otherwise (i.e., for } x \in B_1'). \end{cases}$$

By (2), (3), (5), (6), (11), g is well defined; by (9), (10) $B \subseteq g^{-1}(b_1)$.

To prove that g is a d-morphism, we check some other properties.

(12) $\quad X - \bigcup_{i=1}^{l} A_i \subseteq \mathbf{d}B_j.$

In fact, take an arbitrary $x \notin \bigcup_{i=1}^{l} A_i$ and show that $x \in \mathbf{d}B_j$, i.e.,

(13) $\quad (U - \{x\}) \cap B_j \neq \varnothing.$

for any neighbourhood U of x. First assume that $x \notin B_j$. Take a basic open X_{k+1} such that $x \in X_{k+1} \subseteq U$. Then $X_{k+1} \not\subseteq \bigcup_{i=1}^{l} A_i$, and (8) implies $B_{j,k+1} \cap X_{k+1} \neq \varnothing$. Thus $B_j \cap U \neq \varnothing$. So we obtain (13).

Suppose $x \in B_j$; then $x \in B_{jk}$ for some k. Since $\mathfrak{X}$ is dense-in-itself and $\{X_1, X_2, \dots\}$ is its open base, $\{X_{s+1} \mid s \geq k\}$ is also an open base (note that every ball in $\mathfrak{X}$ contains a smaller ball). So $x \in X_{s+1} \subseteq U$ for some $s \geq k$. Since $x \notin \bigcup_{i=1}^{l} A_i$, we have $X_{s+1} \not\subseteq \bigcup_{i=1}^{l} A_i$, and so $(B_{j,s+1} - B_{js}) \cap X_{s+1} \neq \varnothing$ by (8); thus $(B_j - B_{js}) \cap U \neq \varnothing$. Now $x \in B_{jk} \subseteq B_{js}$ implies (13).

(14) $\quad \mathbf{d}B_j \subseteq X - \bigcup_{i=1}^{l} A_i.$

In fact, $B_j \subseteq -A_i$, by (3), (5), (6). So $\mathbf{d}B_j \subseteq \mathbf{d}(-A_i) \subseteq -A_i$, since A_i is open.

Similarly we obtain

(15) $\quad \mathbf{d}B'_1 \subseteq X - \bigcup_{i=1}^{l} A_i, \qquad \mathbf{d}A_i \subseteq X - \bigcup_{r \neq i} A_r.$

Also note that

(16) $\quad A_i \subseteq \mathbf{d}A_i,$

since A_i is open, $\mathfrak{X}$ is dense-in-itself. Similary to (12) we have
(17) $\quad \alpha_i$ is dense at every point of B_j, B'_1 (and thus $B_j,\ B'_1 \subseteq \mathbf{d}A_i$).

To conclude that g is a d-morphism, note that

$$g^{-1}(a_i) = A_i,\ g^{-1}(b_j) = B_j \text{ (for } j \neq 1),\ g^{-1}(b_1) = B'_1,$$

and so by (15), (16), (17)

$$\mathbf{d}g^{-1}(a_i) = \mathbf{d}A_i = X - \bigcup_{r \neq i} A_r = g^{-1}(R^{-1}(a_i)),$$

and by (12), (14), (15)

$$\mathbf{d}g^{-1}(b_j) = \mathbf{d}B_j = X - \bigcup_{i=1}^{l} A_i = g^{-1}(R^{-1}(b_j)) \quad \text{(for } j \neq 1),$$

$$\mathbf{d}g^{-1}(b_1) = \mathbf{d}B'_1 = X - \bigcup_{i=1}^{l} A_i = g^{-1}(R^{-1}(b_1)). \blacksquare$$

For the proof see Appendix.

Lemma 6.8. *Assume that*

(1) $\mathfrak{X}$ is a dense-in-itself separable metric space,

(2) $B \subset X$ is closed nowhere dense,

(3) $F = C \cup F_1 \cup \dots \cup F_l$ is a **D4***-frame, where $C = \{b_1, \dots, b_m\}$ is a non-degenerate root cluster, $F_1, \dots, F_l$ are the subframes generated by the successors of C,*

(4) for any nonempty open ball U in $\mathfrak{X}$, for any $i \in \{1, \dots, l\}$ there exists a d-morphism $f_i^U : U \twoheadrightarrow^d F_i$.

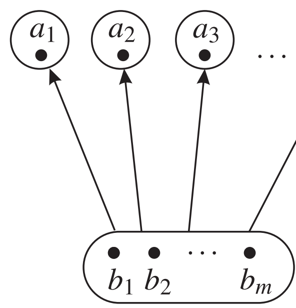


Figure 2: Frame Φ_{ml}.

Then there exists $f : \mathfrak{X} \twoheadrightarrow^d F$ *such that* $f(B) = \{b_1\}$.

Proof. First, we construct $g : \mathfrak{X} \twoheadrightarrow^d \Phi_{ml}$ according to Proposition 6.7. Then $B \subseteq g^{-1}(b_1)$ and $A_i = g^{-1}(a_i)$ is the union of a set α_i of disjoint open balls. Then put

$$f(x) := \begin{cases} g(x) & \text{if } g(x) \in C, \\ f_i^U(x) & \text{if } x \in U,\ U \in \alpha_i. \end{cases} \tag{1}$$

Since g and f_i^U are surjective, the same holds for f. So let us show

$$\mathbf{d}f^{-1}(a) = f^{-1}(R^{-1}(a))$$

(R is the accessibility relation on F). First suppose $a \in C$. Then (since g is a d-morphism)

$$\mathbf{d}f^{-1}(a) = \mathbf{d}g^{-1}(a) = g^{-1}(C) = f^{-1}(C) = f^{-1}(R^{-1}(a)).$$

Now suppose $a \notin C$, $I = \{i \mid a \in F_i\}$, and let R_i be the accessibility relation on F_i. We have:

$$f^{-1}(a) = \bigcup_{i \in I} \bigcup_{U \in \alpha_i} (f_i^U)^{-1}(a), \qquad R^{-1}(a) = C \cup \bigcup_{i \in I} R_i^{-1}(a),$$

and so

$$f^{-1}(R^{-1}(a)) = g^{-1}(C) \cup \bigcup_{i \in I} \bigcup_{U \in \alpha_i} (f_i^U)^{-1}(R_i^{-1}(a)).$$

Since f_i^U is a d-morphism,

$$f^{-1}(R^{-1}(a)) = g^{-1}(C) \cup \bigcup_{i \in I} \bigcup_{U \in \alpha_i} \mathbf{d}_U((f_i^U)^{-1}(a)) \subseteq g^{-1}(C) \cup \mathbf{d}f^{-1}(a). \tag{2}$$

Let us show that

$$g^{-1}(C) \subseteq \mathbf{d}f^{-1}(a). \tag{3}$$

In fact, let $x \in g^{-1}(C)$. Since α_i is dense at x, every neighbourhood of x contains some $U \in \alpha_i$. Since f_i^U is surjective, $f(u) = f_i^U(u) = a$ for some $u \in U$. Therefore, $x \in \mathbf{d}f^{-1}(a)$.

(2) and (3) imply $f^{-1}(R^{-1}(a)) \subseteq \mathbf{d}f^{-1}(a)$. Let us prove the converse:

$$\mathbf{d}f^{-1}(a) \subseteq f^{-1}(R^{-1}(a)). \tag{4}$$

We have $A_j \cap f^{-1}(a) = \varnothing$ for $j \notin I$ and A_j is open, hence $A_j \cap \mathbf{d}f^{-1}(a) = \varnothing$. Thus $\mathbf{d}f^{-1}(a) \subseteq g^{-1}(C) \cup A_i$. Now $g^{-1}(C) \subseteq f^{-1}(R^{-1}(a))$ by (2), so it remains to show that for any $i \in I$

$$\mathbf{d}f^{-1}(a) \cap A_i \subseteq f^{-1}(R^{-1}(a)). \tag{5}$$

To check this, consider any $x \in \mathbf{d}f^{-1}(a) \cap A_i$. Then $x \in U$ for some $U \in \alpha_i$, and thus by 3.2 and (2) $x \in \mathbf{d}f^{-1}(a) \cap U = \mathbf{d}_U(f^{-1}(a) \cap U) = \mathbf{d}_U(f_i^U)^{-1}(a) \subseteq f^{-1}(R^{-1}(a))$. This implies (5) and completes the proof of (4). ■

Recall that ∂ denotes the boundary of a set in a topological space: $\partial A := \mathbf{C}A - \mathbf{I}A$.

Lemma 6.9. (Glueing lemma) *Let $\mathfrak{X}$ be a local T_1-space satisfying*
(a) $X = X_1 \cup Y \cup X_2$ for closed nonempty subsets X_1, Y, X_2 such that

- $X_1 \cap X_2 = X_1 \cap \mathbf{I}Y = X_2 \cap \mathbf{I}Y = \varnothing$,
- $\partial X_1 \cup \partial X_2 = \partial Y$,
- $\mathbf{dI}Y = Y$ *(i.e., Y is regular and dense in-itself).*

or

(b) $X = X_1 \cup X_2$ is a nontrivial closed partition.

Let $F = (W, R)$ be a finite **K4**-*frame, $F_1 = (W_1, R_1)$, $F_2 = (W_2, R_2)$ its generated subframes such that $W = W_1 \cup W_2$ and suppose there are d-morphisms $f_i : \mathfrak{X}_i \twoheadrightarrow^d F_i$, $i = 1, 2$, where $\mathfrak{X}_i$ is the subspace of $\mathfrak{X}$ corresponding to X_i.*

In the case (a) we also assume that F_1, F_2 have a common maximal cluster C, $f_i(\partial X_i) \subseteq R^{-1}(C)$ for $i = 1, 2$ and there is $g : \mathbf{I}Y \twoheadrightarrow^d C$ (where C is regarded as a frame with the universal relation, $\mathbf{I}Y$ as a subspace of $\mathfrak{X}$). Then $f_1 \cup f_2 \cup g : \mathfrak{X} \twoheadrightarrow^d F$ in the case (a), $f_1 \cup f_2 : \mathfrak{X} \twoheadrightarrow^d F$ in the case (b).[9]

Proof. Let $f := f_1 \cup f_2 \cup g$ (or $f := f_1 \cup f_2$), $F_i = (W_i, R_i)$, $\mathbf{d} := \mathbf{d}_X$, $\mathbf{d}_i := \mathbf{d}_{X_i}$. For $w \in W$ there are four options.

(1) $w \in W_1 - W_2$. Then $\mathbf{d}f^{-1}(w) = \mathbf{d}f_1^{-1}(w) = \mathbf{d}_1 f_1^{-1}(w) = f_1^{-1}(R_1^{-1}(w))$ (since X_1 is closed and f_1 is a d-morphism). It remains to note that $R_1^{-1}(w) = R^{-1}(w) \subseteq W_1 - W_2$, and thus $f_1^{-1}(R_1^{-1}(w)) = f^{-1}(R^{-1}(w))$.

(2) $w \in W_2 - W_1$. Similar to the case (1).

(3) $w \in (W_1 \cap W_2) - C$ in the case (a) or $w \in W_1 \cap W_2$ in the case (b). Then $f^{-1}(w) = f_1^{-1}(w) \cup f_2^{-1}(w)$, so similarly to (1),

$$\mathbf{d}f^{-1}(w) = \mathbf{d}_1 f_1^{-1}(w) \cup \mathbf{d}_2 f_2^{-1}(w) = f_1^{-1}(R_1^{-1}(w)) \cup f_2^{-1}(R_2^{-1}(w)) = f^{-1}(R^{-1}(w)).$$

(4) $w \in C$ in case (a). First note that $\mathbf{d}g^{-1}(w) = Y$. In fact, g is a d-morphism onto the cluster C, so $\mathbf{d}_{\mathbf{I}Y} g^{-1}(w) = g^{-1}(C) = \mathbf{I}Y$. Hence $\mathbf{I}Y \subseteq \mathbf{d}g^{-1}(w) \subseteq \mathbf{dI}Y = Y$, and thus

$$Y = \mathbf{dI}Y \subseteq \mathbf{dd}g^{-1}(w) \subseteq \mathbf{d}g^{-1}(w)$$

[9] $f_1 \cup f_2$ is the map f such that $f|X_i = f_i$; similarly for $f_1 \cup f_2 \cup g$.

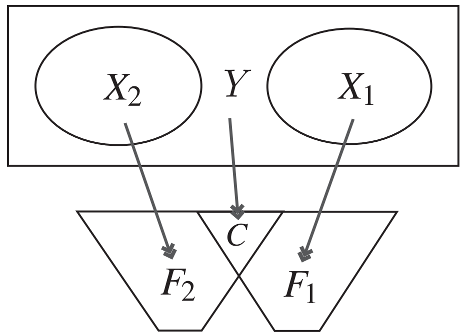


Figure 3: Case (a)

by 3.11(2). Next, since X_1, X_2 are closed and f_1, f_2 are d-morphisms we have

$$\mathbf{d}f^{-1}(w) = \mathbf{d}f_1^{-1}(w) \cup \mathbf{d}f_2^{-1}(w) \cup \mathbf{d}g^{-1}(w) = \mathbf{d}_1 f_1^{-1}(w) \cup \mathbf{d}_2 f_2^{-1}(w) \cup Y = \\ = f_1^{-1}(R_1^{-1}(w)) \cup f_2^{-1}(R_2^{-1}(w)) \cup Y = f^{-1}(R^{-1}(w)).$$ ∎

The case (b) of the previous lemma can be generalized as follows.

Lemma 6.10. *Suppose a topological space* $\mathfrak{X}$ *is the disjoint union of open subspaces:* $\mathfrak{X} = \bigsqcup_{i \in I} \mathfrak{X}_i$. *Suppose a Kripke* **K4**-*frame* F *is the union of its generated subframes:* $F = \bigcup_{i \in I} F_i$ *and suppose* $f_i : \mathfrak{X}_i \twoheadrightarrow^d F_i$. *Then* $\bigcup_{i \in I} f_i : \mathfrak{X} \twoheadrightarrow^d F$.

Definition 6.11. *Let* $\mathfrak{X}$ *be a topological space,* $F = (W, R, R_D)$ *be a frame. Then a surjective map* $f : X \longrightarrow W$ *is called a* dd-morphism *(in symbols,* $f : \mathfrak{X} \twoheadrightarrow^{dd} F$*) if*

(1) $f : \mathfrak{X} \twoheadrightarrow^d (W, R)$ *is a d-morphism ;*

(2) $f : (X, \neq_X) \twoheadrightarrow (W, R_D)$ *is a p-morphism of Kripke frames.*

Lemma 6.12. *If* $f : \mathfrak{X} \twoheadrightarrow^{dd} F$, *then* $\mathbf{Ld}_{\neq}(\mathfrak{X}) \subseteq \mathbf{L}(F)$ *and for any closed 2-modal* A

$$\mathfrak{X} \vDash A \Leftrightarrow F \vDash A.$$

Proof. Similar to 6.2 and 2.3. ∎

Definition 6.13. *A set-theoretic map* $f : X \longrightarrow Y$ *is called* n-fold at $y \in Y$ *if* $|f^{-1}(y)| = n$;[10] f *is called* manifold at y *if it* n*-fold for some* $n > 1$.

Proposition 6.14. *(1) Let* $G = (X, \neq_X)$, $F = (W, S)$ *be Kripke frames such that* $\overline{S} = W^2$, *and let* $f : X \longrightarrow W$ *be a surjective function. Then*

$$f : G \twoheadrightarrow F \quad \text{iff} \quad f \text{ is manifold exactly at } S\text{-reflexive points of } F.$$

[10] $|\ldots|$ denotes the cardinality.

(2) Let $\mathfrak{X}$ be a T_1-space, $F = (W, R, R_D)$ a rooted $\mathbf{KT_1}$*-frame, $f : \mathfrak{X} \twoheadrightarrow^d (W, R)$. Then $f : \mathfrak{X} \twoheadrightarrow^{dd} F$ iff for any strictly R-minimal v*

$$vR_Dv \Leftrightarrow f \text{ is manifold at } v.$$

(3) If $\mathfrak{X}$ is a T_1-space, $f : \mathfrak{X} \twoheadrightarrow^d F = (W, R)$ and $R^{-1}(w) \neq \varnothing$ for any $w \in W$, then $f : \mathfrak{X} \twoheadrightarrow^{dd} F_\forall$, where $F_\forall := (W, R, W^2)$.

Proof. (1) Note that f is a p-morphism iff for any $x \in X$

$$f(X - \{x\}) = S(f(x)) = \begin{cases} W & \text{if } f(x)Sf(x), \\ W - \{f(x)\} & \text{otherwise.} \end{cases}$$

(2) By (1), $f : \mathfrak{X} \twoheadrightarrow^{dd} F$ iff

$$\forall v \in W(vR_Dv \Leftrightarrow \left|f^{-1}(v)\right| > 1).$$

The latter equivalence holds whenever $R^{-1}(v) \neq \varnothing$. In fact, then by Corollary 6.3, $\mathbf{d}f^{-1}(v) = f^{-1}(R^{-1}(v)) \neq \varnothing$, and thus $f^{-1}(v)$ is not a singleton (since $\mathfrak{X}$ is a T_1-space). $R^{-1}(v) \neq \varnothing$ also implies vR_Dv, by Proposition 4.12.

(3) follows from (2). ■

After we have proved the main technical results, in the next sections we will study dd-logics of specific spaces.

7 D4 and $\mathbf{DT_1}$ as logics of zero-dimensional dense-in-themselves spaces

In this section we will prove the d-completeness of **D4** and dd-completeness of $\mathbf{DT_1}$ w.r.t. zero-dimensional spaces. The proof follows rather easily from the previous section and an additional technical fact (Proposition 7.2) similar to the McKinsey–Tarski lemma.

Recall that a (nonempty) topological space $\mathfrak{X}$ is called *zero-dimensional* if clopen sets constitute its open base [3]. Zero-dimensional T_1-spaces with a countable base are subspaces of the Cantor discontinuum, or of the set of irrationals [28].

Lemma 7.1. *Let $\mathfrak{X}$ be a zero-dimensional dense-in-itself Hausdorff space. Then for any n there exists a nontrivial open partition $\mathfrak{X} = \mathfrak{X}_1 \sqcup \ldots \sqcup \mathfrak{X}_n$, in which every $\mathfrak{X}_i$ is also a zero-dimensional dense-in-itself Hausdorff space.*

Proof. It is sufficient to prove the claim for $n = 2$ and then apply induction. A dense-in-itself space cannot be a singleton, so there are two different points $x, y \in X$. Since $\mathfrak{X}$ is T_1 and zero-dimensional, there exists a clopen U such that $x \in U$, $y \notin U$. So $X = U \cup (X - U)$ is a nontrivial open partition. The Hausdorff property, density-in-itself, zero-dimensionality are inherited for open subspaces. ■

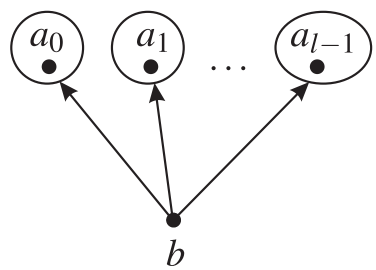


Figure 4: Frame Ψ_l.

Proposition 7.2. *Let $\mathfrak{X}$ be a zero-dimensional dense-in-itself metric space, $y \in X$. Let Ψ_l be the frame consisting of an irreflexive root b and its reflexive successors $a_0, \dots, a_{l-1}$ (Fig. 4).*

Then there exists $f : \mathfrak{X} \twoheadrightarrow^d \Psi_l$ such that $f(y) = b$ and for every i there is an open partition of $f^{-1}(a_i)$, which is dense at y.

Proof. Let $O(a, r) := \{x \in X \mid \rho(a, x) < r\}$, where ρ is the distance in $\mathfrak{X}$.

There exist clopen sets $Y_0, Y_1, \dots$ such that

$$\{y\} \subset \dots \subset Y_{n+1} \subset Y_n \subset \dots Y_1 \subset Y_0 = X$$

and $Y_n \subseteq O(y, 1/n)$ for $n > 0$.

These Y_n can be easily constructed by induction. Then

$$\bigcap_n Y_n = \{y\}, \text{ and } X - \{y\} = \bigsqcup_n X_n,$$

where $X_n = Y_n - Y_{n+1}$. Note that the X_n are nonempty and open, $X_n \subseteq O(y, 1/n)$ for $n > 0$.

Now define a map $f : X \longrightarrow \Psi_l$ as follows:

$$f(x) = \begin{cases} a_{r(n)} & \text{if } x \in X_n; \\ b & \text{if } x = y, \end{cases}$$

where $r(n)$ is the remainder of dividing n by l; it is clear that f is surjective.

Let us show that for any x,

$$x \in \mathbf{d}f^{-1}(u) \text{ iff } f(x)Ru. \qquad (*)$$

(i) Assume that $u = a_j$. Then $f^{-1}(u) = \bigcup_n X_{nl+j}$, and

$$f(x)Ru \text{ iff } (f(x) = b \text{ or } f(x) = u).$$

To prove 'if' in (*), consider two cases.

1. Suppose $f(x) = u$, $x \in X_{nl+j}$. Since X_{nl+j} is nonempty and open, it is dense-in-itself, and thus $x \in \mathbf{d}X_{nl+j} \subseteq \mathbf{d}f^{-1}(u)$.

2. Suppose $f(x) = b$, i.e. $x = y$. Then $x \in \mathbf{d}f^{-1}(u)$, since $X_{nl+j} \subseteq O(y, 1/n)$.

The previous argument also shows that $\{X_{nl+j} \mid n \geq 0\}$ is an open partition of $f^{-1}(a_j)$, which is dense at y.

To prove 'only if', suppose $f(x)Ru$ is not true. Then $f(x) = a_k$ for some $k \neq j$, and so for some n, $x \in X_n$, $X_n \cap f^{-1}(u) = \varnothing$. Since X_n is open, $x \notin \mathbf{d}f^{-1}(u)$.

(ii) Assume that $u = b$. Then $f^{-1}(u) = \{y\}$, and so $\mathbf{d}f^{-1}(u) = \varnothing = f^{-1}(R^{-1}(u))$. ■

Proposition 7.3. *Let $\mathfrak{X}$ be a zero-dimensional dense-in-itself separable metric space, F a finite rooted* **D4***-frame. Then there exists a d-morphism $\mathfrak{X} \twoheadrightarrow^d F$, which is 1-fold at the root of F if this root is irreflexive.*

Proof. By induction on the size of F.

(i) If F is a finite cluster, the claim follows from Proposition 6.7.

(ii) If $F = C \cup F_1 \cup \cdots \cup F_l$, where $C = \{b_1, \dots, b_m\}$ is a non-degenerate root cluster, $F_1, \dots, F_l$ are the subframes generated by the successors of C, we can apply Lemma 6.8. In fact, every open ball U in $\mathfrak{X}$ is zero-dimensional and dense-in-itself.

(iii) Suppose $F = \breve{b} \cup F_0 \cup \cdots \cup F_{l-1}$, where b is an irreflexive root of F, F_i are the subframes generated by the successors of b. There exists $g : X \twoheadrightarrow^d \Psi_l$ by 7.2, with an arbitrary $y \in X$. Then $g^{-1}(a_i)$ is a union of a set α_i of disjoint open sets, and α_i is dense at y. If $U \in \alpha_i$, then by IH, there exists $f_i^U : U \twoheadrightarrow^d F_i$. Put

$$f(x) = \begin{cases} b & \text{if } x = y; \\ f_i^U(x) & \text{if } x \in U,\ U \in \alpha_i. \end{cases}$$

Then similarly to Lemma 6.8 it follows that $f : X \twoheadrightarrow^d F$.

Finally note that if the root of F is irreflexive, the first step of the construction is case (iii), so the preimage of the root is a singleton. ■

Theorem 7.4. *If $\mathfrak{X}$ is a zero-dimensional dense-in-itself separable metric space, then* $\mathbf{Ld}(\mathfrak{X}) = \mathbf{D4}$.

Proof. By Propositions 7.3 and 6.2 $\mathbf{Ld}(\mathfrak{X}) \subseteq \mathbf{L}(F)$ for any finite rooted **D4**-frame F, thus $\mathbf{Ld}(\mathfrak{X}) \subseteq \mathbf{D4}$, since **D4** has the fmp. By Lemma 3.11 $\mathbf{D4} \subseteq \mathbf{Ld}(\mathfrak{X})$. ■

Lemma 7.5. *Let $\mathfrak{X}$ be a zero-dimensional dense-in-itself separable metric space, F a finite* **D4***-frame. Then there exists a d-morphism $\mathfrak{X} \twoheadrightarrow^d F$, which is 1-fold at all strictly minimal points.*

Proof. $F = F_1 \cup \ldots \cup F_n$ for different finite rooted **D4**-frames F_i. By Lemma 7.1, $\mathfrak{X} = \mathfrak{X}_1 \sqcup \ldots \sqcup \mathfrak{X}_n$ for zero-dimensional dense-in-themselves subspaces $\mathfrak{X}_i$, which are also metric and separable. By Proposition 7.3, we construct $f_i : \mathfrak{X}_i \twoheadrightarrow^d F_i$. Then by Lemma 6.10, $\bigcup\limits_{i=1}^{n} f_i : \mathfrak{X} \twoheadrightarrow^d F$. Every strictly minimal point of F is an irreflexive root of a unique F_i, so its preimage is a singleton. ■

Proposition 7.6. *Let $\mathfrak{X}$ be a zero-dimensional dense-in-itself separable metric space, $F \in \mathfrak{F}_0$ a finite* $\mathbf{DT_1}$*-frame. Then there exists a dd-morphism $\mathfrak{X} \twoheadrightarrow^{dd} F$.*

Proof. We slightly modify the proof of the previous lemma. Let $F = (W, R, R_D)$, $G = (W, R)$. Then $G = G_1 \cup \ldots \cup G_n$ for different cones G_i. We call G_i *special* if its root is strictly R-minimal and R_D-reflexive. We may assume that exactly $G_1, \ldots, G_m$ are special. Then we count them twice and present G as $G_1 \cup G_1' \cup \ldots \cup G_m \cup G_m' \cup G_{m+1} \cup \ldots \cup G_n$, where $G_i' = G_i$ for $i \leq m$ (or as $G_1 \cup G_1' \cup \ldots \cup G_m \cup G_m'$ if $m = n$).

Now we can argue as in the proof of Lemma 7.5. By Lemma 7.1, $\mathfrak{X} = \mathfrak{X}_1 \sqcup \mathfrak{X}_1' \sqcup \ldots \sqcup \mathfrak{X}_m \sqcup \mathfrak{X}_m' \sqcup \mathfrak{X}_{m+1} \sqcup \ldots \sqcup \mathfrak{X}_n$ for zero-dimensional dense-in-itself separable metric $\mathfrak{X}_i, \mathfrak{X}_i'$. By Proposition 7.3, we construct the maps $f_i : \mathfrak{X}_i \twoheadrightarrow^d G_i$, $f_i' : \mathfrak{X}_i' \twoheadrightarrow^d G_i'$, which are 1-fold at irreflexive roots; hence by Lemma 6.10, $f : \mathfrak{X} \twoheadrightarrow^d G$ for $f := \bigcup\limits_{i=1}^{n} f_i \cup \bigcup\limits_{i=1}^{m} f_i'$.

Every strictly minimal point $a \in G$ is an irreflexive root of a unique G_i. If a is R_D-irreflexive, then G_i is not special, so $f^{-1}(a) = f_i^{-1}(a)$ is a singleton. If a is R_D-reflexive, then G_i is special, so $f^{-1}(a) = f_i^{-1}(a) \cup (f_i')^{-1}(a)$, and thus f is 2-fold at a. Therefore, $f : \mathfrak{X} \twoheadrightarrow^{dd} F$ by Proposition 6.14. ∎

Lemma 7.7. *Let $M = (W, R, R_D, \varphi)$ be a rooted Kripke model over a basic frame*[11] *validating AT_1, Ψ a set of 2-modal formulas closed under subformulas. Let $M' = (W', R', R_D', \theta')$ be a filtration of M through Ψ described in Lemma 2.4*[12]*. Then the frame (W', R', R_D') is also basic and validates AT_1.*

Proof. In fact, R' is transitive by definition. For any two different $a, b \in W'$ we have $aR_D'b$, since xR_Dy for any $x \in a$, $y \in b$ (as $F \in \mathfrak{F}_0$).

Next, note that if a is R_D'-irreflexive, then $a = \{x\}$ for some R_D-irreflexive x. In this case, since $(W, R, R_D) \vDash AT_1$, there is no y such that yRx (Proposition 4.12), hence $(R')^{-1}(a) = \varnothing$, and thus $(W', R', R_D') \vDash AT_1$.

Finally, $R' \subseteq R_D'$. In fact, all different points in F' are R_D'-related, so it remains to show that every R_D'-irreflexive point is R'-irreflexive. As noted above, such a point is a singleton class $x^\sim = \{x\}$, where x is R_D-irreflexive. Then x is R-minimal, so in W' there is no loop of the form $x^\sim \underline{R} x_1 \underline{R} \ldots \underline{R} x^\sim$, and thus $x^\sim$ is R'-irreflexive. ∎

By a standard argument Lemma 7.7 implies

Theorem 7.8. *Every logic of the form $\mathbf{KT_1} + A$, where A is a closed 2-modal formula, has the finite model property.*

Proof. Let L be such a logic and suppose $L \nvdash B$. By Proposition 5.2 L is Kripke complete, so by the Generation lemma there is a rooted Kripke frame $F = (W, R, R_D)$ such that $F \vDash L$, $F \nvDash B$. Then F is basic by definition. Let $M = (F, \theta)$ be a Kripke model over F refuting B. Let Ψ be the set of all subformulas of A or B, and let us construct the filtration $M' = (W', R', R_D', \theta')$ of M through Ψ as in Lemmas 2.4(2) and 7.7. By the previous lemma, $F' := (W', R', R_D') \vDash \mathbf{KT_1}$.

By the Filtration lemma, $M' \nvDash B$. By the same lemma, the truth of A is preserved in M', so $F' \vDash A$, since A is closed. Therefore, $F' \vDash L$. ∎

[11] Basic frames were defined in Section 4.

[12] Recall that R' is the transitive closure of $\underline{R}$, $R_D' = \underline{R_D}$.

Theorem 7.9. *Let $\mathfrak{X}$ be a zero-dimensional dense-in-itself separable metric space. Then $\mathbf{Ld}_{\neq}(\mathfrak{X}) = \mathbf{DT_1}$.*

Proof. For any finite $\mathbf{DT_1}$-frame F we have $\mathbf{Ld}_{\neq}(\mathfrak{X}) \subseteq \mathbf{L}(F)$ by Proposition 7.6 and Lemma 6.12. By the previous theorem, $\mathbf{DT_1}$ has the fmp, so $\mathbf{Ld}_{\neq}(\mathfrak{X}) \subseteq \mathbf{DT_1}$. Since $\mathfrak{X} \vDash^d \mathbf{DT_1}$ (Proposition 4.11), it follows that $\mathbf{Ld}_{\neq}(\mathfrak{X}) = \mathbf{DT_1}$. ∎

Proposition 7.10. *[8, Lemma 3.1] Every countable*[13] *rooted* $\mathbf{K4}$*-frame is a d-morphic image of a subspace of* $\mathbf{Q}$.

To apply this proposition to the language with the difference modality, we need to examine the preimage of the root for the constructed morphism. Fortunately, in the proof of Proposition 7.10 in [8] the preimage of a root r is a singleton iff r is irreflexive.

Lemma 7.11. *Let F be a countable* $\mathbf{K4}$*-frame. Then there exists a d-morphism from a subspace of* $\mathbf{Q}$ *onto F, which is 1-fold at all strictly minimal points.*

Proof. Similar to Lemma 7.5. We can present F as a countable union of different cones $\bigcup\limits_{i\in I} F_i$ and $\mathbf{Q}$ as a disjoint union $\bigsqcup\limits_{i\in I} \mathfrak{X}_i$ of spaces homeomorphic to $\mathbf{Q}$. By Proposition 7.10 (and the remark after it), for each i there exists $f_i : \mathcal{Y}_i \twoheadrightarrow^d F_i$ for some subspace $\mathcal{Y}_i \subseteq \mathfrak{X}_i$ such that f_i is 1-fold at the root r_i of F_i if r_i is irreflexive. Now by Lemma 6.10 $f := \bigcup\limits_{i\in I} f_i : \bigsqcup\limits_{i\in I} \mathcal{Y}_i \twoheadrightarrow^d F$, and f is 1-fold at all strictly minimal points of F (i.e., the irreflexive r_i) — since every r_i belongs only to F_i, so $f^{-1}(r_i) = f_i^{-1}(r_i)$. ∎

Proposition 7.12. *Let F be a countable* $\mathbf{KT_1}$*-frame. Then there exists a dd-morphism from a subspace of* $\mathbf{Q}$ *onto F.*

Proof. Similar to Proposition 7.6. If $F = (W, R, R_D)$, the frame $G = (W, R)$ is a countable union of different cones. There are two types of cones: non-special G_i $(i \in I)$ and special (with strictly R-minimal and R_D-reflexive roots) H_j $(j \in J)$:

$$G = \bigcup_{i\in I} G_i \cup \bigcup_{j\in J} H_j.$$

Then we duplicate all special cones

$$G = \bigcup_{i\in I} G_i \cup \bigcup_{j\in J} H_j \cup \bigcup_{j\in J} H'_j$$

and as in the proof of 7.11, construct $f : \bigsqcup\limits_{i\in I} \mathcal{Y}_i \sqcup \bigsqcup\limits_{j\in J} \mathcal{Z}_j \sqcup \bigsqcup\limits_{j\in J} \mathcal{Z}'_j \twoheadrightarrow^d F$. This map is 1-fold exactly at all R_D-irreflexive points, so it is a dd-morphism onto F. ∎

Corollary 7.13. $\mathbf{Ld}_{\neq}$(*all T_1-spaces*) $= \mathbf{KT_1}$.

[13] In this chapter, as well as in [8], 'countable' means 'of cardinality at most $\aleph_0$'.

Proof. Note that $\mathbf{KT_1}$ is complete w.r.t. countable frames and every subspace of $\mathbf{Q}$ is T_1. ∎

Proposition 7.14. *Let $\Lambda = \mathbf{KT_1} + \Gamma$ be a consistent logic, where Γ is a set of closed formulas. Then Λ is dd-complete w.r.t. subspaces of $\mathbf{Q}$.*

Proof. Since every closed formula is canonical, Λ is Kripke complete. So for every formula $A \notin \Lambda$ there is a frame F_A such that $F_A \models \Lambda$ and $F_A \not\models A$. By Proposition 7.12, there is a subspace $\mathfrak{X}_A \subseteq \mathbf{Q}$ and $f_A : \mathfrak{X}_A \twoheadrightarrow^{dd} F_A$. Then $\mathfrak{X}_A \not\vDash A$, $\mathfrak{X}_A \vDash \Lambda$ by Lemma 6.12. Therefore $\mathbf{Ld}_{\neq}(\mathcal{K}) = \Lambda$ for $\mathcal{K} := \{\mathfrak{X}_A \mid A \notin \Lambda\}$. ∎

Remark 7.15. A logic of the form described in Proposition 7.14 is dd-complete w.r.t. a set of subspaces of $\mathbf{Q}$. This set may be non-equivalent to a single subspace. For example, there is no subspace $\mathfrak{X} \subseteq \mathbf{Q}$ such that $\mathbf{KT_1} = \mathbf{Ld}_{\neq}(\mathfrak{X})$. In fact, consider

$$A := [\neq]\Box\bot \wedge \Box\bot.$$

Then A is satisfiable in $\mathfrak{X}$ iff $\mathfrak{X} \vDash^d A$ iff $\mathfrak{X}$ is discrete. So A is consistent in $\mathbf{KT_1}$. Now if $\mathbf{KT_1} = \mathbf{Ld}_{\neq}(\mathfrak{X})$, then A must be satisfiable in $\mathfrak{X}$, hence $\mathfrak{X} \vDash^d A$; but $\mathbf{KT_1} \not\vdash A$, and so we have a contradiction.

8 Connectedness

Connectedness was the first example of a property expressible in cu-logic, but not in c-logic. The corresponding connectedness axiom from [38] will be essential for our further studies. In this section we show that it is weakly canonical, i.e., valid in weak canonical frames — a fact not mentioned in [38].

Lemma 8.1. *[38] A topological space $\mathfrak{X}$ is connected iff $\mathfrak{X} \vDash^c AC$, where*

$$AC := [\forall](\Box p \vee \Box\neg p) \to [\forall]p \vee [\forall]\neg p.$$

For the case of Alexandrov topology there is an equivalent definition of connectedness in relational terms.

Definition 8.2. *For a transitive Kripke frame $F = (W, R)$ we define the* comparability *relation $R^{\pm} := R \cup R^{-1} \cup I_W$. F is called* connected *if the transitive closure of $R^{\pm}$ is universal. A subset $V \subseteq W$ is called* connected *in F if the frame $F|V$ is connected.*

A 2-modal frame (W, R, S) is called (R)-connected *if (W, R) is connected.*

Thus F is connected iff every two points x, y can be connected by a *non-oriented path* (which we call just a *path*), a sequence of points $x_0 x_1 \ldots x_n$ such that $x = x_0 R^{\pm} x_1 \ldots R^{\pm} x_n = y$.

From [38] and Proposition 4.9 we obtain

Lemma 8.3. *(1) For an $\mathbf{S4}$-frame F, the associated space $N(F)$ is connected iff F is connected.*
(2) For a $\mathbf{K4}$-frame F, $F_{\forall} \models AC^{\sharp u}$ iff F is connected.

Lemma 8.4. *Let $M = (W, R, R_D, \theta)$ be a rooted generated submodel of m-weak canonical model for a modal logic $\Lambda \supseteq \mathbf{K4D}^+$. Then*

(1) Every R-cluster in M is finite of cardinality at most 2^m.

(2) (W, R) has finitely many R-maximal clusters.

(3) For each R-maximal cluster C in M there exists an m-formula $\beta(C)$ such that:

$$\forall x \in M\ (M, x \vDash \beta(C) \Leftrightarrow x \in \overline{R}^{-1}(C)).$$

The proof is similar to [11, Section 8.6].

Lemma 8.5. *Every rooted generated subframe of a weak canonical frame for a logic $\Lambda \supseteq \mathbf{K4D}^+ + AC^{\sharp u}$ is connected.*

Proof. Let M be a weak canonical model for Λ, M_0 its rooted generated submodel with the frame $F = (W, R, R_D)$, and suppose F is disconnected. Then there exists a nonempty proper clopen subset V in the space $N(W, \overline{R})$. Let Δ be the set of all R-maximal clusters in V and put

$$B := \bigvee_{C \in \Delta} \beta(C).$$

Then B defines V in M_0, i.e., $V = \overline{R}^{-1}(\bigcup \Delta)$. In fact, $\bigcup \Delta \subseteq V$ implies $\overline{R}^{-1}(\bigcup \Delta) \subseteq V$, since V is closed. The other way round, $V \subseteq \overline{R}^{-1}(\bigcup \Delta)$, since for any $v \in V$, $\overline{R}(v)$ contains an R-maximal cluster $C \in \Delta$, and $\overline{R}(v) \subseteq V$ as V is open.

So $w \models B$ for any $w \in V$, and since V is open, $w \models \overline{\Box} B$. By the same reason, $w \models \overline{\Box}\neg B$ for any $w \notin V$. Hence

$$M_0 \models [\forall]\,(\overline{\Box} B \vee \overline{\Box}\neg B).$$

By Proposition 2.6 all substitution instances of AC are true in M_0. So we have

$$M_0 \models [\forall]\,(\overline{\Box} B \vee \overline{\Box}\neg B) \to [\forall]\, B \vee [\forall]\, \neg B,$$

and thus

$$M_0 \models [\forall]\, B \vee [\forall]\, \neg B.$$

This contradicts the fact that V is a nonempty proper subset of W. ■

In d-logic instead of connectedness we can express some its local versions; they will be considered in the next section.

9 Kuratowski formula and local 1-componency

In this section we briefly study Kuratowski formula distinguishing $\mathbf{R}$ from $\mathbf{R}^2$ in d-logic. Here the main proofs are similar to the previous section, so most of the details are left to the reader.

Definition 9.1. *We define* Kuratowski formula *as*

$$Ku := \quad \Box(\overline{\Box} p \vee \overline{\Box}\neg p) \to \Box p \vee \Box\neg p.$$

The spaces validating Ku are characterized as follows [31].

Lemma 9.2. *For a topological space $\mathfrak{X}$, $\mathfrak{X} \vDash^d Ku$ iff*

for any $x \in X$ and any open neighbourhood U of x, if $U - \{x\}$ is a disjoint union $V_1 \cup V_2$ of sets open in the subspace $U - \{x\}$, then there exists a neighbourhood[14] *$V \subseteq U$ of x such that $V - \{x\} \subseteq V_1$ or $V - \{x\} \subseteq V_2$.*

Definition 9.3. *A topological space $\mathfrak{X}$ is called* locally connected *if every neighbourhood of any point x contains a connected neighbourhood of x. Similarly, $\mathfrak{X}$ is called* locally 1-component *if every punctured neighbourhood of any point x contains a connected punctured neighbourhood of x.*

It is well known [3] that in a locally connected space every neighbourhood U of any point x contains a connected *open* neighbourhood of x (e.g. the connected component of x in $\mathbf{I}U$).

Lemma 9.4. *If $\mathfrak{X}$ is locally 1-component, then $\mathfrak{X} \vDash^d Ku$.*

The proof is straightforward, and we leave it to the reader.

Lemma 9.5. *(1) Every space d-validating Ku has the following* non-splitting property*:*

(NSP) If an open set U is connected, $x \in U$ and $U - \{x\}$ is open, then $U - \{x\}$ is connected.

(2) Suppose $\mathfrak{X}$ is locally connected and local T_1. Then (NSP) holds in $\mathfrak{X}$ iff $\mathfrak{X}$ is locally 1-component iff $\mathfrak{X} \vDash^d Ku$.

Proof. (1) We assume $\mathfrak{X} \vDash^d Ku$ and check (NSP). Suppose U is open and connected, $U^\circ := U - \{x\}$ is open, and consider a partition $U^\circ = U_1 \cup U_2$ for open U_1, U_2. By 9.2 there exists an open $V \subseteq U$ containing x such that $V \subseteq \{x\} \cup U_1$ or $V \subseteq \{x\} \cup U_2$. Consider the first option (the second one is similar). We have a partition

$$U = (\{x\} \cup U_1) \cup U_2,$$

and $\{x\} \cup U_1 = V \cup U_1$, so $\{x\} \cup U_1$ is open. Hence by connectedness, $U = \{x\} \cup U_1$, i.e., $U^\circ = U_1$. Therefore, U° is connected.

(2) It suffices to show that (NSP) implies the local 1-componency. Consider $x \in X$ and its neighbourhood U_1. Since $\mathfrak{X}$ is local T_1, U_1 contains an open neighborhood U_2, in which x is closed, i.e., $\mathbf{C}\{x\} \cap U_2 = \{x\}$. By the local connectedness, U_2 contains a connected open neighbourhood U_3, and again $\mathbf{C}\{x\} \cap U_3 = \{x\}$; thus $U_3 - \{x\}$ is open. Eventually, $U_3 - \{x\}$ is connected, by (NSP). ■

Remark 9.6. The *(n-th) generalized Kuratowski formula* is the following formula in variables $p_0, \ldots, p_n$

$$Ku_n := \Box \bigvee_{k=0}^{n} \overline{\Box} Q_k \to \bigvee_{k=0}^{n} \Box\neg Q_k,$$

where $Q_k := p_k \wedge \bigwedge_{j \neq k} \neg p_j$.

[14] In [31] neighbourhoods are supposed open, but this does not matter here, since every neighbourhood contains an open neighbourhood.

The formula Ku_1 is related to the equality found by Kuratowski [27]:

$$(*) \quad \mathbf{d}((x \cap \mathbf{d}(-x)) \cup (-x \cap \mathbf{d}x)) = \mathbf{d}x \cap \mathbf{d}(-x),$$

which holds in every algebra $DA(\mathbf{R}^n)$ for $n > 1$, but not in $DA(\mathbf{R})$. This equality corresponds to the modal formula

$$Ku' := \quad \Diamond((p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond p)) \leftrightarrow \Diamond p \wedge \Diamond\neg p,$$

and one can show that $\mathbf{D4} + Ku' = \mathbf{D4} + Ku_1 = \mathbf{D4} + Ku$.

Remark 9.7. The class of spaces validating Ku_n is described in [31]. In particular, it is valid in all locally n-component spaces defined as follows.

A neighbourhood U of a point x in a topological space is called *n-component at x* if the punctured neighbourhood $U - \{x\}$ has at most n connected components. A topological space is called *locally n-component* if the n-component neighbourhoods at each of its point constitute a local base (i.e., every neighbourhood contains an n-component neighbourhood).

Lemma 9.8. *[31] For a transitive Kripke frame (W, R)*

$(W, R) \vDash Ku$ iff for any R-irreflexive x, the subset $R(x)$ is connected (in the sense of Definition 8.2).

Theorem 9.9. *The logics $\mathbf{K4} + Ku$, $\mathbf{D4} + Ku$ are weakly canonical, and thus Kripke complete.*

A proof of 9.9 based on Lemma 9.8 and a 1-modal version of Lemma 8.4 is straightforward, cf. [37] or [31] (the latter paper proves the same for Ku_n).

Hence we obtain

Theorem 9.10. *The logic $\mathbf{DT_1K} := \mathbf{DT_1} + Ku$ is weakly canonical, and thus Kripke complete.*

Proof. (Sketch.) For the axiom Ku the argument from the proof of 9.9 is still valid due to definability of all maximal clusters (Lemma 8.4). The remaining axioms are Sahlqvist formulas. ∎

Theorem 9.11. *The logic $\mathbf{DT_1CK} := \mathbf{DT_1K} + AC^{\sharp u}$ is weakly canonical, and thus Kripke complete.*

Proof. We can apply the previous theorem and Lemma 8.5. ∎

Completeness theorems from this section can be refined: in the next section we will prove the fmp for the logics considered above.

10 The finite model property of D4K, $\mathbf{DT_1K}$, and $\mathbf{DT_1CK}$

For the logic $\mathbf{D4} + Ku$ the first proof of the fmp was given in [37]. Another proof (also for $\mathbf{D4} + Ku_n$) was proposed by M. Zakharyaschev [42]; it is based on a general and powerful method.

In this section we give a simplified version of the proof from [37]. It is based on a standard filtration method, and the same method is also applicable to 2-modal logics $\mathbf{DT_1K}$, $\mathbf{DT_1CK}$.

Theorem 10.1. *The logics* $\mathbf{DT_1K}$ *and* $\mathbf{DT_1CK}$ *have the finite model property.*

Proof. Let Λ be one of these logics. Consider an m-formula $A \notin \Lambda$. Take a generated submodel $M = (W, R, R_D, \varphi)$ of the m-restricted canonical model of Λ such that $M, u \not\models A$ for some u. As we know, its frame is basic and its R-maximal clusters are definable (Lemma 8.4).

Put

$$\Psi_0 := \{\beta(C) \mid C \text{ is an } R\text{-maximal cluster in } M\},$$
$$\Psi_1 := \{A\} \cup \left\{\overline{\Box}\gamma \mid \gamma \text{ is a Boolean combination of formulas from } \Psi_0\right\},$$
$$\Psi := \text{the closure of } \Psi_1 \text{ under subformulas.}$$

The set Ψ is obviously finite up to equivalence in Λ.

Take the filtration $M' = (W', R', R'_D, \varphi')$ of M through Ψ as in Lemma 7.7. By that lemma, $F' := (W', R', R'_D) \vDash \mathbf{KT_1}$. The seriality of R' easily follows from the seriality of R.

Next, if $\Lambda = \mathbf{DT_1CK}$, the frame (W, R, R_D) is connected by Lemma 8.5. So for any $x, y \in W$ there is an R-path from x to y. aRb implies $a^\sim R' b^\sim$, so there is an R'-path from $x^\sim$ to $y^\sim$ in F'. Therefore $F' \vDash AC^{\sharp u}$. It remains to show that $F' \vDash Ku$. Consider an R'-irreflexive point $x^\sim \in W'$ and assume that $R'(x^\sim)$ is disconnected. Let V be a nonempty proper connected component of $R'(x^\sim)$. Consider

$$\Delta := \{C \mid \exists y(y^\sim \in V \ \&\ C \subseteq R(y) \ \&\ C \text{ is an } R\text{-maximal cluster in } M)\};$$
$$B := \bigvee_{C \in \Delta} \beta(C),$$

where $\beta(C)$ is from Lemma 8.4. Note that

(1) $\quad z \in C \ \&\ C \in \Delta \Rightarrow z^\sim \in V.$

In fact, if $C \in \Delta$, then for some $y^\sim \in V$ we have yRz; hence $y^\sim R' z^\sim$, so $z^\sim \in V$, by the connectedness of V.

Let us show that for any $y^\sim \in R'(x^\sim)$

(2) $\quad M', y^\sim \models B$ iff $M, y \models B$ iff $y^\sim \in V$,

i.e., B defines V in $R'(x^\sim)$.

The first equivalence holds by the Filtration Lemma, since $B \in \Psi_1$.

Let us prove the second equivalence. To show 'if', suppose $y^\sim \in V$. By Lemma 2.9, in the restricted canonical model there is a maximal cluster C R-accessible from y; then $M, y \models \beta(C)$. We have $C \in \Delta$, and thus $M, y \models B$.

To show 'only if', suppose $y^\sim \notin V$, but $M, y \models B$. Then $M, y \models \beta(C)$, for some $C \in \Delta$, hence $C \subseteq R(y)$, i.e., yRz for some (and for all) $z \in C$; so it follows that $y^\sim R' z^\sim$. Thus $y^\sim$ and $z^\sim$ are in the same connected component of $R'(x^\sim)$, which implies $z^\sim \notin V$. However, $z^\sim \in V$ by (1), leading to a contradiction.

By Proposition 2.6 all substitution instances of Ku are true in M. So

$$M \vDash Ku(B) := \Box(\overline{\Box}B \vee \overline{\Box}\neg B) \to \Box B \vee \Box\neg B.$$

Consider an arbitrary $y \in R(x)$. Then for any $z \in R(y)$, $y^\sim$ and $z^\sim$ are in the same connected component of $R'(x^\sim)$. Thus $y^\sim$ and $z^\sim$ are both either in V or not in V, and so by (2), both of them satisfy either B or $\neg B$. Hence $M, y \models \overline{\Box} B \vee \overline{\Box} \neg B$. Therefore, x satisfies the premise of $Ku(B)$. Consequently, x must satisfy the conclusion of $Ku(B)$. Thus $M, x \models \Box B$ or $M, x \models \Box \neg B$. Since $\Box B, \Box \neg B \in \Psi_1$, the Filtration Lemma implies $M', x^\sim \models \Box B$ or $M', x^\sim \models \Box \neg B$. Eventually by (2), $V = R'(x^\sim)$ or $V = \varnothing$, which contradicts the assumption about V.

To conclude the proof, note that $A \in \Psi$, so by the Filtration Lemma $M', u^\sim \not\models A$. As we have proved, $F' \models \Lambda$. Therefore Λ has the fmp. ■

Theorem 10.2. *The logic* **D4K** *has the finite model property.*

Proof. Use the argument from the proof of 10.1 without the second relation. ■

Thanks to the fmp, we have a convenient class of Kripke frames for the logic $\mathbf{DT_1CK}$. This will allow us to prove the topological completeness result in the next section.

11 The dd-logic of $\mathbf{R}^n$, $n \geq 2$.

This section contains the main result of the Chapter. The proof is based on the fmp theorem from the previous section and a technical construction of a dd-morphism presented in the Appendix.

In this section $\|\cdot\|$ denotes the standard norm in $\mathbf{R}^n$, i.e. for $x \in \mathbf{R}^n$

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

We begin with some simple observations on connectedness. For a path $\alpha = w_0 w_1 \ldots w_n$ in a **K4**-frame (W, R) we use the notation $\overline{R}(\alpha) := \bigcup_{i=0}^{n} \overline{R}(w_i)$. A path α is called *global* (in F) if $\overline{R}(\alpha) = W$.

Lemma 11.1. *Let $F = (W, R)$ be a finite connected* **K4**-*frame, $w, v \in W$. Then there exists a global path from w to v.*

Proof. In fact, in the finite connected graph $(W, R^{\pm})$ the vertices w, v can be connected by a path visiting all the vertices (perhaps, several times). ■

Lemma 11.2. *Let $F = (W, R, R_D)$ be a finite rooted* $\mathbf{DT_1CK}$-*frame. Then the set of all R_D-reflexive points in F is connected.*

Proof. Let x, y be two R_D-reflexive points. Since (W, R) is connected, there exists a path connecting x and y. Consider such a path α with the minimal number n of R_D-irreflexive points, and let us show that $n = 0$.

Suppose not. Take an R_D-irreflexive point z in α; then $\alpha = x \ldots uzv \ldots y$, for some u, v, and it is clear that zRu, zRv, since z is strictly R-minimal. By Lemma 9.8, $R(z)$ is connected, so u, v can be connected by a path β in $R(z)$. Thus in α we can replace the part uzv with β, and the combined path $x \ldots \beta \ldots y$ contains $(n-1)$ R_D-irreflexive points, which contradicts the minimality of n. ■

Lemma 11.3. *Let $F = (W, R, R_D)$ be a finite rooted* $\mathbf{DT_1CK}$*-frame and let $w', w'' \in W$ be R_D-reflexive. Then there is a global path $\alpha = w_0 \dots w_n$ in (W, R) such that $w' = w_0$, $w_n = w''$ and all R_D-irreflexive points occur only once in α.*

Proof. Let $\{u_1, \dots, u_k\}$ be the R_D-irreflexive points. By connectedness there exists paths $\alpha_0, \dots, \alpha_k$ respectively from w' to u_1, from u_1 to u_2, $\dots$, from u_k to w''.

By Lemma 11.2, the set $W' := W - \{u_1, \dots, u_k\}$ is connected. Hence we may assume that each α_i does not contain R_D-irreflexive points except its ends. Also there exists a loop β in $F' := F|W'$ from w'' to w'' such that $W - \bigcup_{i=1}^{k-1} \overline{R}(\alpha_i) \subseteq \overline{R}(\beta)$. Then we can define α as the joined path

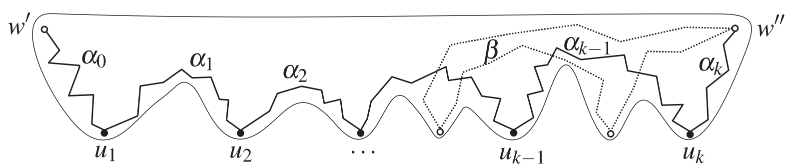


Figure 5: Path α.

$\alpha_0 \dots \alpha_k \beta$, (Fig. 5). ■

Proposition 11.4. *For a finite rooted* $\mathbf{DT_1CK}$*-frame $F = (W, R, R_D)$ and R-reflexive points $w', w'' \in W$, the following holds.*

(a) *If $X = \{x \in \mathbf{R}^n \mid ||x|| \leq r\}$, $n \geq 2$, then there exists $f : X \twoheadrightarrow^{dd} F$ such that $f(\partial X) = \{w'\}$;*

(b) *If $0 \leq r_1 < r_2$ and*

$$X = \{x \in \mathbf{R}^n \mid r_1 \leq ||x|| \leq r_2\},$$
$$Y' = \{x \in \mathbf{R}^n \mid ||x|| = r_1\}, \; Y'' = \{x \in \mathbf{R}^n \mid ||x|| = r_2\},$$

then there exists $f : X \twoheadrightarrow^{dd} F$ such that $f(Y') = \{w'\}$, $f(Y'') = \{w''\}$.

Proof. By induction on $|W|$. Let us prove (a) first. There are five cases:

(a1) $W = R(b)$ (and hence bRb) and $b = w'$. Then there exists $f : X \twoheadrightarrow^{d} (W, R)$. In fact, let C be the cluster of b (as a subframe of (W, R)). Then $(W, R) = C$ or $(W, R) = C \cup F_1 \cup \ldots \cup F_l$, where the F_i are generated by the successors of C. If $(W, R) = C$, we apply Proposition 6.7; otherwise we apply Lemma 6.8 and IH.

By 4.12 it follows that R_D is universal. And so by 6.14(3) f is a dd-morphism.

(a2) $W = R(b)$ and not $w'Rb$. We may assume that $r = 3$. Put

$$X_1 := \{x \mid ||x|| \leq 1\}, \; Y := \{x \mid 1 \leq ||x|| \leq 2\}, \; X_2 := \{x \mid 2 \leq ||x|| \leq 3\}.$$

By the case (a1), there is $f_1 : X_1 \twoheadrightarrow^{dd} F$ with $f_1(\partial X_1) = \{b\}$. Let C be a maximal cluster in $R(w')$. By 6.7 there is $g : \mathbf{I}Y \twoheadrightarrow^{d} C$. Since $R(w') \neq$

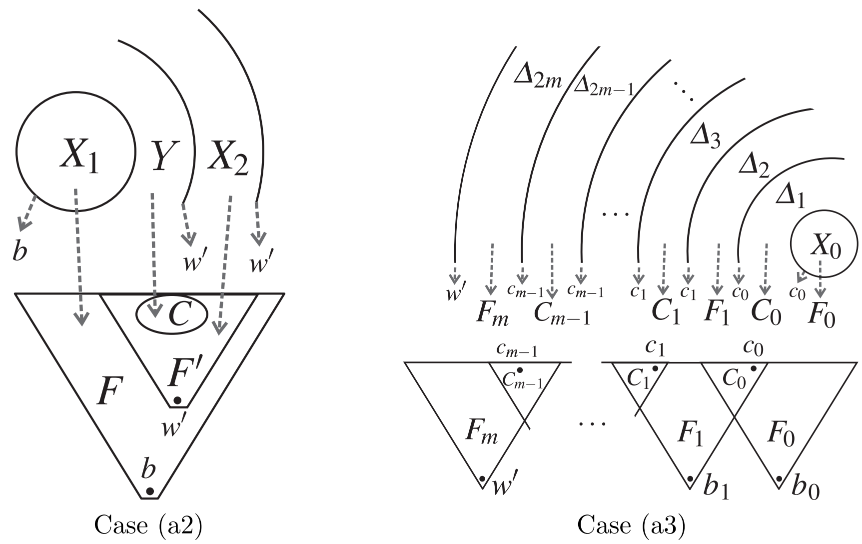


Figure 6: dd-morphism f

W, we can apply IH to the frame $F' := F^{w'}_{\forall}$ and construct a dd-morphism $f_2 : X_2 \twoheadrightarrow^{dd} F'$ with $f_2(\partial X_2) = \{w'\}$. Now since $f_i(\partial X_i) \subseteq R^{-1}(C)$, the Glueing lemma 6.9 is applicable. Thus $f : X \twoheadrightarrow^{d} F$ for $f := f_1 \cup f_2 \cup g$ (See Fig. 6, Case (a2)). Note that $\partial X \subset \partial X_2$, so $f(\partial X) = f_2(\partial X) = \{w'\}$.

As in the case (a1), f is a dd-morphism by 6.14.

(a3) (W, R) is not rooted. By Lemma 11.3 there is a global path α in F with a single occurrence of every R_D-irreflexive point. We may assume that $\alpha = b_0 c_0 b_1 c_1 \dots c_{m-1} b_m$, $b_m = w'$ and for any $i < m$ $c_i \in C_i \subseteq R(b_i) \cap R(b_{i+1})$, where C_i is an R-maximal cluster. Such a path is called *reduced*. For $0 \leq j \leq m$ we put $F_j := F|\overline{R}(b_j)$.

Since (W, R) is not rooted, each F_j is of smaller size than F, so we can apply the induction hypothesis to F_j. We may assume that

$$X = \{x \mid ||x|| \leq 2m+1\}, \; Y = \{x \mid ||x|| = 2m+1\}.$$

Then put

$$X_i := \{x \mid ||x|| \leq i+1\} \text{ for } 0 \leq i \leq 2m,$$

$$Y_i := \partial X_i, \; \Delta_i := \mathbf{C}(X_i - X_{i-1}) \text{ for } 0 \leq i \leq 2m.$$

By IH and Proposition 6.7 there exist

$$f_0 : X_0 \twoheadrightarrow^{dd} F_0 \text{ such that } f_0(Y_0) = \{c_0\},$$

$$f_{2j} : \Delta_{2j} \twoheadrightarrow^{dd} F_j \text{ such that } f_{2j}(Y_{2j}) = \{c_j\}, \; f_{2j}(Y_{2j-1}) = \{c_{j-1}\} \text{ for } 1 \leq j \leq m,$$

$$f_{2j-1} : \mathbf{I}\Delta_{2j+1} \twoheadrightarrow^{d} C_j \text{ for } 0 \leq j \leq m-1.$$

One can check that $f : X \twoheadrightarrow^{dd} F$ for $f := \bigcup_{j=0}^{2m} f_j$ (Fig. 6).

(a4) $W = \overline{R}(b)$, $\neg bR_D b$ (and so $\neg bRb$). We may assume that

$$X = \{x \mid ||x|| \leq 2\}\,,\ Y = \{x \mid ||x|| = 2\}\,.$$

Then similar to case (a3) put

$$X_0 := X,\ Y_0 := Y,\ X_i := \left\{x \mid ||x|| \leq \frac{1}{i}\right\},\ Y_i := \partial X_i,\ \Delta_i := \mathbf{C}(X_i - X_{i+1}),\ (i > 0).$$

Consider the frame $F' := F|W'$, where $W' = W - \{b\}$. Note that $w' \in W'$, since $w'Rw'$, by the assumption of 11.4. By Lemma 9.8 F' is connected, and thus $F' \vDash \mathbf{DT_1CK}$. By Lemma 11.3 there is a reduced global path $\alpha = a_1 \dots a_m$ in F' such that $a_1 = w'$. Let

$$\gamma = a_1 a_2 \dots a_{m-1} a_m a_{m-1} \dots a_2 a_1 a_2 \dots$$

be an infinite path shuttling back and forth through α. Rename the points in γ:

$$\gamma = b_0 c_0 b_1 c_1 \dots b_m c_m b_{m+1} \dots \qquad (6)$$

Again as in the case (a3) we put $F_j := F|\overline{R}(b_j)$, and assume that $c_j \in C_j$ and C_j is an R-maximal cluster. By IH there exist

$$f_0 : \Delta_0 \twoheadrightarrow^{dd} F_0 \text{ such that } f_0(Y_0) = \{b_0\} = \{w'\},\ f_1(Y_1) = \{c_0\},$$
$$f_{2j} : \Delta_{2j} \twoheadrightarrow^{dd} F_j \text{ such that } f_{2j}(Y_{2j}) = \{c_{j-1}\},\ f_{2j}(Y_{2j+1}) = \{c_j\} \text{ for } j > 0,$$

and by Proposition 6.7 there exist $f_{2j+1} : \mathbf{I}\Delta_{2j+1} \twoheadrightarrow^{d} C_j$. Put

$$f(x) := \begin{cases} b & \text{if } x = \mathbf{0}, \\ f_{2j}(x) & \text{if } x \in \Delta_{2j}, \\ f_{2j+1}(x) & \text{if } x \in \mathbf{I}\Delta_{2j+1}, \end{cases}$$

One can check that f is d-morphic (Fig. 7).

(a5) $W = \overline{R}(b)$, $\neg bRb$ and $bR_D b$. Then R_D is universal, $w' \neq b$. Put

$$X' := \{x \mid ||x|| < 1\}\,,\ X_4 := \{x \mid 1 \leq ||x|| \leq 2\}\,,$$

and let X_1, X_2 be two disjoint closed balls in X', $X_3 := X' - X_1 - X_2$.

Let C be a maximal cluster in $R(w')$, $F' := F|R(w')$. Then there exist:

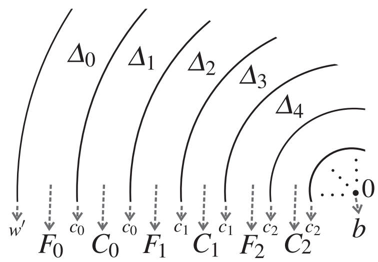
$$f_i : X_i \twoheadrightarrow^{d} (W, R) \text{ for } i = 1, 2 \text{ such that } f_i(\partial X_i) = \{w'\},\ \text{by the case (a4)},$$
$$f_3 : X_3 \twoheadrightarrow^{d} C,\ \text{by Proposition 6.7},$$
$$f_4 : X_4 \twoheadrightarrow^{dd} F' \text{ such that } f_4(\partial X_4) = \{w'\},\ \text{by the induction hypothesis.}$$

Put $f := f_1 \cup f_2 \cup f_3 \cup f_4$ (Fig. 7). Then $f(\partial \mathfrak{X}) = \{w'\}$.

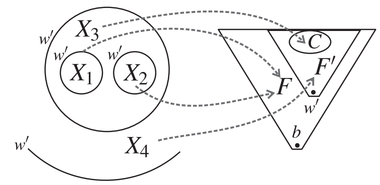
By Lemma 6.9 (b) $f_1 \cup f_2 : X_1 \cup X_2 \twoheadrightarrow^{d} F$, and hence $f : X \twoheadrightarrow^{d} F$ by Lemma 6.9 (a). f is manifold at b, thus it is a dd-morphism by 6.12.

Now we prove (b). There are three cases.

(b1) $w' = w'' = b$ and $W = R(b)$. The argument is the same as in the case (a1), using Proposition 6.7, Lemma 6.8, the induction hypothesis, and Proposition 6.14.



Case (a4)



Case (a5)

Figure 7: dd-morphism f

(b2) $w' = w'' = b$, but $W \neq R(b)$. Consider a maximal cluster $C \subseteq R(b)$. Since all spherical shells for different r_1 and r_2 are homeomorphic, we assume that $r_1 = 1$, $r_2 = 4$. Consider the sets

$$X_1 := \{x \mid 1 \leq ||x|| \leq 2\}\,,\ \ X' := \{x \mid 2 < ||x|| < 3\}\,,\ \ X_3 := \{x \mid 3 \leq ||x|| \leq 4\}\,,$$

and let $X_0 \subset X'$ be a closed ball, $X_2 := X' - X_0$. Let $F' := F|R(b)$. There exist

$$\begin{aligned}
f_1 :& X_1 \twoheadrightarrow^{dd} F' \text{ such that } f_1(\partial X_1) = \{b\}\,, \text{ by the case (b1)},\\
f_2 :& X_2 \twoheadrightarrow^{d} C, \text{ by Proposition 6.7},\\
f_3 :& X_3 \twoheadrightarrow^{dd} F' \text{ such that } \ f_3(\partial X_3) = \{b\}\,, \text{ by the case (b1)},\\
f_0 :& X_0 \twoheadrightarrow^{dd} F \text{ such that } \ f_4(\partial X_0) = \{b\}\,, \text{ by the statement (a) for } F.
\end{aligned}$$

One can check that $f : X \twoheadrightarrow^{dd} F$ for $f := f_0 \cup f_1 \cup f_2 \cup f_3$.

(b3) $w' \neq w''$ and for some $b \in W$, $W = R(b)$, so F has an R-reflexive root. Let

$$F_1 := F|R(w'),\ F_2 := F|R(w''),$$

and let C_i be an R-maximal cluster in F_i for $i \in \{1, 2\}$.

We assume that $r_1 = 1$, $r_2 = 6$ and consider the sets

$$X_i := \{x \mid i \leq ||x|| \leq i+1\}\,,\ i \in \{1, \ldots, 5\}\,.$$

By the case (b1) and Proposition 6.7 we have

$$\begin{aligned}
&f_1 : X_1 \twoheadrightarrow^{dd} F_1 \text{ such that } f_1(\partial X_1) = \{w'\}\,, && f_2 : \mathbf{I}X_2 \twoheadrightarrow^{d} C_1,\\
&f_3 : X_3 \twoheadrightarrow^{dd} F \text{ such that } \ f_3(\partial X_3) = \{b\}\,, && f_4 : \mathbf{I}X_4 \twoheadrightarrow^{d} C_2,\\
&f_5 : X_5 \twoheadrightarrow^{dd} F_2 \text{ such that } f_1(\partial X_5) = \{w''\}\,.
\end{aligned}$$

One can check that $f : X \twoheadrightarrow^{dd} F$ for $f := \bigcup_{i=1}^{5} f_i$ (Fig. 8, Case (b3)).

(b4) $w' \neq w''$ and $W \neq R(b)$ for any $b \in W$. By Lemma 11.2 there is a reduced path $\alpha = b_0 c_0 b_1 \ldots c_{m-1} b_m$ from $b_0 = w'$ to $b_m = w''$ that does not contain R_D-irreflexive points, $c_i \in C_i$, where C_i is an R-maximal cluster. We may also assume that

$$\overline{R}(b_i) \neq W,\ \text{for any } i \in \{1, \ldots, m-1\}. \qquad (7)$$

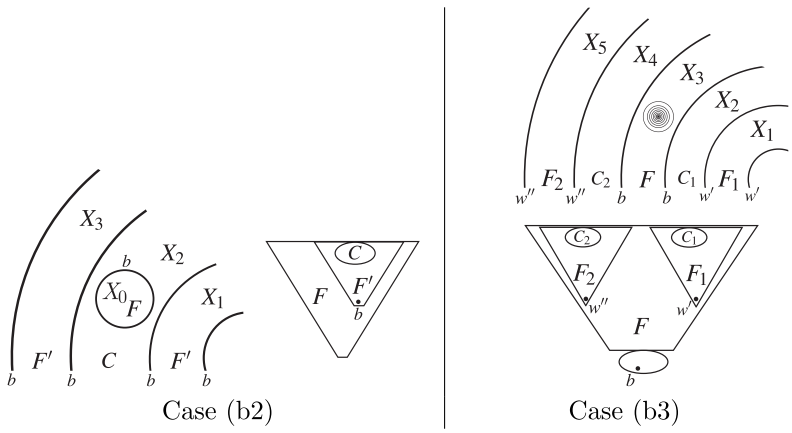


Figure 8: dd-morphism f

In fact, if the frame (W, R) is not rooted, then (7) obviously holds. If (W, R) is rooted, then its root r is irreflexive and by Lemma 9.8, $R(r)$ is connected, so there exists a path α in $R(r)$ satisfying (7). Put

$$F_0 := F,\ F_j := F|R(b_j), 1 \leq j \leq m.$$

Assuming that $r_1 = 1$, $r_2 = 2m + 1$ we define

$$X_i := \{x \mid ||x|| \leq i + 1\},\ Y_i := \partial X_i \text{ (for } 0 \leq i \leq 2m + 1),$$
$$\Delta_i := \mathbf{C}(X_{i+1} - X_i) \text{ (for } 0 \leq i \leq 2m).$$

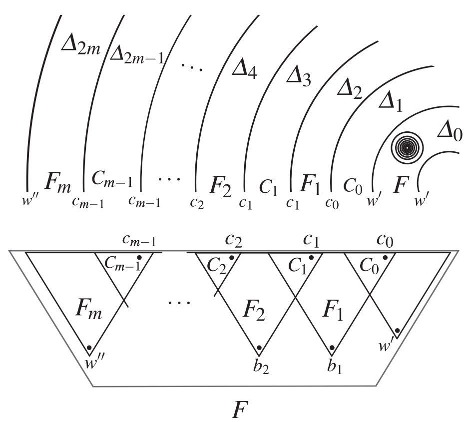


Figure 9: dd-morphism f, case (b4)

By the cases (b2), (b1), Proposition 6.7, and the induction hypothesis there exist

$f_0 : \Delta_0 \twoheadrightarrow^{dd} F = F_0$ such that $f_0(Y_0) = f_0(Y_1) = \{w'\}$;

$f_{2j} : \Delta_{2j} \twoheadrightarrow^{dd} F_j$ such that $f_{2j}(Y_{2j+1}) = \{c_j\}$, $f_{2j}(Y_{2j}) = \{c_{j-1}\}$ $(1 \leq j \leq m)$;

$f_{2j-1} : \mathbf{I}\Delta_{2j-1} \twoheadrightarrow^{d} C_{j-1}$ $(1 \leq j \leq m)$,

$f_{2m} : \Delta_{2m} \twoheadrightarrow^{dd} F_m$ such that $f_{2m}(Y_{2m}) = \{c_m\}$, $f_{2m}(Y_{2m+1}) = \{w''\}$.

We claim that $f : X \twoheadrightarrow^{dd} F$ for $f := \bigcup_{i=0}^{2m} f_i$ (Fig. 9). First, we prove by induction using Lemma 6.9 (see previous cases) that f is a d-morphism. Note that $f(Y') = f(Y_0) = \{w'\}$ and $f(Y'') = f(Y_{2m+1}) = \{w''\}$.

Second, there are no R_D-irreflexive points in α, so all preimages of R_D-irreflexive points are in Δ_0; since f_0 is a dd-morphism, f is 1-fold at any R_D-irreflexive point and manifold at all the others. Thus f is a dd-morphism by Proposition 6.14. ■

Theorem 11.5. *For $n \geq 2$, the dd-logic of $\mathbf{R}^n$ is* $\mathbf{DT_1CK}$.

Proof. Since $\mathbf{R}^n$ is a locally 1-component connected dense-in-itself metric space, $\mathbf{R}^n \models^d \mathbf{DT_1CK}$.

Now consider a formula $A \notin \mathbf{DT_1CK}$. Due to the fmp (Theorem 10.1) there exists a finite rooted Kripke frame $F = (W, R, R_D) \vDash \mathbf{DT_1CK}$ such that $F \nvDash A$. By Proposition 11.4 there exists $f : \mathbf{R}^n \twoheadrightarrow^{dd} F$. Hence $\mathbf{R}^n \nvDash^d A$ by Lemma 6.12. ■

12 Concluding remarks

Hybrid logics. Logics with the difference modality are closely related to hybrid logics. The paper [29] describes a validity-preserving translation from the language with the topological and the difference modalities into the hybrid language with the topological modality, nominals and the universal modality.

Apparently a similar translation exists for dd-logics considered in our chapter. There may be an additional option — to use 'local nominals', propositional constants that may be true not in a single point, but in a discrete set. Perhaps one can also consider 'one-dimensional nominals' naming 'lines' or 'curves' in the main topological space; there may be many other similar options.

Definability. Among several types of topological modal logics considered in this chapter dd-logics are the most expressive. The correlation between all the types are shown in Fig. 10. A language $\mathcal{L}_1$ is *reducible* to $\mathcal{L}_2$ ($\mathcal{L}_1 \leq \mathcal{L}_2$) if every $\mathcal{L}_1$-definable class of spaces is $\mathcal{L}_2$-definable; $\mathcal{L}_1 < \mathcal{L}_2$ if $\mathcal{L}_1 \leq \mathcal{L}_2$ and $\mathcal{L}_2 \nleq \mathcal{L}_1$. The non-strict reductions 1–7 in Fig. 10 are rather obvious. Let us explain, why 1–6 are strict.

The relations 1 and 2 are strict, since the c-logics of $\mathbf{R}$ and $\mathbf{Q}$ coincide [32], while the cu- and d-logics are different [38, 14].

The relation 3 is strict, since in d-logic without the universal modality we cannot express connectedness (this follows from [14]). The relations 4

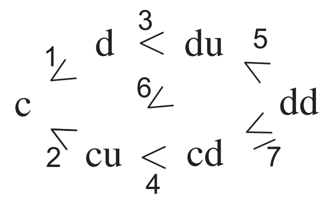


Figure 10: Correlation between topomodal languages.

and 6 are strict, since the cu-logics of $\mathbf{R}$ and $\mathbf{R}^2$ are the same [38], while the cd- and du-logics are different [16, 31].

In cd- and dd-logic we can express *global 1-componency*: the formula

$$[\neq](\overline{\Box}p \vee \overline{\Box}\neg p) \to [\neq]p \vee [\neq]\neg p$$

is c-valid in a space $\mathfrak{X}$ iff the complement of any point in $\mathfrak{X}$ is connected. So we can distinguish the line $\mathbf{R}$ and the circle $\mathbf{S^1}$. In du- (and cu-) logic this is impossible, since there is a local homemorphism $f(t) = e^{it}$ from $\mathbf{R}$ onto S^1. It follows that the relation 5 is strict. Our conjecture is that the relation 7 is strict as well.

Axiomatization. There are several open questions about axiomatization and completeness of certain dd-logics.

1. The first group of questions is about the logic of $\mathbf{R}$. On the one hand, in [24] it was proved that $\mathbf{Lc}_{\neq}(\mathbf{R})$ is not finitely axiomatizable. Probably, the same method can be applied to $\mathbf{Ld}_{\neq}(\mathbf{R})$. On the other hand, $\mathbf{Lc}_{\neq}(\mathbf{R})$ has the fmp [25], and we hope that the same holds for the dd-logic. The decidability of $\mathbf{Ld}_{\neq}(\mathbf{R})$ follows from [10], since this logic is a fragment of the universal monadic theory of $\mathbf{R}$; and by a result from [34] it is PSPACE-complete. However, constructing an explicit infinite axiomatization of $\mathbf{Lc}_{\neq}(\mathbf{R})$ or $\mathbf{Ld}_{\neq}(\mathbf{R})$ might be a serious technical problem.

2. A 'natural' semantical characterization of the logic $\mathbf{DT_1C} + Ku_2$ (which is a proper sublogic of $\mathbf{Ld}_{\neq}(\mathbf{R})$) is not quite clear. Our conjecture is that it is complete w.r.t. 2-dimensional cell complexes, or more exactly, adjunction spaces obtained from finite sets of 2-dimensional discs and 1-dimensional segments.

3. We do not know any syntactic description of dd-logics of 1-dimensional cell complexes (i.e., unions of finitely many segments in $\mathbf{R}^3$ that may have only endpoints as common). Their properties are probably similar to those of $\mathbf{Ld}_{\neq}(\mathbf{R})$.

4. It may be interesting to study topological modal logics with the graded difference modalities $[\neq]_n A$ with the following semantics: $x \models [\neq]_n A$ iff there are at least n points $y \neq x$ such that $y \models A$.

5. The papers [32] and [21] prove completeness and strong completeness of $\mathbf{S4}$ w.r.t. any dense-in-itself metric space. The corresponding result for d-logics is completeness of $\mathbf{D4}$ w.r.t. an arbitrary dense-in-itself separable metric space. Is separabilty essential here? Does strong completeness hold in this case? Similar questions make sense for dd-logics.

6. [16] presents a 2-modal formula cd-valid exactly in T_0-spaces. However, the cd-logic (and the dd-logic) of the class of T_0-spaces is still unknown. Note that the d-logic of this class has been axiomatized in [7]; probably the same technique is applicable to cd- and dd-logics.

7. In footnote 7 we have mentioned that there is a gap in the paper [38]. Still we can prove that for any connected, locally connected metric space $\mathfrak{X}$ such that the boundary of any ball is nowhere dense, $\mathbf{Lc}_\forall(\mathfrak{X}) = \mathbf{S4U}+AC$. But for an arbitrary connected metric space $\mathfrak{X}$ we do not even know if $\mathbf{Lc}_\forall(\mathfrak{X})$ is finitely axiomatizable.

8. Is it possible to characterize finitely axiomatizable dd-logics that are complete w.r.t. Hausdorff spaces? metric spaces? Does there exist a dd-logic complete w.r.t. Hausdorff spaces, but incomplete w.r.t. metric spaces?

9. Suppose we have a c-complete modal logic L, and let $\mathcal{K}$ be the class of all topological spaces where L is valid. Is it always true that $\mathbf{Lc}_\forall(\mathcal{K}) = LU$? and $\mathbf{Lc}_{\neq}(\mathcal{K}) = LD$? Similar questions can be formulated for d-complete modal logics and their du- and dd-extensions.

10. An interesting topic not addressed in this chapter is the complexity of topomodal logics. In particular, the complexity is unknown for the d-logic (and the dd-logic) of $\mathbf{R}^n$ $(n > 1)$.

We would like to thank anonymous referee who helped us improve the first version of the manuscript.

The work on this chapter was supported by RFBR grants 11-01-00281-a, 11-01-00958-a, 11-01-93107-CNRS-a and the Russian President's grant NSh-5593.2012.1.

References

[1] M. A. ABASHIDZE. Ordinally complete normal extensions of the logic of provability. In: Logic, Methodology and Philosophy of Science, Abstracts, v.5, part 1, pp. 9-10. Moscow (1987).

[2] M. A. ABASHIDZE. Ordinal completeness of modal Gödel - Löb system (in Russian). In: Intensional logics and logical structure of theories (Telavi, 1985), pp. 49-73. Tbilisi (1988).

[3] P. S. ALEKSANDROV. Introduction to set theory and general topology (in Russian). Nauka, Moscow (1977).

[4] C. E. AULL, W. J. THRON. Separation axioms between T_0 and T_1. Indag. Math 24, pp. 26-37, (1962).

[5] G. BEZHANISHVILI, L. ESAKIA, D. GABELAIA. Some results on modal axiomatization and definability for topological spaces. Studia Logica, v. 81, 325-355 (2005).

[6] G. BEZHANISHVILI, L. ESAKIA, D. GABELAIA. The modal logic of Stone spaces: diamond as derivative. Review of Symbolic Logic, v.3, 26-40 (2010).

[7] G. BEZHANISHVILI, L. ESAKIA, D. GABELAIA. Spectral and T_0-spaces in d-semantics. In: Lecture Notes in Artificial Intelligence. N. Bezhanishvili, S. Löbner, K. Schwabe, and L. Spada, eds, pp. 16–29. Springer (2011).

[8] G. BEZHANISHVILI, J. LUCERO-BRYAN. More on d-logics of subspaces of the rational numbers. Notre Dame Journal of Formal Logic, v. 53, 319-345 (2012).

[9] A. BLASS. Infinitary combinatorics and modal logic. Journal of Symbolic Logic, v. 55, 761-778 (1990).

[10] J. BURGESS, YU. GUREVICH. The decision problem for linear temporal logic. Notre Dame Journal of Formal Logic, v. 26, 115-128 (1985).

[11] A. CHAGROV, M. ZAKHARYASCHEV. Modal logic. Oxford University Press, 1997.

[12] M. DE RIJKE. Extending modal logic. Ph. D. Thesis. ILLC Dissertation Series, Amsterdam, 1993.

[13] L. ESAKIA. Diagonal constructions, Löb formula, and scattered Cantor spaces (in Russian). In: Logical and semantical investigations, pp. 128-143, Tbilisi (1981).

[14] L. ESAKIA. Weak transitivity — a restitution (in Russian). In: Logical investigations, v. 8, pp. 244-245. Nauka, Moscow (2001).

[15] K. FINE. Logics containing K4, Part II. Journal of Symbolic Logic, v.50 , 619-651 (1985).

[16] D. GABELAIA. Modal definability in topology. Master's thesis, ILLC, 2001.

[17] G. GARGOV, V. GORANKO. Modal logic with names. In: Colloquium on Modal Logic, pp. 81-103. Dutch Network for Language, Logic and Information, Amsterdam (1991).

[18] V. GORANKO, S. PASSY. Using the universal modality: gains and questions. Journal of Logic and Computation, v. 2, pp. 5-30 (1992).

[19] M. AIELLO, I. PRATT-HARTMANN, J. VAN BENTHEM (Eds.) Handbook of Spatial Logics (Springer, 2007).

[20] I. HODKINSON On the Priorean temporal logic with 'around now' over the real line. Submitted, 2012.

[21] P. KREMER. Strong completeness of S4 for any dense-in-itself metric space. Review of Symbolic Logic, forthcoming.

[22] A. KUDINOV. Difference modality in topological spaces. Algebraic and Topological Methods in Non-classical Logics II, Barcelona, abstracts, pp.50-51, 2005.

[23] A. KUDINOV. Topological modal logics with difference modality. In: Advances in Modal Logic, v.6, pp. 319-332. College Publications, London (2006).

[24] A. KUDINOV. Topological modal logic of R with inequality. Russian Mathematical Surveys, v. 63, pp. 163-165 (2008).

[25] A. KUDINOV. Modal logic of R with the difference modality (in Russian). In: Proceedings of the Conference on Information Technologies and Systems, pp. 335-339. Moscow, 2011.

[26] A. KUDINOV, I. SHAPIROVSKY. Some examples of modal logics without a finite axiomatisation (in Russian). In: Proceedings of the conference on Information Technologies and Systems, pp. 258-262. Moscow, 2010.

[27] C. KURATOWSKI. Sur l'opération $\bar{A}$ de l'Analysis Situs. Fundamenta Mathematicae, v. 3, 181-199 (1922).

[28] K. KURATOWSKI. Topology, v. 1. Academic Press (1966).

[29] T. LITAK. Isomorphism via translation. In: Advances in Modal Logic, v. 6, pp. 333–351. College Publications (2006).

[30] J. LUCERO-BRYAN. The d-Logic of the rational numbers: a fruitful construction. Studia Logica, v. 97, 265-295 (2011).

[31] J. LUCERO-BRYAN. The d-logic of the real line. Journal of Logic and Computation, 2011, Oxford University Press, doi: 10.1093/logcom/exr054.

[32] J. C. C. MCKINSEY, A. TARSKI. The algebra of topology. Annals of Mathematics, v. 45, 141-191 (1944).

[33] H. RASIOWA, R. SIKORSKI. The mathematics of metamathematics. Warsaw, 1963.

[34] M. REYNOLDS. The complexity of temporal logic over the reals. Annals of Pure and Applied Logic, 2010. http://dx.doi.org/10.1016/j.apal.2010.01.002.

[35] K. SEGERBERG. An essay in classical modal logic. Filosofiska studier, Uppsala, 1971.

[36] K. SEGERBERG. A note on the logic of elsewhere. Theoria , v. 46, 183-187 (1980).

[37] V. SHEHTMAN. Derived sets in Euclidean spaces and modal logic. ITLI Prepublication Series, X-90-05, University of Amsterdam (1990).

[38] V. SHEHTMAN. "Everywhere" and "here". Journal of Applied Non-Classical Logics, v. 9, 369-380 (1999).

[39] V. SHEHTMAN. Modal logics of topological spaces (in Russian). Habilitation Thesis. Moscow (2000).

[40] V. SHEHTMAN. On neighbourhood semantics thirty years later. In: We Will Show Them! (Sergei N. Artemov et al., eds.) Essays in Honour of Dov Gabbay, v.2, pp.663-692. College Publications (2005).

[41] J. VAN BENTHEM. Modal logic and classical logic. Bibliopolis, Napoli (1983).

[42] M. ZAKHARYASCHEV. A sufficient condition for the finite model property of modal logics above K4. Bulletin of the IGPL, v.1, 13-21 (1993).