Methodol Comput Appl Probab
DOI 10.1007/s11009-015-9461-8

Series Representations for Multivariate Time-Changed Lévy Models

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Received: 14 December 2014 / Revised: 2 August 2015 / Accepted: 14 August 2015

Abstract In this paper, we analyze a Lévy model based on two popular concepts - subordination and Lévy copulas. More precisely, we consider a two-dimensional Lévy process such that each component is a time-changed (subordinated) Brownian motion and the dependence between subordinators is described via some Lévy copula. The main result of this paper is the series representation for our model, which can be efficiently used for simulation purposes.

Keywords Lévy copula · Time-changed Lévy process · Subordination

Mathematics Subject Classification (2010) Primary 60G51 · Secondary 62F99

1 Introduction

Copula is with no doubt the most popular tool for describing the dependence between two random variables. The popularity is partially based on the fact that the dependence between any random variables can be modelled by some copula. This fact is known as Sklar's theorem, which states that for any two random variables Y_1 and Y_2 there exists a *copula* C (a two-dimensional real-valued distribution function with domain $[0, 1]^2$ and uniform margins) such that

$$\mathbb{P}\{Y_1 \leq u_1, Y_2 \leq u_2\} = C\Big(\mathbb{P}\{Y_1 \leq u_1\}, \mathbb{P}\{Y_2 \leq u_2\}\Big), \tag{1}$$

This study (research grant No 14-05-0007) was supported by the National Research University-Higher School of Economics' Academic Fund Program in 2014–2015.

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for any $u_1, u_2 \geq 0$. We refer to Cherubini et al. (2004), Joe (1997), Nelsen (2006) for a comprehensive overview of the copula theory.

Now let us switch from random variables to stochastic processes and try to describe dependence between components of some two-dimensional *Lévy process* $\vec{X}(t) = (X_1(t), X_2(t))$, that is, of some cadlag process with independent and stationary increments. Applying Sklar's theorem for any fixed time moment t, we get that the dependence between X_1 and X_2 can be described by some copula C_t, i.e.,

$$\mathbb{P}\{X_1(t) \leq u_1, X_2(t) \leq u_2\} = C_t\Big(\mathbb{P}\{X_1(t) \leq u_1\}, \mathbb{P}\{X_2(t) \leq u_2\}\Big), \tag{2}$$

for any $u_1, u_2 \geq 0$. Nevertheless, the direct application of the representation (2) to stochastic modeling has a couple of drawbacks. First, it turns out that the copula C_t in most cases essentially depends on t, see Tankov (2004) for examples. Second, since the distribution of $\vec{X}(t)$ is infinitely divisible, (2) is possible only for some subclasses of copulas. In other words, the class C_t depends on the class of marginal laws $X_i(t)$.

To avoid such difficulties, researches are trying to characterize the dependence between the components of Lévy process in *the time-independent fashion*. One of the most popular approaches for this characterization is the so-called Lévy copula (defined below), which was introduced by Tankov (2003), and later studied by Barndorff-Nielsen and Lindner (2004), Cont and Tankov (2004), Kallsen and Tankov (2006), and others. Among many papers in this field, we would like to emphasize some articles about statistical inference for Lévy copulas (mainly with applications to insurance), which include some basic ideas that are widely used in statistical research on this topic, in particular, in the statistical analysis in the current research - Esmaeili and Klüppelberg (2010 and 2011), Avanzi et al. (2011), Bücher and Vetter (2013).

The main objective of this article is the application of the Lévy copula approach to a class of stochastic processes, known as *time-changed Lévy processes*. In the one-dimensional case, the time-changed Lévy process is defined as $Y_s = L_{\mathcal{T}(s)}$, where L is a Lévy process and $\mathcal{T}$ is a non-negative, non-decreasing stochastic process with $\mathcal{T}(0) = 0$ referred as *stochastic time change* or simply *stochastic clock*. If the process $\mathcal{T}$ is also a Lévy process, then it is called the *subordinator*, and the process Y_s is usually refered as the subordinated process. The economical interpretation of the time change is based on the idea that the "business" time $\mathcal{T}(s)$ may run faster than the physical time in some periods, for instance, when the amount of transactions is high, see Clark (1973), Ané and Geman (2000), Veraart and Winkel (2010).

In this paper, we consider one natural generalization of the aforementioned model to the multidimensional case known as *multivariate subordination*. This construction was introduced by Barndorff-Nielsen et al. (2001) and later studied by many researchers, e.g., Semeraro (2008), Luciano and Semeraro (2010). The main contribution of this paper is Theorem 5.1, which gives the series representation of a process $\vec{X}(t)$ from our class in the following form:

$$\vec{X}(t) = \sum_{i=1}^{\infty} H\left(\Gamma_i, \vec{D}_i\right) \cdot I\{R_i \leq t\}, \tag{3}$$

where Γ_i are arrival times in a Poisson process of rate 1, R_i is a sequence of independent random variables, uniformly distributed on [0, 1], $\vec{D}_i$ is a sequence of i.i.d. random vectors from some Borel space S, and $H : \mathbb{R} \times S \to \mathbb{R}^d$.

The proof of Theorem 5.1 is based on the paper by Rosiński (2001), which was also used for some previous results of this type. For instance, Theorem 6.3 from the book of Cont and Tankov (2004) gives the series representation of the type (3) for a bivariate Lévy process *with positive jumps* linked by Lévy copulas. In this respect, our paper can be considered as an attempt to generalize the result by Cont and Tankov (2004) to an important class of multivariate time-changed Lévy models. Note also that other popular simulation tools like Gaussian approximations of small jumps (see Cohen and Rosinsky, 2007) are also not available for multidimensional time-changed models. We refer to the survey by Hilber and Winter (2009) for the overview of the existing methods.

Series representations in the spirit of Eq. 3 are of great interest in the statistical literature, especially in the context of Bayesian statistics. In particular, representation for generalized gamma processes (see Ishwaran and Zarepour, 2009) or for the Poisson-Dirichlet processes (see Leisen and Lijoi, 2011; Leisen et al. 2013; Zhu and Leisen, 2015) are widely used to simulate priors in Bayesian nonparametric methods. Discrete random measures, which allows for the series representations of similar types, draw attention of many researchers, see Kalli et al. (2011), Griffin and Walker (2011), Kolossiatis et al. (2013). We clarify the novelty of our approach in comparison with previously known related methods in Bayesian statistics in Section 6.

The paper is organized as follows. In the next section we give the definitions of Lévy processes, Lévy measures and Lévy copulas, and formulate the most important results from this theory. In Section 3 we shortly explain the notion of stochastic change of time. Afterwards, in Section 4, we introduce our model and discuss some properties of it. Our main results are given in Section 5, where we also provide some examples. The novelty of our main result, Theorem 5.1, is discussed in Section 6. In the last two sections, we provide a simulation study and a real-data example.

2 Brief Introduction to the Theory of Lévy Processes

We start with a very short introduction to the theory of Lévy processes. For the comprehensive study of this topic we refer to Sato (1999).

Definition 2.1 An $\mathbb{R}^d$-valued *Lévy process* $\vec{Z} = \left(\vec{Z}_t,\ t \geq 0\right)$ is a stochastic process with the following properties:

1. $\vec{Z}_0 = 0$ a.s.;
2. independent increments: $\forall\, 0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq \infty$, the random variables $\vec{Z}_{t_2} - \vec{Z}_{t_1},\ \vec{Z}_{t_3} - \vec{Z}_{t_2},\ ...,\ \vec{Z}_{t_n} - \vec{Z}_{t_{n-1}}$ are independent;
3. stationary increments: $\forall\, t > s$, $\vec{Z}_t - \vec{Z}_s \stackrel{\mathcal{L}}{=} \vec{Z}_{t-s}$;
4. stochastic continuity: $\forall\, t > 0,\ \vec{Z}_{t+h} \stackrel{\mathbb{P}}{\to} \vec{Z}_t$ as $h \to 0$.

The class of Lévy processes includes Brownian motion, compound Poisson process, α-stable processes, gamma-process, and others. Below we formulate three basic results from the theory of Lévy processes.

First, it turns out that this class of stochastic processes is closely related to the infinitely divisible distributions, which we define below.

Definition 2.2 The probability distribution F is infinitely divisible, if for any natural n there exist n i.i.d. variables $\xi_1, ..., \xi_n$ such that their sum $\xi_1 + \xi_2 + ... + \xi_n$ has distribution F.

Proposition 2.3 *Any Lévy process $\vec{Z}_t$ at any time point t has an infinitely divisible distribution. Conversely, for any infinitely divisible distribution $\mathscr{L}$ there exists a Lévy process $\vec{Z}_t$ such that $\vec{Z}_1$ has distribution $\mathscr{L}$.*

Second, the distribution of the Lévy process $\vec{Z}_t$ completely determines by the distribution in one point, say, by the distribution of $\vec{Z}_1$. This fact easily follows from the following proposition.

Proposition 2.4 *For any Lévy process $\vec{Z}_t$, there exists a function $\psi : \mathbb{R}^d \to \mathbb{C}$ known as characteristic exponent (or cumulant function), such that*

$$\phi_t(\vec{u}) := \mathbb{E}\left[e^{\mathrm{i}\langle \vec{u}, \vec{Z}_t\rangle}\right] = e^{t\psi(\vec{u})}.$$

Finally, we formulate the proposition, which gives the exact form for the characteristic function. This form is based on the fact that any Lévy process $\vec{Z}_t$ can be decomposed into the sum of a.s. continuous process $\vec{b}t + \Sigma^{1/2}\vec{W}_t$, where $\vec{b} \in \mathbb{R}^d$ and $\vec{W}_t$ is a d-dimensional standard Brownian motion, and a pure jump process J_t, which is described by the Lévy measure. The Lévy measure ν is defined as

$$\nu(B) = \mathbb{E}\left[\sharp\left\{t \in [0,1] : \Delta\vec{Z}_t \in B\right\}\right], \qquad \Delta\vec{Z}_t = \vec{Z}_t - \vec{Z}_{t-}, \qquad B \in \mathcal{B}(\mathbb{R}^d/\{0\}),$$

that is, for any Borel subset B the Lévy measure is the expected number, per unit time, of jumps whose size belongs to B. The triplet $(\vec{b}, \Sigma, \nu)$ is known as Lévy triplet, and it completely determines the distribution of $\vec{Z}_t$.

Proposition 2.5 *The characteristic exponent of a Lévy process $\vec{Z}_t$ allows for the following representation*

$$\psi(\vec{u}) = \mathrm{i}\langle\vec{b}, \vec{u}\rangle - \frac{1}{2}\langle\vec{u}, \Sigma\vec{u}\rangle + \int_{\mathbb{R}} \left(e^{\mathrm{i}\langle\vec{u},\vec{x}\rangle} - 1 - \mathrm{i}\langle\vec{u}, \vec{x}\rangle\mathbb{I}\{|\vec{x}| < 1\}\right)\nu(d\vec{x}), \tag{4}$$

where

- $\vec{b} \in \mathbb{R}^d$, Σ *- non-negatively defined matrix of size $d \times d$,*
- ν *is a Lévy measure, and*

$$\int_{\|\vec{x}\|\le 1} \|\vec{x}\|^2 \nu(dx) < \infty, \qquad \int_{\|\vec{x}\|>1} \nu(d\vec{x}) < \infty.$$

If the process $\vec{Z}_t$ is of bounded variation, then the representation (4) reduces to

$$\psi(\vec{u}) = \mathrm{i}\langle\vec{b}^*, \vec{u}\rangle + \int_{\mathbb{R}} \left(e^{\mathrm{i}\langle\vec{u},\vec{x}\rangle} - 1\right)\nu(d\vec{x}), \tag{5}$$

where $\vec{b}^* = \vec{b} - \int_{\|\vec{x}\|\le 1} \vec{x}\nu(d\vec{x})$.

For every Lévy measure, one can define its tail integral, which plays a crucial role in the construction of the Lévy copula.

Definition 2.6 For a one-dimensional Lévy measure ν, its tail integral is defined as

$$U(x) := \begin{cases} \nu(x, +\infty), & \text{if } x > 0, \\ -\nu(-\infty, x), & \text{if } x < 0. \end{cases}$$

Definition 2.6 can be equivalently written as

$$U(x) := (-1)^{s(x)} \nu\Big(I(x)\Big),$$

where

$$I(x) := \begin{cases} (x, +\infty), & \text{if } x > 0, \\ (-\infty, x), & \text{if } x < 0, \end{cases} \qquad \text{and} \qquad s(x) := \begin{cases} 2, & \text{if } x > 0, \\ 1, & \text{if } x < 0. \end{cases}$$

The reason for this definition is that in the case of infinite measure ν, $U(A)$ is infinite for any set A which contains 0. Analogously, for a d-dimensional Lévy process with Lévy measure ν, the tail integral is defined as

$$U(x_1, ..., x_d) := (-1)^{s(x_1)+...+s(x_d)} \cdot \nu\Big(I(x_1) \times ... \times I(x_d)\Big),$$

and this definition is also correct for any real $x_1, ..., x_d$.

Definition 2.7 A d-dimensional Lévy copula is a function from $\bar{\mathbb{R}}^2$ to $\bar{\mathbb{R}}$ such that

1. F is grounded, that is, $F(\vec{u}) = 0$ if $u_i = 0$ for at least one $i = 1, .., d$.
2. F is d-increasing.
3. F has uniform margins, that is, $F^{(1)}(v) = ... = F^{(d)}(v) = v$, where
$$F^{(j)}(v) = \lim_{u_1,...,u_{j-1},u_{j+1},u_d \to \infty} F\left(u_1, ..., u_{j-1}, v, u_{j+1}, ..., u_d\right), \ j = 1..d,$$
4. $F(u_1, ..., u_d) \neq \infty$ for $(u_1, ..., u_d) \neq (\infty, ..., \infty)$.

The main result on Lévy copulas (an analogue of the Sklar theorem for ordinary copulas) states that for any multidimensional Lévy process $\vec{Z}$ with tail integral U and marginal tail integrals $U_1, ..., U_d$, there exists a Lévy copula F such that

$$U(x_1, ..., x_d) = F(U_1(x_1), ..., U_d(x_d)) \tag{6}$$

and vice versa, for any Lévy copula F and any one-dimensional Lévy process with tail integrals $U_1, ..., U_d$ there exists a d-dimensional Lévy process with tail integral U given by Eq. 6 and marginal tail integrals $U_1, ..., U_d$. The first part of this theorem can be easily verified for the case when the one-dimensional Lévy measures are infinite and have no atoms, because in this case the Lévy copula is equal to

$$F(u_1, ..., u_d) = U\Big(U_1^{-1}(u_1), ..., U_d^{-1}(u_d)\Big), \tag{7}$$

where U is the tail integral of the Lévy measure of $\vec{Z}$, see Kallsen and Tankov (2006).

The most popular type of Lévy copulas is the (positive) Archimedian copulas defined by

$$F^{(A)}(x_1, ..., x_d) = \varphi\left(\varphi^{-1}(x_1) + \varphi^{-1}(x_d)\right),$$

where $\varphi(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is an Archimedian Lévy generator - a continuous decreasing function satisfying $\varphi(0) = \infty$, $\lim_{u\to\infty} \varphi(u) = 0$. In the case $\varphi(u) = u^{-1/\theta}$ with $\theta > 0$ we arrive at the so-called positive Clayton-Lévy copula, which we discuss in more details in Section 5.

It would be a worth mentioning that $d-$dimensional Lévy copulas can be iteratively constructed from $(d-1)$ - dimensional Lévy copulas. One of such approaches known as positive nested Archimedian Lévy copulas (PNALS) was introduced in Grothe and Hofert (2015). The key idea of the construction of PNALS is to replace some arguments of Archimedian Lévy copula with other processes from this class. For instance, taking two two-dimensional

Archimedian Lévy copulas $F_1^{(A)}$ and $F_2^{(A)}$ with generators $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ resp., one can construct a three-dimensional positive Lévy copula

$$F^{(NA)}(x_1, x_2, x_3) := F_2^{(A)}\left(x_1, F_1^{(A)}(x_2, x_3)\right), \qquad x_i \in \mathbb{R}_+,\ i = 1, 2, 3,$$

provided that the function $\left(\varphi_2^{-1} \circ \varphi_1\right)'$ is completely monotone.

Another approach is the so-called vine construction of Lévy copulas introduced in Grothe and Nicklas (2013). For instance, in the 3-dimensional case, this method yields that the following function is a Lévy copula:

$$F(x_1, x_2, x_3) = \int_0^{x_3} C\left(G_1(x_1|v), G_2(x_2|v)\right) d\mu_3(v),$$

where C is a distribution copula, μ_3 is the distribution measure of the third component, $G_1(x_1|v)$ and $G_2(x_2|v)$ are the one-dimensional distribution functions of the random variables $\xi_{i|v}$, $i = 1, 2$, defined from the decomposition of the measures $\mu_{i,3}$, $i = 1, 2$, corresponding to the Lévy copulas between i-th and third components, to the generalized product $\mu_{i,3} = \mu_3 \otimes \xi_{i|v}$. For detailed description of such constructions we refer to Section 3.1 from Grothe and Nicklas (2013).

3 Time-Changed Lévy Models

As it was already mentioned in the introduction, the time-changed Lévy process in the one-dimensional case is defined as

$$Y_s = L_{\mathcal{T}(s)}, \tag{8}$$

where L is a Lévy process, and $\mathcal{T}(s)$ - a non-negative, non-decreasing stochastic process with $\mathcal{T}(0) = 0$. This class of models has strong mathematical background based on the so-called Monroe theorem (Monroe 1978), which stands that any semimartingale can be represented as a time-changed Brownian motion (that is, in the form Eq. 8 with L equal to the Brownian motion W) and vice versa, any time-changed Brownian motion is a semimartingale. Various aspects of this theory are discussed in Barndorff-Nielsen and Shiryaev (2010) and Cherny and Shiryaev (2002). Nevertheless, the first part of the Monroe theorem doesn't hold if one introduces any of the following additional assumptions:

1. Processes W and $\mathcal{T}$ are independent. This assumption is widely used in the statistical literature and is quite convenient for both theoretical and practical purposes.
2. Time change process $\mathcal{T}$ is itself a Lévy process, that is, a subordinator. In this case any resulting process Y_s is also a Lévy process, which is usually called *a subordinated process*.

These drawbacks of the time-changed Brownian motion lead to the idea of considering more general model (8) with any Lévy process instead of the Brownian motion and introducing the assumption that the processes $\mathcal{T}$ and L are independent. This model has been attracting attention of many researches, see, e.g., Belomestny and Panov (2013), Bertoin (1998) Carr et al. (2003), Cherubini et al. (2010), and Schoutens (2003).

Nevertheless, there is no clear understanding in the literature how to extend this model to the multi-dimensional case. The most popular construction is to consider the model (8) with a d-dimensional Lévy process $\vec{L}$ and to provide a time change in each component with the same process $\mathcal{T}$, see Sato (1999).

Interestingly enough, in the case when $\vec{L}$ is a Brownian motion, the correlation coefficient between subordinated processes is upper bounded by the correlation coefficient between the components of the Brownian motion, see Eberlein and Madan (2010). Moreover, these coefficients coincides in some cases, see Cont and Tankov (2004).

4 Multidimensional Subordinated Processes

In this section, we introduce a multidimensional generalization of the model (8). This generalization is based on the notion of the d-dimensional subordinator, which we define below.

Definition 4.1 *A d - dimensional subordinator* $\vec{\mathcal{T}}(s) = (T_1(s), ..., T_d(s))$ is a Lévy process in $\mathbb{R}^d$ such that its components $T_1, ..., T_d$ are one-dimensional subordinators.

In particular, $T_1, ..., T_d$ in the above definition may be independent. Another example is given by the following statement (see Semeraro (2008)): if T_1, T_2, T_3 are 3 independent subordinators, then the processes

$$(T_1(s) + T_3(s), \quad T_2(s) + T_3(s))$$

and

$$(T_1(T_3(s)), \quad T_2(T_3(s)))$$

are two-dimensional subordinators.

Consider now d independent one-dimensional Lévy process $L_1(t), ..., L_d(t)$ and a d-dimensional subordinator $\vec{\mathcal{T}}(s) = (\mathcal{T}_1(s), ..., \mathcal{T}_d(s))$ such that $\mathcal{T}_i(s)$ is independent of $L_i(s)$ for all $i = 1, .., d$. Define the subordinated process by composition

$$\vec{X}(s) = \Big(X_1(s), ..., X_d(s)\Big) := \Big(L_1(\mathcal{T}_1(s)), ..., L_d(\mathcal{T}_d(s))\Big). \tag{9}$$

This construction, known as multivariate subordination, was firstly considered in Barndorff-Nielsen et al. (2001). In what follows, we consider the case when $L_i, i = 1..d$, are independent Brownian motions with drifts.

Theorem 4.2 *Consider the model (9) with $L_i(t) = c_i t + \sigma_i W_i(t), \ i = 1..d$, where $W_i(t), \ i = 1, ..., d$ are d independent one-dimensional Brownian motions, $\mu_i \in \mathbb{R}, \sigma_i \geq 0$, $\vec{\mathcal{T}}(s) = (\mathcal{T}_1(s), ..., \mathcal{T}_d(s))$ is a d-dimensional subordinator. Denote*

$$\psi_{\mathcal{T}}(\vec{u}) := \langle \vec{\rho}, \vec{u} \rangle + \int_{\mathbb{R}} \Big(e^{\langle \vec{u}, \vec{x} \rangle} - 1\Big) \eta(d\vec{x}),$$

where $\vec{\rho} = (\rho_1, ..., \rho_d) \in \mathbb{R}^d$, $\rho_i \geq 0, \ i = 1, ..., d$, and η is a Lévy measure in $\mathbb{R}^d_+$.

Denote by diag $(\vec{x})$ *with $\vec{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ a d-dimensional diagonal matrix with values $x_1, ..., x_d$ on the diagonal.*

Then the process

$$\vec{X}(s) := \Big(L_1(\mathcal{T}_1(s)), ..., L_d(\mathcal{T}_d(s))\Big) \tag{10}$$

is a d-dimensional Lévy process with the characteristic function

$$\phi_{\vec{X}}(u_1, ..., u_d) = \exp\Big\{t\psi_{\mathcal{T}}\Big(\mathrm{i}c_1 u_1 - \sigma_1^2 u_1^2/2, ..., \mathrm{i}c_d u_d - \sigma_1^2 u_d^2/2\Big)\Big\}, \tag{11}$$

and the Lévy triplet $\left(\vec{b}, \Sigma, \nu\right)$ *defined as follows:*

- *vector* $\vec{b} \in \mathbb{R}^d$ *equals to*
$$\vec{b} = (c_1\rho_1, ..., c_d\rho_d)^\top + \int_{\mathbb{R}_+^d} \eta(d\vec{u}) \int_{|\vec{x}|\le 1} \vec{x}\ \mu(d\vec{x}, \vec{u}),$$
where $\mu(\cdot, \vec{s})$ *with* $\vec{s} = (s_1, ..., s_d)$ *stands for the distribution of the random vector* $(L_1(s_1), ..., L_d(s_d))$.
- *matrix* Σ *equals to*
$$\Sigma = \mathrm{diag}\left(\sigma_1^2\rho_1, ..., \sigma_d^2\rho_d\right),$$
- *and the Lévy measure* ν *is given by*
$$\nu(B) := \int_{\mathbb{R}_+^d} \mu\Big(B;\ \mathrm{diag}(y)\Big)\ \eta(dy), \qquad B \subset \mathbb{R}^2, \quad y \in \mathbb{R}^d.$$

In particular, if $c_i = 0$ *and* $\sigma_i = 1$ *for all* $i = 1..d$ *(that is,* $L_i(t)$ *are independent standard Brownian motions), then* $\vec{b} = \vec{0}$ *and* $\Sigma = \mathrm{diag}(\vec{\rho})$.

Proof This theorem is essentially proven in Barndorff-Nielsen et al. (2001). □

5 Series Representation for Subordinated Processes

In this section, we apply the result by Rosiński (2001) to our setup.

Theorem 5.1 *Let* $\vec{X}(s)$ *be a* d*-dimensional Lévy process constructed by multivariate subordination of the standard Brownian motion, see Theorem 4.2 for notation. Denote by* $F(u_1, ..., u_d)$ *a positive Lévy copula between* $\mathcal{T}_1(s), .., \mathcal{T}_d(s)$*, and assume that* $\vec{\rho} = \vec{0}$*. Assume also that* $F(u, v)$ *is continuous and the mixed derivative* $\partial^d F(u_1, ..., u_d)/\partial u_1...\partial u_d$ *exists in* $\mathbb{R}_+^d$*. Moreover, assume that there exists a density function* $p^*(\cdot) : \mathbb{R}^{d-1} \to \mathbb{R}$ *and* $(d-1)$ *functions* $f_j^*(x_j, x_d) : \mathbb{R}_+^2 \to \mathbb{R}_+,\ j = 1..(d-1)$*, such that*

1. *for any* $u, x > 0$*,*
$$\int_{-\infty}^{f_1^*(x_1,x_d)} ... \int_{-\infty}^{f_{d-1}^*(x_{d-1},x_d)} p^*(z_1, ..., z_{d-1}) dz_{d-1}...dz_1 = \frac{\partial F(x_1, ..., x_d)}{\partial x_d}; \quad (12)$$
2. *the functions* $f_j^*(x_j, x_d)$*,* $j = 1..(d-1)$ *monotonically increase in* x_j *for any fixed* x_d*, and moreover, for any* $j = 1..(d-1)$ *and any* $y > 0$*, the equation*
$$f_j^*(x_j, x_d) = y$$
has a closed-form solution with respect to x_j*; we denote this solution by* $h_j^*(y, x_d)$*.*

Next, define a d*-dimensional stochastic process* $\vec{Z}(s) = (Z_1(s), ..., Z_d(s))$*,*

$$Z_k(s) := \sum_{i=1}^{\infty} G_i^{(k)} \sqrt{U_k^{(-1)}\left(h_k^*(Q_i^{(k)}, \Gamma_i)\right)} \cdot I\{R_i \le s\} \quad (13)$$

for $k = 1..(d-1)$*, and*

$$Z_d(s) := \sum_{i=1}^{\infty} G_i^{(d)} \cdot \sqrt{U_d^{(-1)}(\Gamma_i)} \cdot I\{R_i \le s\}, \quad (14)$$

where $U_1, ..., U_d$ *are tail integrals of the subordinators* $T_1, ..., T_d$ *resp.,* $U_1^{(-1)}, ..., U_d^{(-1)}$ *are their generalized inverse functions, that is,*

$$U_i^{(-1)}(y) = \inf\{x > 0 : \quad U_i(x) < y\}, \quad i = 1..d, \quad y \in \mathbb{R}_+,$$

Γ_i *is an independent sequence of jump times of a standard Poisson process,* $G_i^{(1)}, ..., G_i^{(d)}$ *- are d sequences of i.i.d. standard normal r.v.,*

$$Q_i := \left(Q_i^{(1)}, .., Q_i^{(d-1)}\right)$$

- sequence of i.i.d. random vectors with density $p^*(\cdot)$, R_i *- sequence of i.i.d. r.v., uniformly distributed on* $[0, 1]$*, and all sequences of r.v. are independent of each other. Then*

$$\vec{X}(s) \stackrel{\mathcal{L}}{=} \vec{Z}(s), \qquad \forall s \in [0, 1].$$

Example 1. Consider the positive Clayton-Lévy copula

$$F_C(x_1, ..., x_d) = (x_1^{-\theta} + ... + x_d^{-\theta})^{-1/\theta}$$

with some $\theta > 0$. Derivative with respect to x_d is equal to

$$\frac{\partial F_C(x_1, ..., x_d)}{\partial x_d} = \frac{1}{\left(1 + (x_1/x_d)^{-\theta} + ... + (x_{d-1}/x_d)^{-\theta}\right)^{(1+\theta)/\theta}}.$$

Motivated by this representation, we suggest to define the density function $p^*(z)$ as

$$p^*(z_1, ..., z_{d-1}) = \frac{\partial}{\partial z}\left\{\frac{1}{\left(1 + z_1^{-\theta} + ... + z_{d-1}^{-\theta}\right)^{(1+\theta)/\theta}}\right\},$$

and the function $f_j^*(x_j, x_d) := x_j/x_d$, $j = 1..(d-1)$. Both conditions on the functions p^* and f_j^* are fulfilled.

2. Note that the same arguments can be applied to any sufficiently smooth homogeneous Lévy copula, that is, to any copula such that

$$F_H(ku_1, ..., ku_d) = kF_H(u_1..., u_d), \qquad \forall\, u_1..., u_d > 0,\ \forall\, k > 0, \tag{15}$$

see Remark 5.3 about the difference between ordinary copulas and Lévy copulas. Note that taking the derivatives with respect to u from both parts of Eq. 15, yields

$$\frac{\partial}{\partial x_d} F_H(kx_1, ..., kx_d) = \frac{\partial}{\partial x} F_H(x_1..., x_d) = \frac{\partial}{\partial r_d} F_H(r_1, ..., r_d)\Bigg|_{\substack{r_1 = x_1/x_d \\ ... \\ r_{d-1} = x_1/x_d \\ r_d = 1}},$$

and therefore one can define

$$p^*(z) = \frac{\partial}{\partial z}\left\{\frac{\partial}{\partial r_d} F_H(r_1, .., r_d)\Bigg|_{\substack{(r_1, ..., r_{d-1}) = z \\ r_d = 1}}\right\}, \qquad f_j^*(x_j, x_d) := x_j/x_d.$$

For the description of the class of homogeneous Lévy copulas we refer to Section 4 from Barndorff-Nielsen and Lindner (2004).

3. Moreover, we can apply the same approach for any mixtures of homogeneous Lévy copulas. In fact, consider the function

$$F_M(x_1, ..., x_d) = \sum_{r=1}^{n} \beta_r F_r(x_1, ..., x_d),$$

where $\beta_1, ..., \beta_n$ are positive numbers such that $\sum_{r=1}^{n} \beta_r = 1$ and $F_r(x_1, ..., x_d)$ are homogeneous Lévy copulas for any $r = 1..n$. It's easy to see that $F_M(x_1, ..., x_d)$ is also a homogeneous Lévy copula. As it was shown in the previous example, we can take $f_j^*(x_j, x_d) := x_j/x_d$. Note also that in the case of mixture model,

$$p^*(z) = \sum_{r=1}^{n} \beta_r p_r^*(z),$$

where by $p_r^*(\cdot)$ we denote the density functions, constructed by Eq. 12 with Lévy copulas $F_r(x_1, ..., x_d)$.

Proof of Theorem 5.1 In the core of this proof lies the result by Rosiński (2001), which is nicely formulated as Theorem 6.2 in Cont and Tankov (2004). Below we give a (slightly simplified) version of this result, which is needed for our purposes. □

Proposition 5.2 *Let S be a measurable space, and $H : (0, \infty) \times S \to \mathbb{R}^d$ - a measurable function. Let D_i be an i.i.d. sequence of random elements from S. Define 2 measures on $\mathbb{R}^d$:*

$$\begin{aligned} \sigma(r, B) &:= \mathbb{P}\{H(r, D_i) \in B\}, \qquad r > 0,\ B \in \mathcal{B}(\mathbb{R}^d), \\ \nu(B) &:= \int_{\mathbb{R}_+} \sigma(r, B) dr, \qquad B \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

Assume that the measure $\sigma(r, B)$ is symmetric in B, and ν is a Lévy measure, that is, $\int_{\mathbb{R}_d} \min(1, \|x\|^2)\nu(dx) < \infty$. Then the series

$$X(s) = \sum_{i=1}^{\infty} H(\Gamma_i, D_i) \cdot I\{R_i \le s\} \tag{16}$$

converges to a Lévy process with triplet $(0, 0, \nu)$, where $\Gamma_1, \Gamma_2, ...$ is a sequence of jumping times of a standard Poisson process, and $R_1, R_2, ...$ is a sequence of independent random variables, uniformly distributed on $[0, 1]$ and independent from D_i and Γ_i.

The general aim of this proof is to find a d- dimensional distribution π and function $H : (0, \infty) \times S \to \mathbb{R}^d$, such that

$$\nu(B) := \int_{\mathbb{R}_+} \mathbb{P}\{H(r, D) \in B\} dr, \qquad r > 0,\ B \in \mathcal{B}(\mathbb{R}^d),$$

where a r.v. $\vec{D}$ has distribution π, and moreover the measure

$$\sigma(r, B) = \mathbb{P}\{H(r, D) \in B\}, \qquad B \in \mathcal{B}(\mathbb{R}^d) \tag{17}$$

is symmetric.

First note that it is sufficient to consider the sets $B = B_1 \times ... \times B_d$, where $B_k = [x_k, \infty),\ x_k \in \mathbb{R},\ k = 1..d$. For such B,

$$\mu\Big(B;\ \mathrm{diag}(U(r))\Big) = \mu\Big(B_1;\ U_1^{-1}(r_1)\Big) \cdot ... \cdot \mu\Big(B_d\,; U_d^{-1}(r_d)\Big),$$

where by $\mu(\cdot; \sigma)$ in the right-hand side we denote the one-dimensional normal distribution with zero mean and variance equal to σ (since there is no risk of confusion, we use the same letter for d-dimensional and 1-dimensional distributions), and $U(r) = (U_1(r_1), ..., U_d(r_d))$.

Since the Lévy copula F is sufficiently smooth, we can differentiate both parts in Eq. 6 and get that

$$\begin{aligned}\nu(B) &= \int_{\mathbb{R}_+^d} \mu\Big(B\,;\,\mathrm{diag}(y)\Big)\ \frac{\partial^d F}{\partial r_1\,...\,\partial r_d}\Bigg|_{\substack{r_1=U_1(y_1)\\ ...\\ r_d=U_d(y_d)}}\ d\,(U_1(y_1))\,...d\,(U_d(y_d))\\ &= \int_{\mathbb{R}_+^d} \mu\Big(B\,;\,\mathrm{diag}(U^{-1}(r))\Big)\ \frac{\partial^d F(r_1,...,r_d)}{\partial r_1\,...\,\partial r_d} dr_1...dr_d,\end{aligned}$$

where $U^{-1}(r) = \Big(U_1^{-1}(r_1),...,U_d^{-1}(r_d)\Big)$, see Proposition 5.8 from Cont and Tankov (2004). Therefore, for $B = B_1 \times ... \times B_d$ defined above,

$$\nu(B) = \int_{\mathbb{R}_+} \mathbb{E}_{\mathscr{L}(r_d)}\Big[G(B_1,...,B_{d-1})\Big]\mu\Big(B_d\,;\,U_d^{-1}(r_d)\Big)dr_d,$$

where

$$G(B_1,...,B_{d-1}) := \mu\Big(B_1\,;\,U_1^{-1}(\cdot)\Big)\cdot ...\cdot \mu\Big(B_{d-1}\,;\,U_{d-1}^{-1}(\cdot)\Big),$$

and by

$$\begin{aligned}&\mathbb{E}_{\mathscr{L}(r_d)}\Big[G(B_1,...,B_{d-1})\Big]\\ &\qquad= \int_{\mathbb{R}_+^{d-1}} G(B_1,...,B_{d-1})\ \frac{\partial^{d-1}}{\partial r_1\,...\,\partial r_{d-1}}\left(\frac{\partial F(r)}{\partial r_d}\right)\ dr_1...dr_{d-1}\end{aligned}$$

we denote the mathematical expectation with respect to the measure $\mathscr{L}(r_d)$ with the distribution function $\tilde{F}(r_d) = \partial F(r)/\partial r_d$ (the proof of the statement that $\tilde{F}(r_d)$ is in fact a distribution function follows the same lines as the proof of Lemma 5.3 from Cont and Tankov (2004)). By the well-known Fubini theorem,

$$\mathbb{E}_{\mathscr{L}(r_d)}\Big[G(B_1,...,B_{d-1})\Big] = \int_{B_1}...\int_{B_{d-1}} g(v_1,..,v_{d-1}\,;\,r_d)\ dv_{d-1}...dv_1,$$

where

$$\begin{aligned}g(v_1,..,v_{d-1}\,;\,r_d) = &\int_{\mathbb{R}_+^{d-1}} p\Big(v_1\,;\,U_1^{-1}(r_1)\Big)\cdot ...\cdot p\Big(v_{d-1}\cdot\,;\,U_{d-1}^{-1}(r_{d-1})\Big)\\ &\cdot\frac{\partial^{d-1}}{\partial r_1\,...\,\partial r_{d-1}}\left(\frac{\partial F(r)}{\partial r_d}\right)dr_1...dr_{d-1}, \qquad r_d > 0,\end{aligned} \tag{18}$$

and $p\,(\cdot\,;\,\sigma)$ is the density of the normal distribution with zero-mean and variance equal to $\sigma > 0$. Note that g is a density function, see Remark 5.4. Changing the variables we get

$$\begin{aligned}g(v_1,..,v_{d-1}\,;\,r_d) = &\int_{\mathbb{R}_+^{d-1}} p\Big(v_1\,;\,\tilde{r}_1\Big)\cdot ...\cdot p\Big(v_{d-1}\cdot\,;\,\tilde{r}_{d-1}\Big)\\ &\cdot\frac{\partial^{d-1}}{\partial r_1\,...\,\partial r_{d-1}}\left(\frac{\partial F(r)}{\partial r_d}\right)\Bigg|_{\substack{r_1=U_1(\tilde{r}_1)\\ ...\\ r_{d-1}=U_{d-1}(\tilde{r}_{d-1})}}\ d\,(U_1(\tilde{r}_1))\,...d\,(U_{d-1}(\tilde{r}_{d-1}))\end{aligned} \tag{19}$$

The last expression yields that $g(\cdot;\ r_d)$ is in fact a variance mixture of the normal distribution (see Barndorff-Nielsen et al. (1982) or Kelker (1971)). This in particularly gives that the random vector

$$\left(\eta_1\sqrt{\eta_d^{(1)}}, ..., \eta_{d-1}\sqrt{\eta_d^{(d-1)}}\right)$$

has a distribution with density $g(\cdot; r_d)$, where $\eta_1, ..., \eta_{d-1}$ are $(d-1)$ i.i.d. standard normal r.v.'s, and vector $\eta_d = \left(\eta_d^{(1)}, ..., \eta_d^{(d-1)}\right)$ has a distribution with density

$$\breve{p}(v_1, .., v_{d-1}; r_d) = \frac{\partial^{d-1}}{\partial r_1 \ldots \partial r_{d-1}} \left(\frac{\partial F(r)}{\partial r_d}\right)\Bigg|_{\substack{r_1=U_1(v_1)\\ \cdots \\ r_d=U_d(v_d)}} U_1'(v_1)...U_{d-1}'(v_{d-1}),$$

that is, the density of the random variable

$$\left(U_1^{-1}(\tilde{\eta}_d^{(1)}), ..., U_d^{-1}(\tilde{\eta}_d^{(d-1)})\right),$$

where $\tilde{\eta}_d := \left(\tilde{\eta}_d^{(1)}, ..., \tilde{\eta}_d^{(d-1)}\right)$ has a distribution function $\tilde{F}(r_d)$. Since (12) holds, we get that

$$\frac{\partial^d F(r)}{\partial r_1 ... \partial r_d} = \frac{\partial f_1^*(r_1, r_d)}{\partial r_1} ... \frac{\partial f_{d-1}^*(r_{d-1}, r_d)}{\partial r_d} \cdot p^*\left(f_1^*(r_1, r_d), ..., f_{d-1}^*(r_{d-1}, r_d)\right),$$

and therefore η_3 has the same distribution as

$$\left(h_1^*(\bar{\eta}_d^{(1)}, r_d), ..., h_{d-1}^*(\bar{\eta}_d^{(d-1)}, r_d)\right)$$

where $\bar{\eta}_d := \left(\bar{\eta}_d^{(1)}, ..., \bar{\eta}_d^{(d-1)}\right)$ has distribution with density $p^*(\cdot)$.

Finally, we get the following representation for the Lévy measure ν:

$$\nu(B) = \int_{\mathbb{R}_+} \left[\mathbb{P}\left\{\bigcup_{j=1}^{d-1}\left\{\eta_j\sqrt{h_j^*(\bar{\eta}_d^{(j)}, r_d)} \in B_j\right\}\right\} \cdot \int_{B_1} p\left(u\ ;\ U_d^{-1}(r_d)\right)du\right] dr_d.$$

This representation motivates to define the function H as

$$H(r, D) = \begin{pmatrix} D_{1,1}\sqrt{U_1^{-1}(h_1^*(r, D_{1,2}))} \\ \cdots \\ D_{(d-1),1}\sqrt{U_{d-1}^{-1}(h_{d-1}^*(r, D_{(d-1),2}))} \\ D_{d,1}\cdot\sqrt{U_d^{-1}(r)} \end{pmatrix},$$

with $D = (D_{1,1}, D_{1,2}, ..., D_{(d-1),1}, D_{(d-1),2}, D_{d,1})$, where $D_{j,1}$ have standard normal distribution for $j = 1..d$, and r.v. $\left(D_{j,1}, ..., D_{j,d-1}\right)$ has a distribution with density function $p^*(\cdot)$. This observation completes the proof.

Remark 5.3 In the context of ordinary copulas, it is common to introduce the homogeneous copula $C_H^{(k)}$ of order k by

$$C_H^{(k)}(ku, kv) = k^\alpha C_H(u, v), \qquad \forall\ k, u, v > 0, \tag{20}$$

see, e.g., Nelsen (2006) (generalizations for d-dimensional case are straightforward). Substituting $u = v = 1$, we get $C_H(k,k) = k^\alpha$. Therefore, taking into account the Fréchet bounds, we arrive at the inequality

$$\max(2k-1,0) \le k^\alpha \le k,$$

which yields that $\alpha \in [1,2]$. Moreover, it turns out that the class of homogeneous ordinary copulas coincides with Cuadras-Augé family. More precisely,

$$C_H^{(k)}(u,v) = (\min(u,v))^{2-\alpha}\,(uv)^{\alpha-1}, \quad u,v \in [0,1],$$

see Theorem 3.4.2 from Nelsen (2006). Returning to Lévy copulas, we realize that similar to Eq. 20 equality

$$F_H(ku,kv) = k^\alpha F_H(u,v), \qquad \forall\, k,u,v > 0$$

is possible only in case $\alpha = 1$. In fact, taking limit as $v \to \infty$, we get the equality $ku = k^\alpha u,\ \forall u$, which leads to trivial conclusion $\alpha = 1$. This argument yields the definition of homogeneous Lévy copula (15).

Remark 5.4 Let us shortly show that the function $g(v_1, .., v_{d-1}\ ; r_d)$ defined by Eq. 21 is a density function for any r_d. In fact, as it was mentioned before, the function $\tilde{F}(r_d) = \partial F(r)/\partial r_d$ is a distribution function, and moreover $\partial^d F(r)/\partial r_1 ... \partial r_d$ is the density function of this distribution. Therefore, $g(\cdot;\ r_d) \ge 0$, and

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} g(v_1, .., v_{d-1}\ ; r_d) dv_1 ... dv_{d-1}\\
&= \int_{\mathbb{R}_+^{d-1}} \left[\int_{\mathbb{R}^{d-1}} p\Big(v_1\ ; U_1^{-1}(r_1)\Big) \cdot ... \cdot p\Big(v_{d-1} \cdot\ ; U_{d-1}^{-1}(r_{d-1})\Big) dv_1 ... dv_{d-1} \right]\\
&\qquad\qquad \cdot \frac{\partial^{d-1}}{\partial r_1 \ ...\ \partial r_{d-1}} \left(\frac{\partial F(r)}{\partial r_d} \right) dr_1 ... dr_{d-1}\\
&\qquad\qquad = \int_{\mathbb{R}_+^{d-1}} \frac{\partial^{d-1}}{\partial r_1 \ ...\ \partial r_{d-1}} \left(\frac{\partial F(r)}{\partial r_d} \right) dr_1 ... dr_{d-1} = 1
\end{aligned} \tag{21}$$

Remark 5.5 It is a worth mentioning that the right way to truncate series in Eqs. 13–14 (or, more generally speaking, in Eq. 16) is to fix some large h and keep $N(h) = \inf_i \{\Gamma_i \le h\}$ terms. In this case, the truncated series

$$X^h(s) = \sum_{i=1}^{N(h)} H\left(\Gamma_i, D_i\right) \cdot I\left\{R_i \le s\right\}$$

is a compound Poisson process (see Rosiński (2001)) with a Lévy measure

$$\nu^h(B) := \int_{\mathbb{R}_+} \mathbb{P}\left\{H(r, D_i) \cdot \mathbb{I}\{r \le h\} \in B\right\} dr, \qquad\qquad B \in \mathcal{B}(\mathbb{R}^d).$$

In Cohen and Rosiński (2007), it is shown that the quality of the approximation $X(s)$ by $X^h(s)$ can be improved by adding a Brownian motion and a drift. More precisely, denote the remainder part of this approximation by $X_h(s) := X(s) - X^h(s)$, the Lévy measure of X_h by $\nu_h := \nu - \nu^h$, and the covariance function of $X_h(1)$ by $A_h = \int_{\mathbb{R}^d} xx^\top \nu_h(dx)$. Theorem 3.1 from Cohen and Rosiński (2007) yields that under some assumptions,

$$X(s) \stackrel{\mathcal{L}}{=} X^h(s) + A_h W(s) + b_h + Y_h(s), \tag{22}$$

where $W(s)$ is a d-dimensional standard Brownian motion independent of $X^h(s)$, $b_h := b + \int_{\|x\|>1} x\nu_h(dx) - \int_{\|x\|\le 1} x\nu^h(dx)$, and $Y_h(s)$ is a càdlàg -process such that for any time horizon $T > 0$

$$\sup_{s\in[0,T]} \|A_h^{-1} Y_h(s)\| \xrightarrow{\mathbb{P}} 0. \tag{23}$$

One can derive from Eq. 22 that the approximation

$$X(s) \approx X^h(s) + A_h W(s) + b_h \tag{24}$$

has smaller error as the approximation $X^h(s)$.

Remark 5.6 With no doubt, this result can be generalized for the case, when $L_i(t)$, $i = 1..d$, are Brownian motions with drift and ρ_i are positive, see Theorem 4.2 for notation. In this case, the series representation (13) – (14) also gives the representation for the jump part, and therefore

$$\vec{X}(s) \stackrel{\mathcal{L}}{=} \vec{Z}(s) + s\vec{b} + \Sigma^{1/2}\vec{W}(s),$$

where

$$\begin{aligned} \vec{b} &= \int_{\mathbb{R}_+^d} \eta(d\vec{u}) \int_{|\vec{x}|\le 1} \vec{x}\; \mu(d\vec{x}, \vec{u}), \\ \Sigma^{1/2} &= \operatorname{diag}\left(\sigma_1\rho_1^{1/2}, ..., \sigma_d\rho_d^{1/2}\right), \end{aligned}$$

and $\vec{W}(s)$ is a d-dimensional standard Brownian motion, independent from $\vec{Z}(s)$.

6 Discussion

In this section, we discuss the novelty of the representation (13) – (14) in comparison with other series representations for Lévy processes and related models. For instance, Ishwaran and Zarepour (2009) used a scale invariance principle to construct the series representation for bivariate generalized gamma processes $(\Gamma_1(t), \Gamma_2(t))$. This representation is in the form

$$\Gamma_1(\cdot) = \sum_{j=1}^{\infty} J_1(V_{j,1}\Gamma_j)\varepsilon_{X_j}(\cdot), \qquad \Gamma_2(\cdot) = \sum_{j=1}^{\infty} J_2(V_{j,2}\Gamma_j)\varepsilon_{X_j}(\cdot), \tag{25}$$

where $(V_{j,1}, V_{j,2})$ and X_j are two sequences of i.i.d. random variables, Γ_j are jumping times of a standard Poisson process, J_1, J_2 are some functions which specify the jump heights of the process, and $\varepsilon_{X_j}(\cdot)$ is a discrete measure concentrated on X_j.

This result can be applied to modeling the bivariate Dirichlet process and some generalizations. Such constructions are of great interest in Bayesian nonparametric statistics, and some further works in this direction are known. For instance, the papers (Leisen and Lijoi 2011; Leisen et al. 2013; Zhu and Leisen 2015) are devoted to the study of the multivariate Poisson-Dirichlet process. As in the present paper, the dependence between components is described via the Lévy copula.

There are at least two essential differences between the aforementioned papers and our approach. First, we consider any one-dimensional Lévy processes as marginal distributions, not only gamma (as in Ishwaran and Zarepour (2009)) or stable (as in Leisen and Lijoi (2011), Leisen et al. (2013), Zhu and Leisen (2015)). Second, we consider the model of time-changed Brownian motions, which has both mathematical and economical motivation, see Sections 1 and 3. In Ishwaran and Zarepour (2009), one partial case known as variance gamma model (Brownian motion subordinated by a gamma process) is considered. Nevertheless, the representation given in Ishwaran and Zarepour (2009), essentially uses the special property of the variance gamma model (the possibility to represent this process as a difference between two independent Gamma-processes), which has no analogues for another time-changed models.

7 Simulation Study

In this section we show the performance our approach for a kind of multivariate variance gamma model.

Consider the two-dimensional process $\vec{X}(s) := (W_1(\mathcal{G}_1(s)), W_2(\mathcal{G}_2(s)))$, where W_1 and W_2 are independent Brownian motions, $\mathcal{G}_1(s)$ and $\mathcal{G}_2(s)$ are dependent gamma-processes with parameters a_1, b_1 and a_2, b_2 (that is, Lévy processes such that $\mathcal{G}_i(s)$ has gamma distribution with parameters $a_i s$ and b_i, i=1,2). Dependence between the subordinators $\mathcal{G}_1(s)$ and $\mathcal{G}_2(s)$ is expressed through the Clayton-Lévy copula $F(x_1, x_2) = (x_1^{-\delta} + x_2^{-\delta})^{-1/\delta}$. Moreover, it is assumed that W_i is independent of $\mathcal{G}_i$, $i = 1, 2$.

For this model, series representation (13)–(14) reduces to

$$Z_1(s) := \sum_{i=1}^{\infty} G_i^{(1)} \sqrt{U_1^{(-1)}\left(Q_i \Gamma_i\right)} \cdot I\left\{R_i \leq s\right\}, \tag{26}$$

$$Z_2(s) := \sum_{i=1}^{\infty} G_i^{(2)} \cdot \sqrt{U_2^{(-1)}(\Gamma_i)} \cdot I\left\{R_i \leq s\right\}, \tag{27}$$

where

- for $j = 1, 2$,
  $$U_j(x) = \int_x^{\infty} a_j e^{-b_j u} u^{-1} du$$
  are tail integrals of the gamma-processes $\mathcal{G}_j$;
- R_i is a sequence of i.i.d. r.v., uniformly distributed on [0, 1];
- Γ_i is an independent sequence of jump times of a standard Poisson process;
- $G_i^{(1)}, G_i^{(2)}$ - are two sequences of i.i.d. standard normal r.v.;
- Q_i is a sequence of i.i.d. random variables with distribution function
  $$F(z) = \left(1 + z^{-\delta}\right)^{-(\delta+1)/\delta},$$
  which is modelled by $\left(-1 + \xi_i^{-\delta/(\delta+1)}\right)^{-1/\delta}$ with i.i.d. random variables ξ_i, $i = 1..d$, uniformly distributed on [0, 1].

In this numerical example, we truncate series in Eqs. 26–27 according to Remark 5.5 with different h. Denote by $\vec{Z}^h(s) = \left(Z_1^h(s), Z_2^h(s)\right)$ the process, obtained from Eqs. 26–27 by keeping only $N(h)$ terms in the infinite sums.

For the simulation study, we set $\delta = 1, a_1 = 10000, b_1 = 0.1, a_2 = 500, b_2 = 1$, and simulate data for $M = 100$ time points on the equidistant grid with step $\Delta = 0.01$.

A typical trajectory of this simulation procedure is displayed on Fig. 1. Since the exact distribution function of the considered model cannot be calculated in closed form, we propose to check the closeness of the fitted model $\vec{Z}^h(s) = \left(Z_1^h(s), Z_2^h(s)\right)$ to the true model

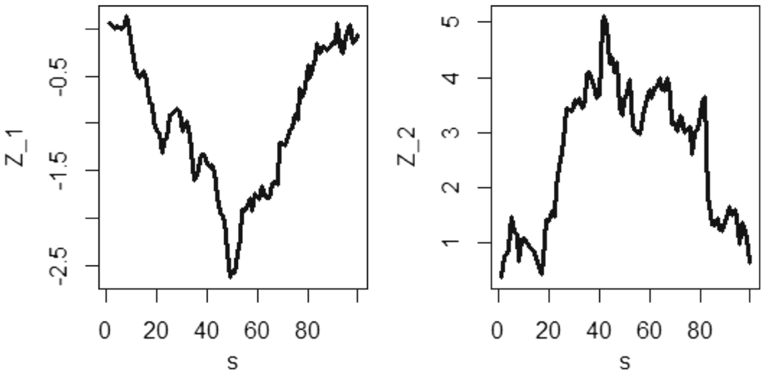

Fig. 1 Typical trajectories for the simulated data

$\vec{X}(s) := (W_1(\mathcal{G}_1(s)), W_2(\mathcal{G}_2(s))$ by comparing moments of these processes. In what follows, we will take into account that $\vec{X}(s)$ is a Lévy process (see Theorem 4.2), and therefore the increments

$$\Delta\vec{X}^{(j)} = \left(\Delta X_1^{(j)}, \Delta X_2^{(j)}\right) := \vec{X}(\Delta j) - \vec{X}(\Delta(j-1)), \;\; j = 1..M$$

form a sequence of i.i.d. random variables. Empirical increments are denoted by

$$\Delta\vec{Z}^h(j) := \vec{Z}^h(\Delta j) - \vec{Z}^h(\Delta(j-1)), \;\; j = 1..M.$$

Theoretical moments are equal to

$$\mathbb{E}\left[\Delta\vec{X}\right] = \vec{0}, \quad \mathbb{E}\left[\Delta X_1 \Delta X_2\right] = 0, \quad \mathbb{E}\left[\Delta X_1^2\right] = 1000, \qquad \mathbb{E}\left[\Delta X_2^2\right] = 5,$$

where the values for the second moments are calculated by the formula $\mathbb{E}\left[\Delta X_i^2\right] = \mathbb{E}\left[T_i(\Delta)\right] = \Delta a_i / b_i, \;\; i = 1, 2$. For simulations, we truncate series in Eqs. 26–27 according to Remark 5.5 with different h corresponding to $N(h) = 5000, 10000, 12000, 20000, 50000$. The values for its empirical moments

$$\widehat{\mathbb{E}}\left[\Delta\vec{Z}^h\right] = M^{-1}\sum_{j=1}^{M}\Delta\vec{Z}^h(j), \quad \widehat{\mathbb{E}}\left[\Delta\vec{Z}_1^h \Delta\vec{Z}_2^h\right] = M^{-1}\sum_{j=1}^{M}\Delta Z_1^h(j)\Delta Z_2^h(j),$$

$$\widehat{\mathbb{E}}\left[(\Delta\vec{Z}_i^h)^2\right] = M^{-1}\sum_{j=1}^{M}(\Delta Z_i^h(j))^2, \;\; i = 1, 2,$$

are given in Table 1. This table indicates that the theoretical moments good matched by the empirical moments.

An interesting point, which was raised by one of the referees, is that the speed of convergence crucially depends on the strength of dependence between the components, expressed in the parameter δ of the Lévy copula. If the parameter δ is relatively small (e.g., $\delta = 1$), the first component converges slower than the second. Otherwise, if δ is relatively large (e.g., $\delta = 10$), then the speed of convergence is approximately the same. To illustrate this worth mentioning, we compare the absolute values of relative errors for the second moments, that is,

$$\frac{\left|\widehat{\mathbb{E}}[(\Delta\vec{Z}_i^h)^2] - \mathbb{E}[\Delta X_i^2]\right|}{\mathbb{E}\left[\Delta X_i^2\right]}, \qquad i = 1, 2,$$

Table 1 Empirical moments of the simulated data

$N = N(h)$	h	$\widehat{\mathbb{E}}\left[\Delta\vec{Z}_1^h\right]$	$\widehat{\mathbb{E}}\left[\Delta\vec{Z}_2^h\right]$	$\widehat{\mathbb{E}}\left[\Delta\vec{Z}_1^h\Delta\vec{Z}_2^h\right]$	$\widehat{\mathbb{E}}[(\Delta\vec{Z}_1^h)^2]$	$\widehat{\mathbb{E}}[(\Delta\vec{Z}_2^h)^2]$
5000	4980	1.255	0.010	−8.718	440.159	5.221
10000	9830	−0.519	−0.082	−4.399	493.826	4.946
12000	11975	−0.262	−0.261	−3.511	676.704	5.096
20000	19954	−0.273	0.073	1.524	628.867	5.43
50000	50425	−0.162	0.018	0.43	889.857	5.082

for different values of the parameter δ of Lévy copula, see Fig. 2. Definitely, for $\delta = 1$ (leftmost plot) first component convergence much slower than the second, while for $\delta = 10$ (rightmost plot) the rates are approximately the same.

8 Real Data Example

In this section, we provide an example of the real data analysis, which uses the methodology described in Section 5. The full description of this study is given in the preprint by Panov and Sirotkin (2015).

Setup Consider the following model:

$$\vec{X}(s) = (X_1(s), X_2(s)) := \left(\widetilde{W}_1(\mathcal{T}_1(s)), \widetilde{W}_2(\mathcal{T}_2(s))\right) \tag{28}$$

where

$$\widetilde{W}_i(t) = \mu_i t + \sigma_i W_i(t), \qquad i = 1, 2, \tag{29}$$

$W_1(t), W_2(t)$ are two independent Brownian motions, $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_+$, and $(\mathcal{T}_1(s), \mathcal{T}_2(s))$ is a two-dimensional subordinator. The dependence between $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$ is described via the Clayton-Lévy copula

$$F(x_1, x_2; \delta) = (x_1^{-\delta} + x_2^{-\delta})^{-1/\delta}$$

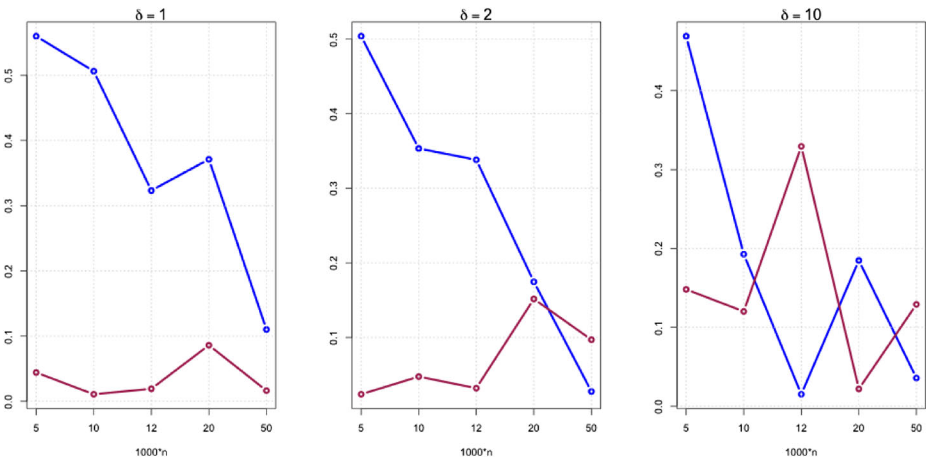

Fig. 2 Absolute values of the relative errors for the second moments for $\delta = 1, 2, 10$. Blue lines corresponds to the first component, maroon line - to the second component

with some $\delta > 0$.The marginal subordinators $\mathcal{T}_1(s)$ and $\mathcal{T}_2(s)$ belong to the class of compound Poisson processes with exponential jumps, that is,

$$\mathcal{T}_1(s) = \sum_{i=i}^{N_1(s)} X_i, \qquad \mathcal{T}_2(s) = \sum_{j=1}^{N_2(s)} Y_j, \tag{30}$$

where X_i and Y_i are i.i.d random variables with densities $f_1(x;\theta_1) = \theta_1 \exp(-\theta_1 x)$ and $f_2(x;\theta_2) = \theta_2 \exp(-\theta_2 x)$ for $x > 0$, $N_1(s)$ and $N_2(s)$ are Poisson processes with intensities λ_1 and λ_2 resp.

Data In what follows, we will apply this setup to modeling the stock returns. In this context, $\vec{X}(s)$ represents the returns of two stocks, and $(\mathcal{T}_1(s), \mathcal{T}_1(s))$ are cumulative numbers of trades of these stocks. Our approach can be considered as a generalization of the paper (Ané and Geman 2000), where the one-dimensional time-changed Brownian motion is used for representing some one-dimensional stock returns.

We examine 10- and 30-minutes Cisco, Intel and Microsoft prices traded on the Nasdaq over the period from the 25. August 2014 till the 21. November 2014. In addition to the prices, the number of trades is available for each equity. The length of time series is 832 observations for 30- minutes data and 2496 observations for 10- minutes data.

Step 1. *Estimation of the parameters* $\delta, \theta_1, \theta_2, \lambda_1, \lambda_2$. First, we aim to estimate the parameter of Lévy copula δ and the parameters of marginal subordinators (30). We follow recent papers (Esmaeli and Klüppelberg 2010, 2011) and apply a maximum likelihood estimation approach. In Section 6.2 from Panov and Sirotkin (2015), it is shown that the likelihood function of the continuously observed two-dimensional CPP process $(\mathcal{T}_1(s), \mathcal{T}_2(s))$ can be written in the following form, assuming that jumps occur at each moment for both components:

$$\begin{aligned} &L(\lambda_1, \lambda_2, \theta_1, \theta_2, \delta) \\ &= \left((1+\delta)\theta_1\theta_2(\lambda_1\lambda_2)^{\delta+1}\right)^n \exp\left\{-\lambda^{\parallel} T - (1+\delta)\left(\theta_1 \sum_{i=1}^{n} x_i + \theta_2 \sum_{i=1}^{n} y_i\right)\right\} \\ &\qquad \cdot \prod_{i=1}^{n} \left(\lambda_1^{\delta} \exp(-\theta_1 \delta x_i) + \lambda_2^{\delta} \exp(-\theta_2 \delta y_i)\right)^{-\frac{1}{\delta}-2}, \end{aligned} \tag{31}$$

where x_i and y_i, $i = 1..n$, are jumps of the first and the second components occuring up to some fixed time T, and $\lambda^{\parallel} = F(\lambda_1, \lambda_2; \delta)$. The results of the numerical maximization of $L(\lambda_1, \lambda_2, \theta_1, \theta_2, \delta)$ are presented in Table 2.

Step 2. *Estimation of the parameters of the processes* $\widetilde{W}_1(t)$ *and* $\widetilde{W}_2(t)$. On this stage, we estimate the parameters the parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$ applying the method of moments separately to the first and second components. Since for $i = 1, 2$, $\mathcal{T}_i(s)$ is a Lévy process, the increments $\Delta\mathcal{T}_i(s) := \mathcal{T}_i(s) - \mathcal{T}_i(s-1)$ form an i.i.d. sample. Next, taking into account that for $i = 1, 2$, $\mathcal{T}_i(s)$ is the CPP with intensity λ_i and jumps distributed by exponential law with parameter θ_i, we get

$$E[\Delta\mathcal{T}_i(s)] = \mu_i \frac{\lambda_i}{\theta_i}\Delta, \qquad \mathrm{Var}[\Delta\mathcal{T}_i(s)] = \frac{\sigma_i^2 \lambda_i \Delta}{\theta_i} + \frac{2\mu_i^2 \lambda_i \Delta}{\theta_i^2},$$

where Δ is the length of the observed equity. Solving the system of equations

$$E[\Delta\mathcal{T}_i(s)] = E[\widehat{\Delta\mathcal{T}_i(s)}], \qquad \mathrm{Var}[\Delta\mathcal{T}_i(s)] = \mathrm{Var}[\widehat{\Delta\mathcal{T}_i(s)}],$$

Table 2 MLE for the parameters of copula and marginal distributions

Pair	θ_1	θ_2	δ	λ_1	λ_2	Log likelihood value
30-minutes returns						
Csco vs Int	0,29	0,14	2,21	24,91	14,69	5161,43
Csco vs Msf	0,23	0,14	2,71	16,39	17,21	5196,93
Int vs Msf	0,14	0,17	2,38	14,41	24,68	5579,32
10-minutes returns						
Csco vs Int	0,85	0,43	1,76	74,18	48,60	10299,11
Csco vs Msf	0,71	0,42	2,11	52,66	55,78	10406,96
Int vs Msf	0,42	0,49	2,00	46,22	72,48	11511,90

where $E[\widehat{\Delta\mathcal{T}_i(s)}]$ and $\mathrm{Var}[\widehat{\Delta\mathcal{T}_i(s)}]$ are the sample mean and sample variance calculated by $\Delta\mathcal{T}_i(s), i = 1..n$, we arrive at the following estimates of the parameters μ_i and σ_i^2:

$$\hat{\mu}_i = \frac{\theta_i E[\widehat{\Delta\mathcal{T}_i(s)}]}{\lambda_i \Delta}, \qquad \hat{\sigma}_i^2 = \frac{\mathrm{Var}[\widehat{\Delta\mathcal{T}_i(s)}] - 2\hat{\mu}_i^2 \lambda_i \Delta/\theta_i}{\lambda_i \Delta}.$$

The results of this estimation procedure are presented in Table 2. Some further details on this part of the empirical analysis can be found in Section 6.3 from Panov and Sirotkin (2015).

Step 3. *Applying simulation techniques.*

Below we describe the simulation algorithm.

1. Fix some truncation level $h > 0$ and simulate i.i.d. standard exponential random variables (T_j) until $\Gamma_i := \sum_{j=1}^{i} T_j < h$. The maximal i is denoted by $N(h)$, see Remark 5.5.
2. Simulate $N(h)$ independent standard normal random variables $G_i^{(1)}$ and $G_i^{(2)}$, $i = 1, \ldots, N(h)$.

Table 3 Estimated values of the parameters of Brownian motions

Pair	μ_1	μ_2	σ_1^2	σ_2^2
30-minutes data				
csco intc	1,94E-08	7,25E-09	3,45E-09	4,93E-09
csco msft	2,31E-08	1,16E-08	4,10E-09	3,05E-09
intc msft	7,33E-09	1,00E-08	4,99E-09	3,84E-10
10-minutes data				
csco intc	-6,00E-09	-7,64E-10	5,17E-10	9,56E-10
csco msft	-7,05E-09	-2,63E-09	6,07E-10	4,14E-10
intc msft	-7,85E-10	-2,35E-09	9,82E-10	3,69E-10

3. Simulate $N(h)$ independent uniform random variables R_i on $[0, 1]$, $i = 1, \ldots, N(h)$.
4. Simulate $N(h)$ independent random variables Q_i with distribution function $H(z) = (z^{-\hat{\delta}} + 1)^{-(1+\hat{\delta})/\hat{\delta}}$ by the method of inverse function, that is, $Q_i = H^{-1}(\xi_i)$, where ξ_i are independent uniform random variables on $[0, 1]$, $i = 1, \ldots, N(h)$.
5. Simulate two subordinated Brownian motions by (truncated) series representation:

$$Z_1^h(s) := \sum_{i=1}^{k} \sqrt{U_1^{-1}(\Gamma_i)} \cdot G_i^{(1)} \cdot I\{R_i \le s\}, \tag{32}$$

$$Z_2^h(s) := \sum_{i=1}^{k} G_i^{(2)} \sqrt{U_2^{-1}(\Gamma_i Q_i)} \cdot I\{R_i \le s\}, \tag{33}$$

where the generalized inverse functions of $U_i(\cdot)$, $i = 1, 2$ are equal to

$$U_i^{(-1)}(x) = \begin{cases} -\frac{1}{\hat{\theta}_i} \log(\frac{x}{\hat{\lambda}_i}), & \text{for } x \le \hat{\lambda}_i, \\ 0, & \text{for } x > \hat{\lambda}_i. \end{cases} \tag{34}$$

6. Simulate a two-dimensional subordinator $(\mathcal{T}_1(s), \mathcal{T}_2(s))$ by the series representation for subordinators, see Algorithm 6.13 from Cont and Tankov (2004).
7. Resulting trajectory is a linear transform of subordinator and subordinated Brownian motion:

$$X_1^h(s) := \hat{\mu}_1 \mathcal{T}_1(s) + \hat{\sigma}_1^2 Z_1^h(s), \tag{35}$$

$$X_2^h(s) := \hat{\mu}_2 \mathcal{T}_2(s) + \hat{\sigma}_2^2 Z_2^h(s). \tag{36}$$

Discussion Our procedure has three steps. The first and the second step consist in the estimation of the parameters of Lévy copula δ, parameters of the marginal subordinators $\theta_1, \theta_2, \lambda_1, \lambda_2$, and the parameters of the Brownian motions with drifts $\mu_1, \sigma_1, \mu_2, \sigma_2$. On the third step, we simulate the data from the two-dimensional time-changed Lévy model based on the methodology described in Section 5.

As the result of this procedure we get the estimates of all parameters of our model (see Tables 2 and 3), and the two-dimensional trajectory $\vec{X}^h(s) := \left(X_1^h(s), X_2^h(s)\right)$ drawn by Eqs. 35–36 with the truncated series (32)–(33). The distribution of $\vec{X}^h(s)$ is close to the distribution of our model in the sense described in Section 5. The presented methodology can be further applied to predictive modeling or to calculating priors in Bayesian nonparametric methods.

Typical trajectories of simulated processes are presented in the Appendix. Figures 3 and 6 display trajectories for the time-changed Brownian motions modeled by Eqs. 32–33 for 30- and 10- minutes data. Figures 4 and 7 show typical trajectories for subordinators modeled as compound Poisson processes with exponential jumps. Finally, Figures 5 and 8 display resulting trajectories for the two-dimensional processes $\vec{X}^h(s)$.

Acknowledgment The author is grateful to Igor Sirotkin, the student of the Higher School of Economics, for his help with the preparation of Section 8. Moreover, the author thanks both referees for various constructive and quite valuable comments and suggestions on the paper.

Appendix

Graphs

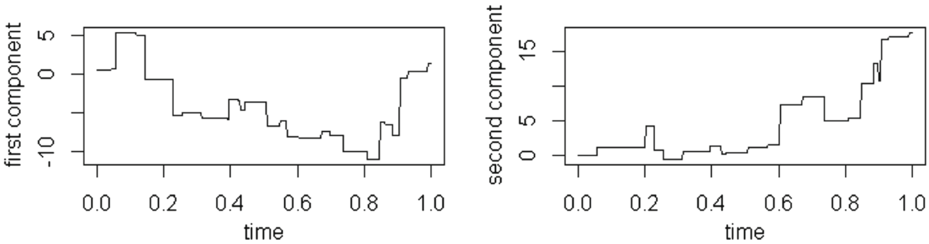

Fig. 3 Time-changed Brownian motion. Subordinators are CPP with exponential jumps. Parameters are estimated from the Cisco and Intel 30-minutes data

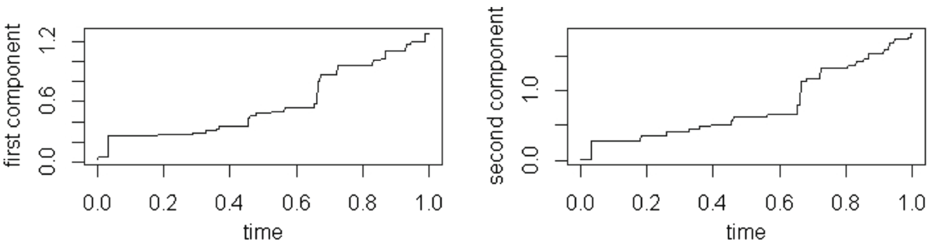

Fig. 4 Subordinators for 30 minute data. Parameters are estimated from the Cisco and intel 30-minute data

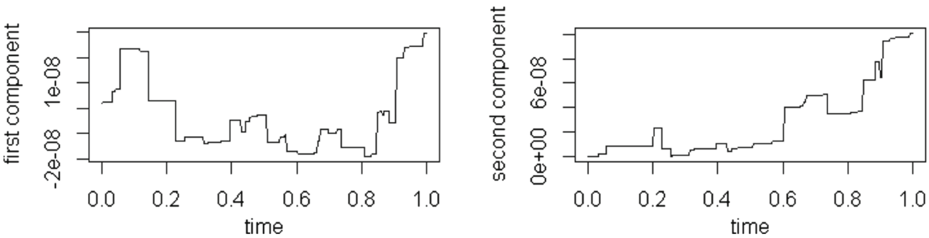

Fig. 5 Resulting trajectory of process $\vec{X}^h(t)$ for 30-minute data

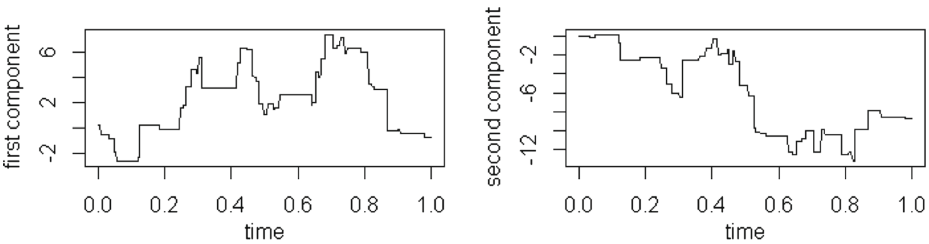

Fig. 6 Time-changed brownian motion. Subordinators are CPP with exponential jumps. Parameters are estimated from the Cisco and Microsoft 10-minutes data

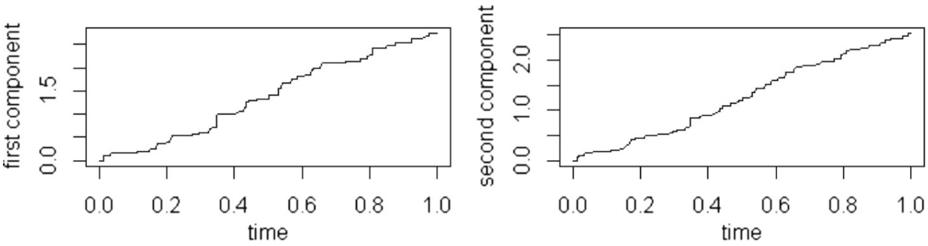

Fig. 7 Subordinators for 10-minute data. Parameters are estimated from Cisco and Microsoft 10-minute data

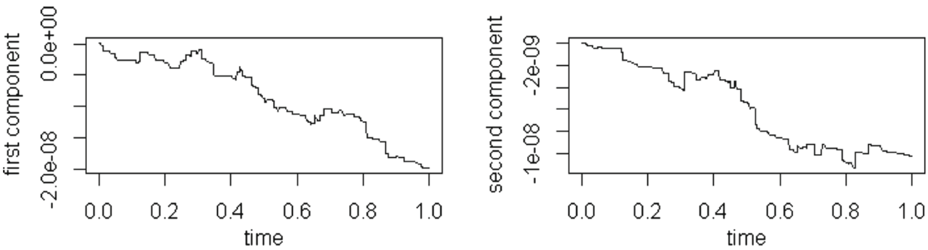

Fig. 8 Resulting trajectory of process $\vec{X}^h(s)$ for 10- minute data

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