

A PROOF OF THE GAUDIN BETHE ANSATZ CONJECTURE.

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ABSTRACT. Gaudin algebra is the commutative subalgebra in $U(\mathfrak{g})^{\otimes N}$ generated by higher integrals of the quantum Gaudin magnet chain attached to a semisimple Lie algebra \mathfrak{g} . This algebra depends on a collection of pairwise distinct complex numbers z_1, \dots, z_N . We prove that this subalgebra has a cyclic vector in the space of singular vectors of the tensor product of any finite-dimensional irreducible \mathfrak{g} -modules, for all values of the parameters z_1, \dots, z_N . The corollary of this result is the Bethe Ansatz conjecture in the Feigin-Frenkel form which states that the joint eigenvalues of the higher Gaudin Hamiltonians on the tensor product of irreducible finite-dimensional \mathfrak{g} -modules are in 1-1 correspondence with monodromy-free opers on the projective line with regular singularities at the points z_1, \dots, z_N, ∞ and prescribed residues at singular points.

1. INTRODUCTION

1.1. Gaudin model. The Gaudin model was introduced in [12] as a spin model related to the Lie algebra sl_2 , and generalized to the case of arbitrary semisimple Lie algebras in [13], 13.2.2. The generalized Gaudin model has the following algebraic interpretation. Let V_λ be an irreducible representation of a semisimple (reductive) Lie algebra \mathfrak{g} with the highest weight λ . For any collection of integral dominant weights $(\lambda) = \lambda_1, \dots, \lambda_n$, let $V_\lambda = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$. For any $x \in \mathfrak{g}$, consider the operator $x^{(i)} = 1 \otimes \dots \otimes 1 \otimes x \otimes 1 \otimes \dots \otimes 1$ (x stands at the i th place), acting on the space V_λ . Let $\{x_a\}$, $a = 1, \dots, \dim \mathfrak{g}$, be an orthonormal basis of \mathfrak{g} with respect to Killing form, and let $\underline{z} := (z_1, \dots, z_N)$ be a collection of pairwise distinct complex numbers. The Hamiltonians of the Gaudin model are the following commuting operators acting in the space V_λ :

$$(1) \quad H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}.$$

We can treat the H_i as elements of the universal enveloping algebra $U(\mathfrak{g})^{\otimes N}$. In [4], the existence of a large commutative subalgebra $\mathcal{A}(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$ containing H_i was proved with the help of the critical level phenomenon for the affine Lie algebra $\widehat{\mathfrak{g}}$. For $\mathfrak{g} = sl_2$, the algebra $\mathcal{A}(\underline{z})$ is generated by H_i and the central elements of $U(\mathfrak{g})^{\otimes N}$. In other cases, the algebra $\mathcal{A}(\underline{z})$ has also some new generators known as higher Gaudin Hamiltonians. We will call $\mathcal{A}(\underline{z})$ the *Gaudin algebra*.

1.2. Bethe Ansatz. The main problem in Gaudin model is the problem of simultaneous diagonalization of (higher) Gaudin Hamiltonians. It follows from the [4] construction that all elements of $\mathcal{A}(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$ are invariant with respect

to the diagonal action of \mathfrak{g} , and therefore it is sufficient to diagonalize the algebra $\mathcal{A}(\underline{z})$ in the subspace $V_\lambda^{sing} \subset V_\lambda$ of singular (highest) vectors with respect to $\text{diag}(\mathfrak{g})$ (i.e., with respect to the diagonal action of \mathfrak{g}). In many important cases, the Gaudin eigenproblem is solved by the *algebraic Bethe Ansatz* method which provides an explicit (but complicated) construction of joint eigenvectors for $\mathcal{A}(\underline{z})$ in V_λ^{sing} from the solutions of some explicit systems of algebraic equations, called Bethe Ansatz equations, on some auxiliary variables. The famous Bethe Ansatz conjecture states that this method always works, i.e. gives an eigenbasis for $\mathcal{A}(\underline{z})$ in V_λ^{sing} . In particular, the conjecture says that, for generic \underline{z} , the algebra $\mathcal{A}(\underline{z})$ has simple spectrum in V_λ^{sing} and there is a 1-1 correspondence between the eigenvectors and the solutions of the corresponding system of Bethe Ansatz equations. The latter was proved in [?] for $\mathfrak{g} = \mathfrak{sl}_n$. More precisely, it is proved that the space V_λ^{sing} is always cyclic as $\mathcal{A}(\underline{z})$ -module, and hence $\mathcal{A}(\underline{z})$ has simple spectrum whenever acts by semisimple operators.

1.3. Opers and the eigenproblem for the Gaudin model. In [4], the Bethe Ansatz equations were interpreted as a “no-monodromy” condition on certain space of *opers* on the projective line \mathbb{P}^1 . Namely, it was shown that $\mathcal{A}(\underline{z})$ is isomorphic to the algebra of functions on a certain space of ${}^L G$ -opers on the projective line \mathbb{P}^1 . More precisely, this is the space of ${}^L G$ -opers on \mathbb{P}^1 with regular singularities at the points z_1, \dots, z_N and ∞ . Here ${}^L G$ is the *Langlands dual group* of G (${}^L G$ is taken to be of adjoint type), and ${}^L G$ -opers are connections on a principal ${}^L G$ -bundle over \mathbb{P}^1 satisfying a certain transversality condition, as defined in [1]. The appearance of the Langlands dual group is not accidental, but is closely related to the geometric Langlands correspondence, through a description of the center of the completed enveloping algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level in terms of ${}^L G$ -opers on the punctured disc [3, 8, 10].

Thus, we obtain that the spectra of $\mathcal{A}(\underline{z})$ on a tensor product of \mathfrak{g} -modules $M_1 \otimes \dots \otimes M_N$ are encoded by ${}^L G$ -opers on \mathbb{P}^1 satisfying the above properties. Furthermore, in [4] it was shown that if each M_i is V_{λ_i} , the irreducible finite-dimensional \mathfrak{g} -module with dominant integral highest weight λ_i , then these ${}^L G$ -opers satisfy two additional properties: they have fixed residues at the points z_i , determined by λ_i (we denote the space of such opers by $\text{Op}_{{}^L G}(\mathbb{P}^1)_{\underline{z}}^\lambda$, it is an affine space), and they have *trivial monodromy*. The latter is a finite number of polynomial conditions on the coefficients of opers which is generically equivalent to Bethe Ansatz equations.

1.4. Bethe ansatz conjecture. The conjecture of completeness of Bethe Ansatz was reformulated in [4] as there is a bijection between the joint eigenvalues of $\mathcal{A}(\underline{z})$ on V_λ and the set of monodromy-free opers from $\text{Op}_{{}^L G}(\mathbb{P}^1)_{\underline{z}}^\lambda$. In this paper we prove this conjecture. In fact, we prove the following statement:

Main Theorem. *The space of singular vectors in the tensor product of \mathfrak{g} -modules V_λ^{sing} is cyclic as an $\mathcal{A}(\underline{z})$ -module. The annihilator of V_λ^{sing} in $\mathcal{A}(\underline{z})$ is generated by the no-monodromy conditions on opers from $\text{Op}_{{}^L G}(\mathbb{P}^1)_{\underline{z}}^\lambda$.*

It was shown in [5] that for real values of the parameters z_1, \dots, z_N , the algebra $\mathcal{A}(\underline{z})$ acts on $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N}$ by Hermitian operators and hence is diagonalizable.

Main Theorem then implies that the spectrum of $\mathcal{A}(\underline{z})$ on $V_{\underline{\lambda}}^{sing}$ is simple. Since the property of having simple spectrum is an open condition on the parameters z_1, \dots, z_N we have the following

Main Corollary. *For generic z_1, \dots, z_N and any dominant integral $\lambda_1, \dots, \lambda_N$, the Gaudin subalgebra $\mathcal{A}(z_1, \dots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ is diagonalizable and has simple spectrum on space of singular vectors in the tensor product irreducible \mathfrak{g} -modules $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N}$. Moreover, its joint eigenvalues (and hence eigenvectors, up to a scalar) are in one-to-one correspondence with monodromy-free opers from $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^{\lambda}$.*

The main idea of our proof is to use the *inhomogeneous* Gaudin algebra $\mathcal{A}_{\mu}(\underline{z})$ (or Gaudin algebra with irregular singularities from [6]). This algebra depends on an additional parameter $\mu \in \mathfrak{g} = \mathfrak{g}^*$ and for regular μ is a maximal commutative subalgebra in $U(\mathfrak{g})^{\otimes N}$. On the one hand, the analog of Bethe Ansatz conjecture for non-homogeneous Gaudin algebras turns to be easier than for homogeneous ones. Namely, by the results of [5], for any regular $\mu \in \mathfrak{g}$ and any collection of pairwise distinct complex numbers $\underline{z} = (z_1, \dots, z_N)$ the algebra $\mathcal{A}_{\mu}(\underline{z})$ has a cyclic vector in any tensor product of irreducible finite-dimensional \mathfrak{g} -modules, and the annihilator is generated by no-monodromy conditions. On the other hand, in the case when $\mu = f$, a regular *nilpotent* element of $\mathfrak{g}^* \simeq \mathfrak{g}$, the algebra $\mathcal{A}_f(\underline{z})$ has a decreasing filtration such that the 0-degree component of the associated (negatively) graded algebra is $\mathcal{A}(\underline{z})$, and it is possible to check that passing to the associated graded does not destroy the cyclicity property.

1.5. Covering of $\overline{M_{0,N+1}}$ and relation to crystals. The family $\mathcal{A}(\underline{z})$, as defined, is parameterized by a noncompact complex algebraic variety of configurations of pairwise distinct points on the complex line. On the other hand, every subalgebra is (in appropriate sense) a point of some Grassmann variety which is compact. Hence there is a family of commutative subalgebras which extends the family $\mathcal{A}(\underline{z})$ and is parameterized by some compact variety. In [18], we show that the closure of the family $\mathcal{A}(\underline{z})$ is parameterized by the Deligne-Mumford compactification $\overline{M_{0,N+1}}$ of the moduli space of stable rational curves with $N + 1$ marked points. Moreover, we show that the natural topological operad structure on $\overline{M_{0,n+1}}$ is compatible with that on commutative subalgebras of $U(\mathfrak{g})^{\otimes N}$ and describe explicitly the algebras corresponding to boundary points of $\overline{M_{0,N+1}}$. In section 4 we deduce from the Main Theorem that the subalgebras corresponding to boundary points of $\overline{M_{0,N+1}}$ have a cyclic vector in $V_{\underline{\lambda}}^{sing}$ as well. We deduce from this the simple spectrum property for the subalgebras attached to *all real points* of $\overline{M_{0,N+1}}$.

This allows us to regard the eigenbasis (or, more precisely, the set of 1-dimensional eigenspaces) of $\mathcal{A}(\underline{z})$ in $V_{\underline{\lambda}}^{sing}$ as a *covering* of the space $\overline{M_{0,N+1}}(\mathbb{R})$. Denote the fiber of this covering at a point $\underline{z} \in \overline{M_{0,n+1}}(\mathbb{R})$ by $B_{\underline{\lambda}}(\underline{z})^{sing}$. The fundamental group of $\overline{M_{0,n+1}}(\mathbb{R})$ (called *pure cactus group* PJ_N) acts on this set. On the other hand, there is an action of the same group on the set $\mathcal{B}_{\underline{\lambda}}^{sing}$ of highest elements in the tensor product $\mathcal{B}_{\lambda_1} \otimes \dots \otimes \mathcal{B}_{\lambda_N}$ of the \mathfrak{g} -crystals with highest

weights $\lambda_1, \dots, \lambda_N$ from the coboundary category formalism. Note that $\mathcal{B}_\lambda^{sing}$ has the same cardinality as $B_\lambda(z)^{sing}$. In [18] we have stated the following

Monodromy Conjecture. (Pavel Etingof) *The actions of PJ_N on $B_\lambda(z)^{sing}$ and on $\mathcal{B}_\lambda^{sing}$ are isomorphic.*

The contribution of the present paper to this conjecture is that now the covering $B_\lambda(z)^{sing}$ is well defined for any \mathfrak{g} .

1.6. The paper is organized as follows. In section 2 we collect basic facts on Gaudin models andopers. In section 3 we prove the main result of the paper. In the last section 4 we prove the cyclicity and simple spectrum property for limiting Gaudin algebras.

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2. PRELIMINARIES

2.1. Gaudin algebras. Gaudin model was introduced in [12] as a spin model related to the Lie algebra \mathfrak{sl}_2 , and generalized to the case of an arbitrary semisimple Lie algebra in [13], Section 13.2.2. For any $x \in \mathfrak{g}$, set

$$x^{(i)} = 1 \otimes \dots \otimes 1 \otimes x \otimes 1 \otimes \dots \otimes 1 \in U(\mathfrak{g})^{\otimes N}$$

(x at the i th place). Let $\{x_a\}$, $a = 1, \dots, \dim \mathfrak{g}$, be an orthonormal basis of \mathfrak{g} with respect to Killing form, and let z_1, \dots, z_N be pairwise distinct complex numbers. The Hamiltonians of Gaudin model are the following mutually commuting elements of $U(\mathfrak{g})^{\otimes N}$:

$$(2) \quad H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}.$$

In [4], a large commutative subalgebra $\mathcal{A}(z)$ containing the H_i 's was constructed with the help of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$, which is the universal central extension of $\mathfrak{g}((t))$. Let us briefly describe the construction.

Consider the infinite-dimensional ind-nilpotent Lie algebra $\mathfrak{g}_- := \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ – it is a "half" of the corresponding affine Kac–Moody algebra $\widehat{\mathfrak{g}}$. The universal enveloping algebra $U(\mathfrak{g}_-)$ has a natural (PBW) filtration by the degree with respect to the generators. The associated graded algebra is the symmetric algebra $S(\mathfrak{g}_-)$ by the Poincaré–Birkhoff–Witt theorem.

There is a natural grading on the associative algebras $S(\mathfrak{g}_-)$ and $U(\mathfrak{g}_-)$ determined by the derivation L_0 defined by

$$(3) \quad L_0(g \otimes t^m) = mg \otimes t^m \quad \forall g \in \mathfrak{g}, m = -1, -2, \dots$$

There is also a derivation L_{-1} of degree -1 with respect to this grading:

$$(4) \quad L_{-1}(g \otimes t^m) = mg \otimes t^{m-1} \quad \forall g \in \mathfrak{g}, m = -1, -2, \dots$$

Let $i_{-1} : S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_-)$ be the embedding, which maps $g \in \mathfrak{g}$ to $g \otimes t^{-1}$. The algebra of invariants, $S(\mathfrak{g})^{\mathfrak{g}}$, is known to be a free commutative algebra with $\text{rk } \mathfrak{g}$ generators. Let Φ_j , $j = 1, \dots, \ell = \text{rk } \mathfrak{g}$ be some set of free generators of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$. The degrees of Φ_j are $d_j + 1$ where d_j are the exponents of \mathfrak{g} . The following result is due to Boris Feigin and Edward Frenkel, see [8] and references therein.

Theorem 2.2. *There exist commuting elements $S_j \in U(\mathfrak{g}_-)$, homogeneous with respect to L_0 , such that $\text{gr } S_j = i_{-1}(\Phi_j)$. Moreover, the elements $L_{-1}^k S_j$ pairwise commute for all $k \in \mathbb{Z}_+$ and $j = 1, \dots, \ell$.*

Let $U(\mathfrak{g})^{\otimes N}$ be the tensor product of N copies of $U(\mathfrak{g})$. We denote the subspace $1 \otimes \dots \otimes 1 \otimes \mathfrak{g} \otimes 1 \otimes \dots \otimes 1 \subset U(\mathfrak{g})^{\otimes N}$, where \mathfrak{g} stands at the i th place, by $\mathfrak{g}^{(i)}$. Respectively, for any $x \in U(\mathfrak{g})$ we set

$$(5) \quad x^{(i)} = 1 \otimes \dots \otimes 1 \otimes x \otimes 1 \otimes \dots \otimes 1 \in U(\mathfrak{g})^{\otimes N}.$$

Let $\text{diag} : U(\mathfrak{g}_-) \hookrightarrow U(\mathfrak{g}_-)^{\otimes N}$ be the diagonal embedding (i.e. for $x \in \mathfrak{g}_-$, we have $\text{diag}(x) = \sum_{i=1}^n x^{(i)}$). To any nonzero $w \in \mathbb{C}$, we assign the homomorphism $\varphi_w : U(\mathfrak{g}_-) \rightarrow U(\mathfrak{g})$ of evaluation at the point w (i.e., for $g \in \mathfrak{g}$, we have $\varphi_w(g \otimes t^m) = w^m g$). For any collection of pairwise distinct nonzero complex numbers $w_i, i = 1, \dots, n$, we have the following homomorphism:

$$(6) \quad \varphi_{w_1, \dots, w_N} = (\varphi_{w_1} \otimes \dots \otimes \varphi_{w_N}) \circ \text{diag} : U(\mathfrak{g}_-) \rightarrow U(\mathfrak{g})^{\otimes N}.$$

More explicitly, we have

$$\varphi_{w_1, \dots, w_N}(g \otimes t^m) = \sum_{i=1}^n w_i^m g^{(i)}.$$

Consider the following $U(\mathfrak{g})^{\otimes N}$ -valued functions in the variable w

$$S_j(w; z_1, \dots, z_N) := \varphi_{w-z_1, \dots, w-z_N}(S_j).$$

We define the Gaudin subalgebra $\mathcal{A}(z) \subset U(\mathfrak{g})^{\otimes N}$ as a subalgebra generated by $S_j(w; z_1, \dots, z_N)$ for all $w \in \mathbb{C} \setminus \{z_1, \dots, z_N\}$. Due to Theorem 2.2, this subalgebra is commutative. The subalgebra $\mathcal{A}(z) \subset U(\mathfrak{g})^{\otimes N}$ is also known as *Bethe algebra*.

Let $S_j^{i,m}(z_1, \dots, z_N)$ be the coefficients of the principal part of the Laurent series of $S_j(w; z_1, \dots, z_N)$ at the point z_i , i.e.,

$$S_j(w; z_1, \dots, z_N) = \sum_{m=1}^{m=\text{deg } \Phi_j} S_j^{i,m}(z_1, \dots, z_N)(w - z_i)^{-m} + O(1) \text{ as } w \rightarrow z_i.$$

Taking the generator S_j corresponding to the quadratic Casimir element on $S(\mathfrak{g})$, one gets the quadratic Gaudin Hamiltonians (2) as the residues of $S_j(w; z_1, \dots, z_N)$ at the points z_1, \dots, z_N . The following result is well-known (see e.g. [2] for the proof).

Proposition 2.3. [2]

- (1) *The elements $S_j^{i,m}(z_1, \dots, z_N) \in U(\mathfrak{g})^{\otimes N}$ are homogeneous under simultaneous affine transformations of the parameters $z_i \mapsto az_i + b$ (i.e. $S_j^{i,m}(az_1 + b, \dots, az_N + b)$ is proportional to $S_j^{i,m}(z_1, \dots, z_N)$).*

- (2) The subalgebra $\mathcal{A}(z)$ is a free commutative algebra generated by the elements $S_j^{i,m}(z_1, \dots, z_N) \in U(\mathfrak{g})^{\otimes N}$, where $i = 1, \dots, n-1$, $l = 1, \dots, \text{rk } \mathfrak{g}$, $m = 1, \dots, \text{deg } \Phi_j$, and by $S_j^{n, \text{deg } \Phi_j}(z_1, \dots, z_N) \in U(\mathfrak{g})^{\otimes n}$, where $l = 1, \dots, \text{rk } \mathfrak{g}$.
- (3) All the elements of $\mathcal{A}(z)$ are invariant with respect to the diagonal action of \mathfrak{g} .
- (4) The center of the diagonal $\text{diag}(U(\mathfrak{g})) \subset U(\mathfrak{g})^{\otimes N}$ is contained in $\mathcal{A}(z)$.

It is easy to see that one can replace $S_j^{n, d_j+1}(z_1, \dots, z_n)$ in (2) by the generators of the center of $\text{diag}(U(\mathfrak{g}))$.

2.4. Inhomogeneous Gaudin algebras. In [17] and [6], the construction of [4] was generalized. Namely, for any collection z_1, \dots, z_N and $\mu \in \mathfrak{g}^*$, there is a commutative subalgebra $\mathcal{A}_\mu(z)$ in $U(\mathfrak{g})^{\otimes n}$ which is invariant with respect to the diagonal action of the centralizer of μ in \mathfrak{g} and is maximal with this property. In particular, $\mathcal{A}(z) = \mathcal{A}_0(z)$ corresponding to $\mu = 0$. Let us describe the generators of this algebra analogously to Proposition 2.3.

To any $\mu \in \mathfrak{g} = \mathfrak{g}^*$, we assign the character $\psi_\mu : U(\mathfrak{g}_-) \rightarrow \mathbb{C}$ such that for $g \in \mathfrak{g}$, we have $\psi_\mu(g \otimes t^m) = \delta_{-1, m} \mu(g)$. For any collection of pairwise distinct nonzero complex numbers $w_i, i = 1, \dots, N$, we can twist the homomorphism $\varphi_{w_1, \dots, w_N}$ by μ :

$$(7) \quad \varphi_{w_1, \dots, w_N; \mu} := (\varphi_{w_1} \otimes \dots \otimes \varphi_{w_N} \otimes \psi_\mu) \circ \text{diag} : U(\mathfrak{g}_-) \rightarrow U(\mathfrak{g}_-)^{\otimes (N+1)} \rightarrow U(\mathfrak{g})^{\otimes N} \otimes \mathbb{C} = U(\mathfrak{g})^{\otimes N}.$$

More explicitly, we have

$$\varphi_{w_1, \dots, w_N; \mu}(g \otimes t^m) = \delta_{-1, m} \mu(g) + \sum_{i=1}^N w_i^m g^{(i)}.$$

Consider the following $U(\mathfrak{g})^{\otimes N}$ -valued functions in the variable w

$$S_j(w; z_1, \dots, z_N; \mu) := \varphi_{w-z_1, \dots, w-z_N; \mu}(S_j).$$

We define the non-homogeneous Gaudin subalgebra $\mathcal{A}_\mu(z) \subset U(\mathfrak{g})^{\otimes N}$ as a commutative subalgebra generated by $S_j(w; z_1, \dots, z_N; \mu)$ for all $w \in \mathbb{C} \setminus \{z_1, \dots, z_N\}$.

Let $S_j^{i,m}(z_1, \dots, z_N; \mu)$ be the coefficients of the principal part of the Laurent series of $S_j(w; z_1, \dots, z_N; \mu)$ at the point z_i , i.e.,

$$S_j(w; z_1, \dots, z_N; \mu) = \sum_{m=1}^{m=\text{deg } \Phi_j} S_j^{i,m}(z_1, \dots, z_N; \mu)(w - z_i)^{-m} + O(1) \text{ as } w \rightarrow z_i.$$

Taking the generator S_j corresponding to the quadratic Casimir element on $S(\mathfrak{g})$, one gets the non-homogeneous quadratic Gaudin Hamiltonians as the residues of $S_j(w; z_1, \dots, z_N)$ at the points z_1, \dots, z_N .

$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k} + \sum_{a=1}^{\dim \mathfrak{g}} \mu(x_a) x_a^{(i)}.$$

From [5] we have the following description of the generators.

- Proposition 2.5.** (1) The elements $S_j^{i,m}(z_1, \dots, z_N; \mu) \in U(\mathfrak{g})^{\otimes N}$ are homogeneous under simultaneous affine transformations of the parameters $z_i \mapsto az_i + b$ (i.e. $S_j^{i,m}(az_1 + b, \dots, az_N + b, a^{-1}\mu)$ is proportional to $S_j^{i,m}(z_1, \dots, z_N; \mu)$).
- (2) The subalgebra $\mathcal{A}_\mu(\underline{z})$ is a free commutative algebra generated by the elements $S_j^{i,m}(z_1, \dots, z_N; \mu) \in U(\mathfrak{g})^{\otimes N}$, where $i = 1, \dots, n$, $j = 1, \dots, \ell$, $m = 1, \dots, d_j + 1$.

Denote by $S_j^{\infty,m}(z_1, \dots, z_N; \mu) \in U(\mathfrak{g})^{\otimes N}$ the coefficients of the principal part of the Laurent series of $S_j(w; z_1, \dots, z_N; \mu)w^{d_j+1}$ at the point ∞ .

Corollary 2.6. One can take as free generators of $\mathcal{A}_\mu(\underline{z})$ the elements $S_j^{i,m}(z_1, \dots, z_N; \mu)$ with $i = 1, \dots, n - 1$, $j = 1, \dots, \ell$, $m = 1, \dots, d_j + 1$ AND the elements $S_j^{\infty,m}(z_1, \dots, z_N; \mu)$ with $j = 1, \dots, \ell$, $m = 1, \dots, d_j + 1$.

The following fact on the limit points of the family $\mathcal{A}_\mu(\underline{z})$ is well known, but missed in the literature.

Proposition 2.7. We have

$$\lim_{s \rightarrow 0} \mathcal{A}_\mu(sz_1, \dots, sz_N) = \lim_{s \rightarrow 0} \mathcal{A}_{s\mu}(\underline{z}) = \text{diag}(\mathcal{A}_\mu) \cdot \mathcal{A}(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$$

for any regular $\mu \in \mathfrak{g}$. More precisely, both $\text{diag}(\mathcal{A}_\mu)$ and $\mathcal{A}(\underline{z})$ contain the center Z of the diagonal $U(\mathfrak{g})$ in $U(\mathfrak{g})^{\otimes N}$, and the product $\text{diag}(\mathcal{A}_\mu) \cdot \mathcal{A}(\underline{z})$ is in fact the tensor product $\text{diag}(\mathcal{A}_\mu) \otimes_Z \mathcal{A}(\underline{z})$.

Proof. Indeed, the RHS has the same number of algebraically independent generators of the same degrees as that of $\mathcal{A}_\mu(z_1, \dots, z_N)$. So we just have to check that the limiting subalgebra contains both $\text{diag}(\mathcal{A}_\mu)$ and $\mathcal{A}(\underline{z})$. The first is generated by the limits of $S_j^{\infty,m}(z_1, \dots, z_N; \mu)$ and the second is generated by the limits of $S_j^{i,m}(z_1, \dots, z_N; \mu)$. \square

2.8. Opers on the projective line. Consider the Langlands dual Lie algebra ${}^L\mathfrak{g}$ whose Cartan matrix is the transpose of the Cartan matrix of \mathfrak{g} . By ${}^L G$ we denote the group of inner automorphisms of ${}^L\mathfrak{g}$. We fix a Cartan decomposition

$${}^L\mathfrak{g} = {}^L\mathfrak{n}_+ \oplus {}^L\mathfrak{h} \oplus {}^L\mathfrak{n}_-.$$

The Cartan subalgebra ${}^L\mathfrak{h}$ is naturally identified with \mathfrak{h}^* . We denote by ${}^L\Delta$, ${}^L\Delta_+$, and ${}^L\Pi$ the root system of ${}^L\mathfrak{g}$, the set of positive roots, and the set of simple roots, respectively. We denote by ${}^L\mathfrak{b}_+$ the Borel subalgebra ${}^L\mathfrak{n}_+ \oplus {}^L\mathfrak{h}$.

Set

$$p_{-1} = \sum_{\alpha^\vee \in \Pi^\vee} e_{-\alpha^\vee} \in {}^L\mathfrak{g}.$$

Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \in \mathfrak{h}^* = {}^L\mathfrak{h}.$$

The operator $\text{Ad } \rho$ defines the principal gradation on ${}^L\mathfrak{g}$, with respect to which we have a direct sum decomposition ${}^L\mathfrak{b}_+ = \bigoplus_{i \geq 0} {}^L\mathfrak{b}_i$. Let p_1 be the unique element of

degree 1 in ${}^L\mathfrak{n}_+$ such that $\{p_{-1}, 2\rho, p_1\}$ is an \mathfrak{sl}_2 -triple (note that p_{-1} has degree -1). Let

$$V_{\text{can}} = \bigoplus_{i=1}^{\ell} V_{\text{can},i}$$

be the space of $\text{Ad } p_1$ -invariants in ${}^L\mathfrak{n}_+$, decomposed according to the principal gradation. Here $V_{\text{can},i}$ has degree d_i , the i th exponent of ${}^L\mathfrak{g}$ (and of \mathfrak{g}). In particular, $V_{\text{can},1}$ is spanned by p_1 . Now choose a linear generator p_j of $V_{\text{can},j}$.

Consider the Kostant slice in ${}^L\mathfrak{g}$,

$${}^L\mathfrak{g}_{\text{can}} = \left\{ p_{-1} + \sum_{j=1}^{\ell} y_j p_j, \quad y \in \mathbb{C} \right\}.$$

By [14], the adjoint orbit of any regular element in the Lie algebra ${}^L\mathfrak{g}$ contains a unique element which belongs to ${}^L\mathfrak{g}_{\text{can}}$. Thus, we have canonical isomorphisms

$$(8) \quad {}^L\mathfrak{g}_{\text{can}} \xrightarrow{\sim} {}^L\mathfrak{g}/{}^L G = {}^L\mathfrak{h}/{}^L W = \mathfrak{h}^*/W = \mathfrak{g}^*/G.$$

Denote by π the factorization map ${}^L\mathfrak{g} \rightarrow {}^L\mathfrak{g}/{}^L G = {}^L\mathfrak{g}_{\text{can}}$. For an affine curve $U = \text{Spec } R$ and an étale coordinate t on U , the space $\text{Op}_{{}^L G}(U)$ of ${}^L G$ -opers on U is isomorphic to the quotient of the space of connections of the form

$$d + (p_{-1} + \mathbf{v}(t))dt, \quad \mathbf{v}(t) \in {}^L\mathfrak{b}_+ \otimes R$$

by the free action of the group ${}^L N_+ \otimes R$ of regular algebraic maps $U \rightarrow {}^L N_+$, where ${}^L N_+$ is the maximal unipotent subgroup of ${}^L G$ such that $\text{Lie } {}^L N_+ = {}^L\mathfrak{n}_+$. Each oper has a unique representative of the form

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} v_j(t) p_j \right) dt, \quad v_j(t) \in R.$$

In particular, the space $\text{Op}_{{}^L G}(D^\times)$ of ${}^L G$ -opers on the formal punctured disc $D^\times = \text{Spec } \mathbb{C}((t))$ is isomorphic to the space of connections of the form

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} v_j(t) p_j \right) dt, \quad v_j(t) \in \mathbb{C}((t)).$$

We will consider opers on \mathbb{P}^1 with regular singularities at a finite number of points. The oper has regular singularity at $z \in \mathbb{P}^1$ if has the following form in local coordinate t at z :

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} (v_j(t-z)^{-(d_j+1)} + o((t-z)^{-(d_j+1)})) p_j \right) dt.$$

The residue of such oper at the point z is $\pi(p_{-1} - \rho + \sum_{j=1}^{\ell} v_j p_j) \in {}^L\mathfrak{g}/{}^L G$.

We denote by $\text{Op}_{{}^L G}(\underline{z})$ the space of ${}^L G$ -opers on $\mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$ with regular singularities at the points $z_i, i = 1, \dots, N$, and at ∞ . Each oper from this

space may be uniquely represented in the following form:

$$d + \left(p_{-1} + \sum_{i=1}^N \sum_{j=1}^{\ell} \sum_{n=0}^{d_j} v_{j,n}^{(i)} (t - z_i)^{-n-1} p_j \right) dt,$$

where

$$(9) \quad \sum_{i=1}^N \sum_{n=0}^m v_{j,n}^{(i)} \binom{m}{n} z_i^{m-n} = 0$$

for any $j = 1, \dots, \ell$ and $m = 0, \dots, d_j - 1$. Thus, $\text{Op}_{LG}(\underline{z})$ is an affine space of dimension $\frac{1}{2}(\dim \mathfrak{g} + \text{rk } \mathfrak{g})(N - 1) + \text{rk } \mathfrak{g}$.

Following [6] and [5], we denote by $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-\mu)}$ the space of LG -opers on $\mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$ with regular singularities at the points $z_i, i = 1, \dots, N$, and with irregular singularity of order 2 at the point ∞ with the 2-residue $\pi(-\mu) \in {}^L\mathfrak{g}/{}^L G = \mathfrak{g}^*/G$, where $\pi: \mathfrak{g}^* \rightarrow \mathfrak{g}^*/G$ is the projection. Each oper from this space may be uniquely represented in the following form:

$$d + \left(p_{-1} + \sum_{j=1}^{\ell} \bar{\mu}_j p_j + \sum_{i=1}^N \sum_{j=1}^{\ell} \sum_{n=0}^{d_j} u_{j,n}^{(i)} (t - z_i)^{-n-1} p_j \right) dt,$$

where

$$p_{-1} + \sum_{j=1}^{\ell} \bar{\mu}_j p_j$$

is the (unique) element of ${}^L\mathfrak{g}_{\text{can}}$ contained in the LG -orbit corresponding to the G -orbit of $\mu \in \mathfrak{g}^*$ under the isomorphism (8). Thus, $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-\mu)}$ is an affine space of dimension $\frac{1}{2}(\dim \mathfrak{g} + \text{rk } \mathfrak{g})N$.

We will be interested in the special case of $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-\mu)}$ when μ is the principal nilpotent element $f \in \mathfrak{g}$. Since $\pi(f) = 0$, the space $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-f)}$ is the space of opers with singularities as above but with *zero* 2-residue at $\infty \in \mathbb{P}^1$. Hence the space $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}$ is naturally a subspace in $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-f)}$. On the punctured disc D_{∞}^{\times} at ∞ (with the coordinate $s = t^{-1}$) each element of $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-f)}$ may be represented by a connection of the form (see [6, Section 5.4]):

$$d - \left(p_{-1} - \sum_{j=1}^{\ell} s^{-2d_j-1} u_j(s) p_j \right) ds, \quad u_j(s) = \sum_{n=0}^{\infty} u_{j,n} s^n.$$

The equations (9) which define subspace $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}$ inside $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-f)}$ can be rewritten as $u_{j,n} = 0$ for $j = 1, \dots, \ell, n = 0, \dots, d_j - 1$.

2.9. Gaudin algebra and opers. We have the following relation between Gaudin algebras and spaces of opers, see [4, 6, 5].

Proposition 2.10. *There is an isomorphism $\mathcal{A}(\underline{z}) \simeq \mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}]$. For regular μ there is an isomorphism $\mathcal{A}_{\mu}(\underline{z}) \simeq \mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-\mu)}]$. The generators $\Phi_1, \dots, \Phi_{\ell}$ of $S(\mathfrak{g})^{\mathfrak{g}}$ can be chosen such that these isomorphisms take $S_j(t; z_1, \dots, z_N)$ (resp. $S_j(t; z_1, \dots, z_N; \mu)$) to $\mathbf{v}_j(t)$.*

Next, it is proved in [6, Theorem, 5.7], that for any collection of \mathfrak{g} -modules M_i with highest weights λ_i , $i = 1, \dots, N$, the natural homomorphism

$$\mathcal{A}_\mu(\underline{z}) \rightarrow \text{End}(M_1 \otimes \cdots \otimes M_N)$$

factors through the algebra of functions on the subspace $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-\mu)}^\lambda \subset \text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-\mu)}$ which consists of the opers with the 1-residue $\pi(-\lambda_i - \rho)$ at z_i . Moreover, for integral dominant λ_i the action of $\mathcal{A}_\mu(\underline{z})$ on the tensor product of the finite-dimensional modules V_{λ_i} factors through the algebra of functions on *monodromy-free* opers from $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-\mu)}^\lambda$.

For every integral dominant weight λ , the set of monodromy-free opers on the punctured disc D_z^\times at z with the regular singularity with the residue $-\lambda - \rho$ at z is defined by finitely many polynomial relations. These polynomial relations are indexed by positive roots of \mathfrak{g} , see [11].

For any collection of integral dominant weights $\lambda_1, \dots, \lambda_N$ attached to the points z_1, \dots, z_N , we denote by $P_{z_k; \alpha}^{z; \mu; \lambda}$ the polynomial in $v_{j,n}^{(i)}$ expressing the "no-monodromy" condition at z_k corresponding to the root $\alpha \in \Delta_+$. We have the following result:

Theorem 2.11. [5] *For any N -tuple of pairwise distinct complex numbers $z_1, \dots, z_N \in \mathbb{C}$ and any regular μ the subalgebra $\mathcal{A}_\mu(z_1, \dots, z_N) \subset U(\mathfrak{g})^{\otimes N}$ has a cyclic vector in $V_{(\lambda_i)} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$. The annihilator of $V_{(\lambda_i)}$ in $\mathcal{A}_\mu(z_1, \dots, z_N)$ is generated by the no-monodromy conditions $P_{z_k; \alpha}^{z; \mu; \lambda}$. In particular, the joint eigenvalues of $\mathcal{A}_\mu(z_1, \dots, z_N)$ in V_λ (without multiplicities) are in one-to-one correspondence with the monodromy-free opers from $\text{Op}_{LG}(\mathbb{P}^1)_{(z_i); \pi(-\mu)}^{(\lambda_i)}$.*

Remark. This theorem can be regarded as an inhomogeneous version of the Bethe Ansatz conjecture. It turns to be much easier than the homogeneous one since the polynomials $P_{z_k; \alpha}^{z; \mu; \lambda}$ form a regular sequence in the polynomial ring $\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{(z_i); \pi(-\mu)}^{(\lambda_i)}]$ and hence one can check that the dimension of the quotient ring coincides with the dimension of V_λ . There is no analogous argument for the homogeneous Gaudin model.

The set of common zeroes of the restrictions $P_{z_k; \alpha}^{z; 0; \lambda}$ of the polynomials $P_{z_k; \alpha}^{z; f; \lambda}$ to the subspace $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}} \subset \text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}; \pi(-f)}^\lambda$ is the set of monodromy-free opers from $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda$.

3. MAIN RESULTS

3.1. Formulation of the Main Theorem.

Theorem 3.2. *The space of singular vectors in the tensor product of \mathfrak{g} -modules V_λ^{sing} is cyclic as an $\mathcal{A}(\underline{z})$ -module. The annihilator of V_λ^{sing} in $\mathcal{A}(\underline{z})$ is generated by the no-monodromy conditions $P_{z_k; \alpha}^{z; 0; \lambda}$ on opers from $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda$.*

This Theorem has an obvious corollary which is often referred to as Bethe ansatz conjecture (in the Feigin-Frenkel form):

Corollary 3.3. *There is a bijection between the set of joint eigenvalues of $\mathcal{A}(\underline{z})$ (without multiplicities) on $V_{\underline{\lambda}}^{sing}$ and the set of monodromy-free opers from $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^{\underline{\lambda}}$.*

Moreover, generically (e.g. for all real values of the parameters) there are no multiplicities.

Corollary 3.4. *For generic $z_1, \dots, z_N \in \mathbb{C}$ the subalgebra $\mathcal{A}(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$ has simple spectrum in $V_{\underline{\lambda}}^{sing} = (V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N})^{sing}$. Hence the joint eigenvectors for higher Gaudin Hamiltonians in $V_{\underline{\lambda}}^{sing}$ are in one-to-one correspondence with monodromy-free opers from $\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^{\underline{\lambda}}$. All real z_1, \dots, z_N are generic in this sense.*

Proof. It is proved in [5] that for real values of the parameters \underline{z} and μ , the algebra $\mathcal{A}_{\mu}(\underline{z})$ acts on $V_{\underline{\lambda}}$ by Hermitian operators hence diagonalizable. On the other hand, by the Main Theorem, $\mathcal{A}(\underline{z})$ has a cyclic vector in $V_{\underline{\lambda}}^{sing}$. Hence for real \underline{z} the algebra $\mathcal{A}(\underline{z})$ acts with simple spectrum on $V_{\underline{\lambda}}^{sing}$. The latter is a Zariski open condition on the parameters hence the assertion. \square

3.5. Proof of the Main Theorem. Let e, f, h be a principal \mathfrak{sl}_2 -triple in \mathfrak{g} such that $h \in \mathfrak{h}$ and $e \in \mathfrak{n}_+$. Then f is a regular element of \mathfrak{g} . Consider the subalgebra $\mathcal{A}_f(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$. The operator $\text{ad } \text{diag}(h)$ defines a grading on $U(\mathfrak{g})^{\otimes N} = \bigoplus_{k \in \mathbb{Z}} U(\mathfrak{g})_k^{\otimes N}$, where $U(\mathfrak{g})_k^{\otimes N} := \{x \in U(\mathfrak{g})^{\otimes N} \mid \text{diag}(h)x - x \text{diag}(h) = 2kx\}$. This grading induces an increasing filtration on $U(\mathfrak{g})^{\otimes N}$ such that $U(\mathfrak{g})^{(k)} := \bigoplus_{i \leq k} U(\mathfrak{g})_i^{\otimes N}$,

$k \in \mathbb{Z}$. Note that $\mathcal{A}_f(\underline{z}) \subset U(\mathfrak{g})^{(0)}$, hence we get a *bounded* increasing filtration on $\mathcal{A}_f(\underline{z})$. The associated graded $\text{gr } \mathcal{A}_f(\underline{z})$ is naturally a commutative subalgebra in $\text{gr } U(\mathfrak{g})^{\otimes N} = U(\mathfrak{g})^{\otimes N}$ and is clearly the same as the limit $\lim_{t \rightarrow 0} \mathcal{A}_{tf}(\underline{z})$. The 0-th graded component of this subalgebra is $\mathcal{A}(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$.

Consider the action of $\mathcal{A}_f(\underline{z})$ on $V_{\underline{\lambda}}$. By [5], $V_{\underline{\lambda}}$ is cyclic as a $\mathcal{A}_f(\underline{z})$ -module. The operator $\text{diag}(h)$ defines a grading on $V_{\underline{\lambda}}$ which agrees with the grading of $U(\mathfrak{g})^{\otimes N}$ defined above and with the decomposition into \mathfrak{g} -isotypic components with respect to the diagonal \mathfrak{g} -action $V_{\underline{\lambda}} = \bigoplus_{\nu} I_{\nu}$ (here $I_{\nu} = m_{\nu} V_{\nu}$ are isotypic components with respect to diagonal \mathfrak{g}). Clearly we have $V_{\underline{\lambda}}^{n+} = \bigoplus_{\nu} I_{\nu}^{top}$ (here the superscript *top* means the top degree part with respect to the $\text{diag}(h)$ -grading). Now we want to deduce the cyclicity of I_{ν}^{top} with respect to $\mathcal{A}(\underline{z})$ from the cyclicity of $V_{\underline{\lambda}}$ with respect to $\mathcal{A}_f(\underline{z})$. The main difficulty is that the decomposition $V_{\underline{\lambda}} = \bigoplus_{\nu} I_{\nu}$ does not agree with the action of $\mathcal{A}_f(\underline{z})$. However, the following holds (and it is sufficient for our purposes):

Lemma 3.6. *There is a decomposition $V_{\underline{\lambda}} = \bigoplus_{\nu} J_{\nu}$ such that it is preserved by $\mathcal{A}_f(\underline{z})$ and $\text{gr } J_{\nu} = I_{\nu}$.*

Proof. The isotypic components I_{ν} are the joint eigenspaces for the center Z of the diagonal $U(\mathfrak{g})$. Since the direct sum is finite $V_{\underline{\lambda}} = \bigoplus_{\nu} I_{\nu}$, it is in fact the decomposition into the eigenspaces for a single (sufficiently generic) element $C \in Z$. Since $Z \subset \mathcal{A}(\underline{z}) \subset \text{gr } \mathcal{A}_f(\underline{z})$, for any $C \in Z$ there exists $\tilde{C} \in \mathcal{A}_f(\underline{z})$

such that $\tilde{C} = C + N$ where $\deg N < 0$. Consider $\pi_\lambda(\tilde{C})$, the image of \tilde{C} in $\text{End}(V_\lambda)$. We have the Jordan decomposition $\pi_\lambda(\tilde{C}) = \pi_\lambda(\tilde{C})_s + \pi_\lambda(\tilde{C})_n$ where both $\pi_\lambda(\tilde{C})_s$ and $\pi_\lambda(\tilde{C})_n$ are polynomials of $\pi_\lambda(\tilde{C})$, hence lie in $\pi_\lambda(\mathcal{A}_f(\underline{z}))$. Since $\pi_\lambda(C)$ is semisimple and $\pi_\lambda(\tilde{C})_n$ is a nilpotent operator expressed as a polynomial of $\pi_\lambda(C + N)$, we have $\deg \pi_\lambda(\tilde{C})_n < 0$. Hence $\pi_\lambda(\tilde{C})_s = \pi_\lambda(C)$ modulo lower degree. Hence the projectors to the eigenspaces of $\pi_\lambda(\tilde{C})_s$ are the projectors to the eigenspaces of C modulo lower degree. Thus the desired decomposition of V_λ is the decomposition into the eigenspaces with respect to $\pi_\lambda(\tilde{C})_s$ for some generic $C \in Z$. \square

By [5], V_λ is cyclic as a $\mathcal{A}_f(\underline{z})$ -module. Hence each J_ν is cyclic with respect to $\mathcal{A}_f(\underline{z})$. In particular, the quotient $J_\nu^{(top)}/J_\nu^{(top-1)}$ is cyclic with respect to $\mathcal{A}_f(\underline{z})/\mathcal{A}_f(\underline{z})_{(-1)} = \mathcal{A}(\underline{z})$. On the other hand, according to Lemma 3.6 $J_\nu^{(top)}/J_\nu^{(top-1)}$ is isomorphic to $I_\nu^{top} = m_\nu V_n u^{n+}$ as $\mathcal{A}(\underline{z})$ -module. Hence the space of highest vectors of each diagonal \mathfrak{g} -isotypic component in V_λ is cyclic as $\mathcal{A}(\underline{z})$ -module, and hence the whole space V_λ^{sing} is a cyclic $\mathcal{A}(\underline{z})$ -module. The first assertion of the Theorem is proved.

To prove the second assertion of the Theorem, we note that the filtration on the coordinate ring $\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z};\pi(-f)}]$ defined by the isomorphism with $\mathcal{A}_f(\underline{z})$ has, according to Proposition 2.10, the following meaning in terms of opers: the degrees of the generators are $\deg u_{j,n} = -d_j + n$, $\deg v_{j,n}^{(i)} = 0$. Hence the 0-th component of $\text{gr } \mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z};\pi(-f)}]$ with respect to this filtration is $\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}]$. The same holds for the quotients by the ‘‘no-monodromy’’ relations: the 0-th component of $\text{gr } \mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z};\pi(-f)}^\lambda]$ is $\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda]$. The natural homomorphism from $\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda]$ to the image of $\mathcal{A}(\underline{z})$ in $\text{End}(V_\lambda^{sing})$ is surjective by definition. For the second assertion of the Theorem, it suffices to show the injectivity of this homomorphism.

Lemma 3.7. *Let $L \subset J_\nu$ be any lifting of $J_\nu^{(top)}/J_\nu^{(top-1)}$ to J_ν . Then we have $J_\nu = L + \mathcal{A}_f(\underline{z})_{(-1)}L$.*

Proof. It is sufficient to check that $I_\nu = I_\nu^{top} + \text{gr } \mathcal{A}_f(\underline{z})_{(-1)}I_\nu^{top}$, i.e. that the isotypic component of V_ν is generated from its highest weight space by the negatively graded part of $\text{gr } \mathcal{A}_f(\underline{z})$. We have $\text{gr } \mathcal{A}_f(\underline{z}) \subset \text{diag}(\mathcal{A}_f)$, the generators of \mathcal{A}_f are either central or have negative degree, and by the results of [5] each V_ν is generated by \mathcal{A}_f from its highest vector. \square

By Lemma 3.6, we have $\dim \pi_\lambda(\mathcal{A}_f(\underline{z}))/\pi_\lambda(\mathcal{A}_f(\underline{z})_{(-1)}) \leq \dim V_\lambda^{sing}$. On the other hand, $\pi_\lambda(\mathcal{A}_f(\underline{z})) = \mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z};\pi(-f)}^\lambda]$ by [5]. Hence $\dim \mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda] = \dim(\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda]/\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda]_{(-1)}) \leq \dim V_\lambda^{sing}$. On the other hand, by the first assertion of the Theorem, the dimension of the image of $\mathcal{A}(\underline{z})$ in $\text{End}(V_\lambda^{sing})$ is greater or equal to $\dim V_\lambda^{sing}$. Thus $\mathbb{C}[\text{Op}_{LG}(\mathbb{P}^1)_{\underline{z}}^\lambda]$ and the image of $\mathcal{A}(\underline{z})$ in $\text{End}(V_\lambda^{sing})$ have the same dimension and we are done. \square

4. LIMITING GAUDIN SUBALGEBRAS.

The family $\mathcal{A}(\underline{z})$, as defined, is parameterized by a noncompact complex algebraic variety of configurations of pairwise distinct points on the complex line. On the other hand, every subalgebra is (in appropriate sense) a point of some Grassmann variety which is compact. Hence there is a family of commutative subalgebras which extends the family $\mathcal{A}(\underline{z})$ and is parameterized by some compact variety. In [18], we show that the closure of the family $\mathcal{A}(\underline{z})$ is parameterized by the Deligne-Mumford compactification $\overline{M_{0,N+1}}$ of the moduli space of stable rational curves with $N + 1$ marked points. The commutative subalgebras corresponding to boundary points of $\overline{M_{0,N+1}}$ are described in [18] as follows.

For any partition of the set $\{1, 2, \dots, N\} = M_1 \cup \dots \cup M_k$, define the homomorphism

$$D_{M_1, \dots, M_k} : U(\mathfrak{g})^{\otimes k} \hookrightarrow U(\mathfrak{g})^{\otimes n},$$

taking $x^{(i)} \in U(\mathfrak{g})^{\otimes k}$, for $x \in \mathfrak{g}$, $i = 1, \dots, k$, to $\sum_{j \in M_i} x^{(j)}$.

For any subset $M = \{j_1, \dots, j_m\} \subset \{1, 2, \dots, N\}$, with $j_1 < \dots < j_m$, let $I_M : U(\mathfrak{g})^{\otimes m} \hookrightarrow U(\mathfrak{g})^{\otimes n}$ be the embedding of the tensor product of the copies of $U(\mathfrak{g})$ indexed by M , i.e. $I_M(x^{(i)}) := x^{(j_i)} \in U(\mathfrak{g})^{\otimes N}$ for any $x \in \mathfrak{g}$, $i = 1, \dots, m$. Clearly, all these homomorphisms are \mathfrak{g} -equivariant and every element in the image of D_{M_1, \dots, M_k} commutes with every element of $I_{M_i}([U(\mathfrak{g})^{\otimes m_i}]^{\mathfrak{g}})$ for $i = 1, \dots, k$. This gives us the following ‘‘substitution’’ homomorphism defining an operad structure on the spaces $[U(\mathfrak{g})^{\otimes n}]^{\mathfrak{g}}$

$$(10) \quad \gamma_{k; M_1, \dots, M_k} = D_{M_1, \dots, M_k} \otimes \bigotimes_{i=1}^k I_{M_i} : [U(\mathfrak{g})^{\otimes k}]^{\mathfrak{g}} \otimes \bigotimes_{i=1}^k [U(\mathfrak{g})^{\otimes m_i}]^{\mathfrak{g}} \xrightarrow{U(\mathfrak{g})^{\otimes N} \mathfrak{g}} .$$

Proposition 4.1. [18] *All subalgebras corresponding to boundary points of $\overline{M_{0,N+1}}$ are images of products of some $\mathcal{A}(\underline{z})$ under compositions of the homomorphisms γ .*

The proof of Theorem 3.18 of [18] shows that our Main Corollary implies that the action of $\mathcal{A}(\underline{z})$ for any $\underline{z} \in \overline{M_{0,N+1}}$ on $V_{\underline{\lambda}}^{sing}$ is cyclic. Hence $\mathcal{A}(\underline{z})$ has simple spectrum on $V_{\underline{\lambda}}^{sing}$ for any point $\underline{z} \in \overline{M_{0,N+1}(\mathbb{R})}$. We summarize this in the following

Proposition 4.2. *For any $\underline{z} \in \overline{M_{0,N+1}}$ the subalgebra $\mathcal{A}(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$ has a cyclic vector in $V_{\underline{\lambda}}^{sing} = (V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N})^{sing}$. For any $\underline{z} \in \overline{M_{0,n+1}(\mathbb{R})}$ the subalgebra $\mathcal{A}(\underline{z}) \subset U(\mathfrak{g})^{\otimes N}$ has simple spectrum in $V_{\underline{\lambda}}^{sing} = (V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N})^{sing}$.*

Corollary 4.3. *For any collection $\underline{\lambda}$ of dominant integral weights, the spectra of the algebras $\mathcal{A}(\underline{z})$ with real \underline{z} in the space $V_{\underline{\lambda}}^{sing}$ form a unbranched covering of $\overline{M_{0,n+1}(\mathbb{R})}$.*

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