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ABSTRACT SUPERPOSITION OPERATORS ON MAPPINGS OF BOUNDED VARIATION OF TWO REAL VARIABLES. I

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Abstract: We define and study the metric semigroup $\mathrm{BV}_2(I_a^b; M)$ of mappings of two real variables of bounded total variation in the Vitali–Hardy–Krause sense on a rectangle I_a^b with values in a metric semigroup or abstract convex cone M. We give a complete description for the Lipschitzian Nemytskii superposition operators acting from $\mathrm{BV}_2(I_a^b; M)$ to a similar semigroup $\mathrm{BV}_2(I_a^b; N)$ and, as a consequence, characterize set-valued superposition operators. We establish a connection between the mappings in $\mathrm{BV}_2(I_a^b; M)$ with the mappings of bounded iterated variation and study the iterated superposition operators on the mappings of bounded iterated variation. The results of this article develop and generalize the recent results by Matkowski and Miś (1984), Zawadzka (1990), and the author (2002, 2003) to the case of (set-valued) superposition operators on the mappings of two real variables.

Keywords: mappings of two variables, total variation, metric semigroup, Nemytskii superposition operator, set-valued operator, Banach algebra type property, Lipschitz condition

§ 1. Introduction

Let I, M, and N be some nonempty sets. Denote by M^I the family of all mappings from I to M. Given a mapping $h : I \times N \to M$, the operator $\mathscr{H} : N^I \to M^I$ acting by the rule $(\mathscr{H}g)(x) = h(x, g(x))$ for $x \in I$ and $g \in N^I$ is called the (*Nemytskii*) *superposition operator with generator* h.

The superposition operator is a classical "simplest" nonlinear operator between function spaces, and the bibliography devoted to its properties is rather extensive. This operator is studied sufficiently in the classes of measurable and continuous functions and the ideal spaces as well as the Lebesgue, Orlicz, Hölder, and Sobolev spaces (see [1–6] and the references therein). It turns out that the "nice" properties of h are not necessarily inherited by $\mathscr{H}$. For example, this is so with the behavior of the superposition operator in the Lebesgue spaces: smoothness and even analyticity of its generator in no way imply smoothness of the original superposition operator (these and other phenomena are presented in [4]).

Although the action of the superposition operator in most classical function spaces is described completely, little is known about this operator in the spaces BV of functions of bounded variation even in the case of a single real variable on an interval $I = [a, b] \subset \mathbb{R}$ for $N = M = \mathbb{R}$ [4, § 6.5; 7]. In this case the important class has been more thoroughly studied of the superposition operators satisfying the Lipschitz condition both univalent [8] and set-valued [9] (in the latter case M is a family of compact convex subsets of a normed space). In the case of a single variable, the Lipschitzian superposition operators in the classes of functions and mappings of bounded generalized variation and the classes of Lipschitz functions and mappings were characterized in [10–23]. For the real functions of bounded variation of two variables (in the Vitali–Hardy–Krause sense), the Lipschitzian superposition operators were described completely in [24, 25].

The first result on the characterization of the Lipschitzian superposition operators belongs to Matkowski [10]. Suppose that $I = [a, b]$, $M = N = \mathbb{R}$, and $B(I) \subset \mathbb{R}^I$ is some Banach function space with norm $\|\cdot\|$. We are interested in the conditions on the generator $h : I \times \mathbb{R} \to \mathbb{R}$ under which the corresponding superposition operator $\mathscr{H} : B(I) \to B(I)$ is Lipschitzian; i.e., there is a constant $L > 0$

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such that $\|\mathscr{H}g_1 - \mathscr{H}g_2\| \le L\|g_1 - g_2\|$ for all $g_1, g_2 \in B(I)$. Recall that (for $L < 1$) this operator $\mathscr{H}$ is closely connected with a solution $g \in B(I)$ to the functional equation $\mathscr{H}g = g$ by the Banach Contraction Mapping Theorem. In [10] it was demonstrated that if $B(I) = \mathrm{Lip}(I)$ is the space of Lipschitz functions on I with the usual Lipschitz norm then $\mathscr{H}$ satisfies the Lipschitz condition if and only if $h(x,u) = f(x)u + h_0(x)$ for all $x \in I$ and $u \in \mathbb{R}$, where f and h_0 are some functions in $\mathrm{Lip}(I)$. Note that such a representation for h does not hold in the space $B(I) = C(I)$ of continuous functions on I with the usual sup-norm and in the Lebesgue space $B(I) = L^p(I)$ of p-summable with $p \ge 1$ functions on I with the standard norm (for example, $h(x,u) = \cos u$, $x \in I$, $u \in \mathbb{R}$).

This result by Matkowski can be interpreted twofold. On the one hand, it demonstrates that the set of Lipschitzian operators on $\mathrm{Lip}(I)$ is rather poor (the generators h of these operators are linear in the second variable by necessity). On the other hand, the functional equation $\mathscr{H}g = g$ cannot be solved in $\mathrm{Lip}(I)$ by Banach's Theorem if the generator h depends nonlinearly on the second argument $u \in \mathbb{R}$ (and we should apply a more powerful fixed theorem, for example, Schauder's Theorem, etc.; see [26]). Practically the same situation holds for the space $B(I) = \mathrm{BV}(I)$ of functions on I of bounded Jordan variation (see [8] and Remark 6 at the end of § 3).

The goal of the present article is an exhausting description for the abstract Lipschitzian Nemytskii superposition operators in the spaces of mappings of bounded variation of several real variables with values in metric semigroups and abstract convex cones and also, as a consequence, a description for the generators of set-valued superposition operators (in this or another context, the mappings of bounded variation with values in metric spaces were studied in [9; 14; 23; 27, Chapter 4; 28] (the case of a single variable) and in [22, 29–31] (the case of several variables). In this article we only consider the mappings of bounded variation of two variables as introduced in [22, 31], since here the principal distinction from the one-dimensional case is most transparent. Moreover, our results extend these of [24, 25] and [32, § 8.3]; in a short form they were published in [33] and reported in [34]. A much more bulky general case of mappings of bounded variation of arbitrarily many variables and superposition operators on them will be published elsewhere.

The present article consists of five sections and splits into two parts. The first comprises § 1–§ 3. In § 2 we introduce and study the space of mappings of bounded variation of the Vitali–Hardy–Krause type with values in a metric semigroup (abstract convex cone) and show that this space itself is a metric semigroup (abstract convex cone). In § 3 we establish a necessary condition for the Lipschitz continuity of the superposition operator (Theorem 1) which is a two-dimensional analog of a condition in [8]; moreover, our results are new even for the superposition operators on the mappings of bounded variation of a single variable (see Remark 6 at the end of the first part). The second part includes § 4 and § 5. In § 4 we present a sufficient condition (Theorems 2 and 3) that generalizes the Banach algebra type condition of [25]. In the final § 5 we propose another description for the space of mappings of bounded variation of two variables and study the iterated Nemytskii superposition operator on the mappings of bounded iterated variation (Theorem 4).

Observe that, in general, our results are not valid in the broader BV space of [30], since this space does not possess the Banach algebra type property (cf. Theorem 2 of part II). Consideration of our space of mappings of bounded variation is natural for the reason that it is essentially connected with the integral representation of continuous linear functionals on the space of continuous functions on a rectangle [35, Chapter 2].

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§ 2. Semigroups and Cones of Mappings

Definition 1. A *metric semigroup* [22] is a triple $(M, d, +)$, where (M, d) is a metric space with metric d, while $(M, +)$ is an additive commutative semigroup with addition operation $+$, and d is translation-invariant: $d(u + w, v + w) = d(u, v)$ for all $u, v, w \in M$. A metric semigroup $(M, d, +)$ is *complete* if

(M, d) is a complete metric space. If M contains the *zero* element $0 \in M$ (so that $u + 0 = 0 + u = u$ for all $u \in M$) then we put $|u|_d = d(u, 0)$ for $u \in M$.

For arbitrary elements $u, v, \bar{u}, \bar{v} \in M$ of a metric semigroup $(M, d, +)$ we have

$$d(u, v) \leq d(u + \bar{u}, v + \bar{v}) + d(\bar{u}, \bar{v}), \tag{1}$$

$$d(u + \bar{u}, v + \bar{v}) \leq d(u, v) + d(\bar{u}, \bar{v}). \tag{2}$$

It follows from (2) that if some sequences $\{u_k\} = \{u_k\}_{k\in\mathbb{N}}$, $\{v_k\}$, $\{\bar{u}_k\}$, and $\{\bar{v}_k\}$ of elements of M converge to elements u, v, $\bar{u}$, and $\bar{v}$ of M as $k \to \infty$ then

$$\lim_{k\to\infty} d(u_k + \bar{u}_k, v_k + \bar{v}_k) = d(u + \bar{u}, v + \bar{v}); \tag{3}$$

in particular, the addition operation $(u, v) \mapsto u + v$ is a continuous mapping from $M \times M$ to M.

Definition 2. The quadruple $(M, d, +, \cdot)$ is an *abstract convex cone* if $(M, d, +)$ is a metric semigroup with zero $0 \in M$ and the operation $\cdot : \mathbb{R}^+ \times M \to M$ of multiplication of elements of M by nonnegative numbers defined by the rule $(\lambda, u) \mapsto \lambda u$ possesses the following properties for all $u, v \in M$ and $\lambda, \mu \in \mathbb{R}^+$: $\lambda(u + v) = \lambda u + \lambda v$, $(\lambda + \mu)u = \lambda u + \mu u$, $\lambda(\mu u) = (\lambda\mu)u$, $1 \cdot u = u$, and $d(\lambda u, \lambda v) = \lambda d(u, v)$. If (M, d) is complete then this cone is called *complete* and, as in Definition 1, given $u \in M$, we put $|u|_d = d(u, 0)$.

Observe that the following equality holds in an abstract convex cone $(M, d, +, \cdot)$:

$$d(\lambda u + \mu v, \lambda v + \mu u) = |\lambda - \mu| d(u, v), \quad \lambda, \mu \in \mathbb{R}^+, \ u, v \in M. \tag{4}$$

Consequently, $d(\lambda u, \mu v) \leq \lambda d(u, v) + |\lambda - \mu| \cdot |v|_d$ and so the operation of multiplication by nonnegative numbers in M is continuous.

The simplest example of a metric semigroup and an abstract convex cone is an arbitrary normed vector space $(Y, |\cdot|)$ with the induced metric $d(u, v) = |u - v|$, $u, v \in Y$, and the operations $+$ and $\cdot$ from Y. If $K \subset Y$ is a convex cone (i.e., $u + v, \lambda u \in K$ for all $u, v \in K$ and $\lambda \geq 0$) then $(K, d, +, \cdot)$ is an abstract convex cone complete whenever Y is a Banach space and K is closed in Y.

Let $(Y, |\cdot|)$ be a real normed vector space. Denote by $\mathrm{cbc}(Y)$ the family of all nonempty closed bounded convex subsets of Y with the Hausdorff metric D generated by the norm in Y:

$$D(P, Q) = \max\{\sup_{p\in P} \inf_{q\in Q} |p - q|, \sup_{q\in Q} \inf_{p\in P} |p - q|\}, \quad P, Q \in \mathrm{cbc}(Y).$$

For $P, Q \in \mathrm{cbc}(Y)$, we put $P + Q = \{p + q \mid p \in P, q \in Q\}$, $\lambda P = \{\lambda p \mid p \in P\}$, $\lambda \in \mathbb{R}^+$, and $P \overset{*}{+} Q = \mathrm{cl}(P + Q)$, where cl stands for the closure in Y. Then the following equalities [36, 37] hold in $\mathrm{cbc}(Y)$: $P \overset{*}{+} Q = \mathrm{cl}(\mathrm{cl}\, P + \mathrm{cl}\, Q)$, $\lambda(P \overset{*}{+} Q) = \lambda P \overset{*}{+} \lambda Q$, $(\lambda + \mu)P = \lambda P \overset{*}{+} \mu P$, $\lambda(\mu P) = (\lambda\mu)P$, and $D(\lambda P, \lambda Q) = \lambda D(P, Q)$ for all $\lambda, \mu \in \mathbb{R}^+$. Moreover, since (see [38, Lemma 3; 39, Lemma 2.2])

$$D(P \overset{*}{+} R, Q \overset{*}{+} R) = D(P + R, Q + R) = D(P, Q), \quad P, Q, R \in \mathrm{cbc}(Y),$$

$(\mathrm{cbc}(Y), D, \overset{*}{+}, \cdot)$ is an abstract convex cone; and this cone is complete if Y is a Banach space (which follows from the properties of the metric D; for example, see [40, Theorems II–9 and II–14]). Note that, by the above, the following two set-valued mappings are continuous: the $\overset{*}{+}$-addition operation in $\mathrm{cbc}(Y)$ and multiplication by numbers of $\mathbb{R}^+$. Other relevant examples of metric semigroups and abstract convex cones will appear below.

Suppose that (M, d) is a metric space and $[a, b] \subset \mathbb{R}$ is a closed interval. Recall that the (*Jordan*) *variation of a mapping* $\varphi : [a, b] \to M$ is the quantity

$$V_a^b(\varphi) = \sup_{\xi} \sum_{i=1}^{m} d(\varphi(t_i), \varphi(t_{i-1}))$$

(for example, see [27, Chapter 4, § 9; 28]), where the supremum is taken over all partitions $\xi = \{t_i\}_{i=0}^m$ of the interval $[a,b]$ (i.e., $m \in \mathbb{N}$ and $a = t_0 < t_1 < \dots < t_{m-1} < t_m = b$). If this quantity is finite then we say that φ is a *mapping of bounded variation* on $[a,b]$ and write $\varphi \in \mathrm{BV}_1([a,b];M)$. In the case when $(M,d,+)$ is a (complete) metric semigroup (abstract convex cone), we can introduce (see [9, 20, 23]) the structure of a (complete) metric semigroup (abstract convex cone) on $\mathrm{BV}_1([a,b];M)$, by defining the addition (and multiplication by nonnegative numbers) pointwise and the translation-invariant metric d_1 by the rule

$$d_1(\varphi,\psi) = d\big(\varphi(a),\psi(a)\big) + W_a^b(\varphi,\psi), \quad \varphi,\psi \in \mathrm{BV}_1([a,b];M),$$

where the semimetric $W_a^b(\varphi,\psi)$ called the *joint variation of* φ *and* ψ is

$$W_a^b(\varphi,\psi) = \sup_\xi \sum_{i=1}^m d(\varphi(t_i)+\psi(t_{i-1}), \psi(t_i)+\varphi(t_{i-1})). \tag{5}$$

Correctness of the above construction follows from the properties of $W_a^b(\varphi,\psi)$ (cf. [32, Lemmas 2.14 and 2.15]).

Lemma 1. *Let* $\varphi,\psi \in \mathrm{BV}_1([a,b];M)$. *Then*

(a) $|d(\varphi(t),\psi(t)) - d(\varphi(s),\psi(s))| \le d(\varphi(t)+\psi(s),\psi(t)+\varphi(s)) \le W_a^b(\varphi,\psi)$, $t,s \in [a,b]$;

(b) $d(\varphi(t),\psi(t)) \le d_1(\varphi,\psi)$ *for all* $t \in [a,b]$;

(c) $\big|V_a^b(\varphi) - V_a^b(\psi)\big| \le W_a^b(\varphi,\psi) \le V_a^b(\varphi) + V_a^b(\psi)$;

(d) $W_a^t(\varphi,\psi) + W_t^b(\varphi,\psi) = W_a^b(\varphi,\psi)$ *if* $t \in [a,b]$;

(e) *if sequences* $\{\varphi_k\}$ *and* $\{\psi_k\}$ *in* $\mathrm{BV}_1([a,b];M)$ *converge pointwise on* $[a,b]$ *to mappings* φ *and* ψ *then*

$$W_a^b(\varphi,\psi) \le \liminf_{k\to\infty} W_a^b(\varphi_k,\psi_k).$$

We turn to considering the mappings of bounded total variation of two real variables.

We write the coordinate representations of points $x, y \in \mathbb{R}^2$ in the form $x = (x_1,x_2)$ and $y = (y_1,y_2)$ and assume that $x \le y$ or $x < y$ (in $\mathbb{R}^2$) if these inequalities hold coordinatewise. Suppose that $a = (a_1,a_2) < b = (b_1,b_2)$ in $\mathbb{R}^2$ and $I_a^b = I_{a_1,a_2}^{b_1,b_2} = [a_1,b_1] \times [a_2,b_2]$ is a *basic* rectangle on the plane (the domain of most mappings). Given a mapping $f : I_a^b \to M$ and points $x_1 \in [a_1,b_1]$ and $x_2 \in [a_2,b_2]$, define the two mappings $f(\cdot,x_2) : [a_1,b_1] \to M$ and $f(x_1,\cdot) : [a_2,b_2] \to M$ of a single variable by the rules: $f(\cdot,x_2)(t) = f(t,x_2)$ for $t \in [a_1,b_1]$ and $f(x_1,\cdot)(s) = f(x_1,s)$ for $s \in [a_2,b_2]$.

Suppose that $(M,d,+)$ is a metric semigroup and I_a^b is a basic rectangle.

Definition 3. The (*Vitali*) *mixed difference* of a mapping $f : I_a^b \to M$ on a rectangle $I_x^y = [x_1,y_1] \times [x_2,y_2] \subset I_a^b$, where $x,y \in I_a^b$, $x \le y$, is defined by [22, 31]

$$\mathrm{md}\big(f, I_x^y\big) = \mathrm{md}\big(f, I_{x_1,x_2}^{y_1,y_2}\big) = d(f(x_1,x_2)+f(y_1,y_2), f(x_1,y_2)+f(y_1,x_2)).$$

We say that a pair (ξ,η) is a (net) *partition* of I_a^b if there exist $m,n \in \mathbb{N}$ such that $\xi = \{t_i\}_{i=0}^m$ is a partition of $[a_1,b_1]$ (see above) and $\eta = \{s_j\}_{j=0}^n$ is a partition of $[a_2,b_2]$. Then the mixed difference $\mathrm{md}(f,I_{ij})$ on the rectangles

$$I_{ij} = I_{t_{i-1},s_{j-1}}^{t_i,s_j} = [t_{i-1},t_i] \times [s_{j-1},s_j], \quad i = 1,\dots,m,\ j = 1,\dots,n, \tag{6}$$

which constitute this partition is calculated by the formula

$$\mathrm{md}\big(f, I_{t_{i-1},s_{j-1}}^{t_i,s_j}\big) = d(f(t_{i-1},s_{j-1})+f(t_i,s_j), f(t_{i-1},s_j)+f(t_i,s_{j-1})).$$

Definition 4. The *double variation* of a mapping $f : I_a^b \to M$ is defined by the rule (Vitali [41] for $M = \mathbb{R}$)

$$V_2\big(f, I_a^b\big) = \sup_{(\xi,\eta)} \sum_{i=1}^m \sum_{j=1}^n \mathrm{md}(f, I_{ij}),$$

where the supremum is taken over all partitions (ξ,η) of the rectangle I_a^b of the above form. The *total variation* (in the modification of Hardy and Krause, see [42] if $M=\mathbb{R}$) of a mapping f is the quantity

$$TV_d\big(f,I_a^b\big)=V_{a_1}^{b_1}(f(\cdot,a_2))+V_{a_2}^{b_2}(f(a_1,\cdot))+V_2\big(f,I_a^b\big), \tag{7}$$

and the class of all mappings of finite total variation is called the *space of mappings of bounded variation* (in the Vitali–Hardy–Krause sense) and denoted by $\mathrm{BV}_2\big(I_a^b;M\big)$.

Observe that the notion of total variation (7) was effectively applied to proving the Helly selection principle in $\mathrm{BV}_n\big(I_a^b;M\big)$ in [43, § III.6.5; 44, Theorem 3.2] (for $n=2$ and $M=\mathbb{R}$), [45, Theorem 4] (for $n\in\mathbb{N}$ and $M=\mathbb{R}$), and [31, Theorem 2] (for $n=2$ and a metric semigroup M).

The main properties of the double variation $V_2(\cdot,\cdot)$ are as follows: *Additivity* in the second argument; i.e., for every above-indicated partition (ξ,η) of the rectangle I_a^b generating subrectangles $\{I_{ij}\}_{i,j=1}^{m,n}$, we have

$$V_2\big(f,I_a^b\big)=\sum_{i=1}^{m}\sum_{j=1}^{n}V_2(f,I_{ij}); \tag{8}$$

and (sequential) *lower semicontinuity* in the first argument; i.e., if a sequence of mappings $f_k:I_a^b\to M$ converges pointwise on I_a^b in the metric d to a mapping $f:I_a^b\to M$ then the following inequality is valid:

$$V_2\big(f,I_a^b\big)\le\liminf_{k\to\infty}V_2\big(f_k,I_a^b\big). \tag{9}$$

Using additivity in the second argument and lower semicontinuity in the first argument of the Jordan variation $V_a^b(\cdot,\cdot)$ (for example, see [28, § 2]), we find that (9) remains valid if we replace $V_2(\cdot,\cdot)$ with $TV_d(\cdot,\cdot)$.

If $f\in\mathrm{BV}_2(I_a^b;M)$ then $f(\cdot,s)\in\mathrm{BV}_1([a_1,b_1];M)$ for all $s\in[a_2,b_2]$ and, similarly, $f(t,\cdot)\in\mathrm{BV}_1([a_2,b_2];M)$ for all $t\in[a_1,b_1]$; moreover, the following inequalities hold [25, 31]:

$$V_{x_1}^{y_1}(f(\cdot,s))\le V_{x_1}^{y_1}(f(\cdot,a_2))+V_2\big(f,I_{x_1,a_2}^{y_1,s}\big),\quad x_1,y_1\in[a_1,b_1],\ x_1\le y_1, \tag{10}$$

$$V_{x_2}^{y_2}(f(t,\cdot))\le V_{x_2}^{y_2}(f(a_1,\cdot))+V_2\big(f,I_{a_1,x_2}^{t,y_2}\big),\quad x_2,y_2\in[a_2,b_2],\ x_2\le y_2. \tag{11}$$

Given $f\in\mathrm{BV}_2\big(I_a^b;M\big)$, the function $\nu_f(x)=TV_d\big(f,I_a^x\big)$, $x\in I_a^b$, is called the *function of total variation* of f on I_a^b and possesses the following properties [25, 31]:

$$d(f(y),f(x))\le TV_d\big(f,I_x^y\big)\le\nu_f(y)-\nu_f(x),\quad x,y\in I_a^b,\ x\le y; \tag{12}$$

$$V_2\big(\nu_f,I_a^b\big)=V_2\big(f,I_a^b\big)\quad\text{and}\quad TV\big(\nu_f,I_a^b\big)=TV_d\big(f,I_a^b\big);$$

$$\text{the function } \nu_f:I_a^b\to\mathbb{R} \text{ is } \textit{completely monotone}; \tag{13}$$

i.e., $\nu_f(\cdot,a_2)$ is nondecreasing on the interval $[a_1,b_1]$, $\nu_f(a_1,\cdot)$ is nondecreasing on $[a_2,b_2]$, and $\nu_f(x_1,x_2)+\nu_f(y_1,y_2)-\nu_f(x_1,y_2)-\nu_f(y_1,x_2)\ge 0$ for all points $x,y\in I_a^b$, $x\le y$.

In the case when $(M,d,+)$ is a metric semigroup (abstract convex cone) the structure of a metric semigroup (abstract convex cone) on $\mathrm{BV}_2\big(I_a^b;M\big)$ is defined as follows [32, § 8.3]:

Definition 5. Let $f,g\in\mathrm{BV}_2\big(I_a^b;M\big)$. The *addition operation* $+$ (multiplication by a nonnegative number λ) in $\mathrm{BV}_2\big(I_a^b;M\big)$ is introduced pointwise: $(f+g)(x)=f(x)+g(x)$ $((\lambda f)(x)=\lambda f(x))$, $x\in I_a^b$, and the *translation-invariant metric* d_2 on $\mathrm{BV}_2\big(I_a^b;M\big)$ is defined by the rule

$$d_2(f,g)=d(f(a),g(a))+TW_d\big(f,g,I_a^b\big),$$

where the *joint total variation* of f and g is

$$TW_d\big(f,g,I_a^b\big)=W_{a_1}^{b_1}(f(\cdot,a_2),g(\cdot,a_2))+W_{a_2}^{b_2}(f(a_1,\cdot),g(a_1,\cdot))+W_2\big(f,g,I_a^b\big).$$

Here the first summand on the right-hand side is the quantity (5) calculated in the metric d for the mappings $t \mapsto f(t, a_2)$ and $t \mapsto g(t, a_2)$ on the interval $[a_1, b_1]$, the second summand has a similar meaning, and the *joint double variation* $W_2(f, g, I_a^b)$ of the mappings f and g is defined in the notations of (6) by the rule

$$W_2(f, g, I_a^b) = \sup_{(\xi,\eta)} \sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{md}_2(f, g, I_{ij}),$$

where the supremum is taken over all partitions $\xi = \{t_i\}_{i=0}^m$ and $\eta = \{s_j\}_{j=0}^n$ of the respective intervals $[a_1, b_1]$ and $[a_2, b_2]$ $(m, n \in \mathbb{N})$ and the *joint mixed difference* $\operatorname{md}_2(f, g, I_x^y)$ on the subrectangle $I_x^y = [x_1, y_1] \times [x_2, y_2] \subset I_a^b$ is

$$\operatorname{md}_2(f, g, I_{x_1,x_2}^{y_1,y_2}) = d(f(x_1, x_2) + f(y_1, y_2) + g(x_1, y_2) + g(y_1, x_2),$$
$$g(x_1, x_2) + g(y_1, y_2) + f(x_1, y_2) + f(y_1, x_2)).$$

Correctness of the definition of operations in $\mathrm{BV}_2(I_a^b; M)$ is verified immediately by using (2):

$$TV_d(f + g, I_a^b) \le TV_d(f, I_a^b) + TV_d(g, I_a^b) \quad (TV_d(\lambda f, I_a^b) = \lambda TV_d(f, I_a^b)).$$

The further verification of correctness of Definition 5 relies upon the main properties of the semimetric $TW_d(\cdot, \cdot, I_a^b)$ as given in the following

Lemma 2. *If $(M, d, +)$ is a metric semigroup and $f, g \in \mathrm{BV}_2(I_a^b; M)$ then*
(a) *$|d(f(y), g(y)) - d(f(x), g(x))| \le TW_d(f, g, I_x^y)$ for all $x, y \in I_a^b$, $x \le y$;*
(b) *$|TV_d(f, I_a^b) - TV_d(g, I_a^b)| \le TW_d(f, g, I_a^b) \le TV_d(f, I_a^b) + TV_d(g, I_a^b)$;*
(c) *if $\{f_k\}, \{g_k\} \subset \mathrm{BV}_2(I_a^b; M)$ and $d(f_k(x), f(x)) \to 0$, $d(g_k(x), g(x)) \to 0$ as $k \to \infty$ for all $x \in I_a^b$ then $TW_d(f, g, I_a^b) \le \liminf_{k\to\infty} TW_d(f_k, g_k, I_a^b)$.*

Proof. (a) Applying (1) thrice, using the invariance of d under translations, and given $x, y \in I_a^b$, $x \le y$, we find that

$$|d(f(y_1, y_2), g(y_1, y_2)) - d(f(x_1, x_2), g(x_1, x_2))|$$
$$\le d(f(y_1, y_2) + g(x_1, x_2), g(y_1, y_2) + f(x_1, x_2))$$
$$\le d(f(y_1, y_2) + g(x_1, x_2) + f(y_1, x_2) + g(y_1, y_2), g(y_1, y_2) + f(x_1, x_2) + f(y_1, y_2)$$
$$+g(y_1, x_2)) + d(f(y_1, x_2) + g(y_1, y_2), f(y_1, y_2) + g(y_1, x_2))$$
$$\le d(f(y_1, x_2) + g(x_1, x_2), g(y_1, x_2) + f(x_1, x_2))$$
$$+d(f(x_1, y_2) + g(x_1, x_2), g(x_1, y_2) + f(x_1, x_2)) + d(g(x_1, x_2) + g(y_1, y_2) + f(x_1, y_2) + f(y_1, x_2),$$
$$f(x_1, x_2) + f(y_1, y_2) + g(x_1, y_2) + g(y_1, x_2))$$
$$\le W_{x_1}^{y_1}(f(\cdot, x_2), g(\cdot, x_2)) + W_{x_2}^{y_2}(f(x_1, \cdot), g(x_1, \cdot)) + W_2(f, g, I_x^y) = TW_d(f, g, I_x^y).$$

(b) Note first that

$$|V_2(f, I_a^b) - V_2(g, I_a^b)| \le W_2(f, g, I_a^b) \le V_2(f, I_a^b) + V_2(g, I_a^b). \tag{14}$$

Indeed, for every subrectangle $I_x^y \subset I_a^b$ we have

$$\operatorname{md}(f, I_x^y) \le \operatorname{md}(g, I_x^y) + \operatorname{md}_2(f, g, I_x^y), \tag{15}$$

since, by (1),

$$\operatorname{md}(f, I_x^y) = d(f(x_1, x_2) + f(y_1, y_2), f(x_1, y_2) + f(y_1, x_2))$$
$$\le d(g(x_1, x_2) + g(y_1, y_2), g(x_1, y_2) + g(y_1, x_2))$$
$$+d(f(x_1, x_2) + f(y_1, y_2) + g(x_1, y_2) + g(y_1, x_2),$$
$$g(x_1, x_2) + g(y_1, y_2) + f(x_1, y_2) + f(y_1, x_2)) = \operatorname{md}(g, I_x^y) + \operatorname{md}_2(f, g, I_x^y)$$

and, similarly, by (2),

$$\mathrm{md}_2\big(f, g, I_x^y\big) \le \mathrm{md}\big(f, I_x^y\big) + \mathrm{md}\big(g, I_x^y\big). \tag{16}$$

From (15) and (16) we derive (14). From (7), Lemma 1(c), and (14) we obtain

$$\begin{aligned}
\big|TV_d\big(f, I_a^b\big) - TV_d\big(g, I_a^b\big)\big| &\le \big|V_{a_1}^{b_1}(f(\cdot, a_2)) - V_{a_1}^{b_1}(g(\cdot, a_2))\big| \\
&+ \big|V_{a_2}^{b_2}(f(a_1, \cdot)) - V_{a_2}^{b_2}(g(a_1, \cdot))\big| + \big|V_2\big(f, I_a^b\big) - V_2\big(g, I_a^b\big)\big| \\
&\le W_{a_1}^{b_1}(f(\cdot, a_2), g(\cdot, a_2)) + W_{a_2}^{b_2}(f(a_1, \cdot), g(a_1, \cdot)) + W_2\big(f, g, I_a^b\big) = TW_d\big(f, g, I_a^b\big) \\
&\le V_{a_1}^{b_1}(f(\cdot, a_2)) + V_{a_1}^{b_1}(g(\cdot, a_2)) + V_{a_2}^{b_2}(f(a_1, \cdot)) + V_{a_2}^{b_2}(g(a_1, \cdot)) + V_2\big(f, I_a^b\big) + V_2\big(g, I_a^b\big) \\
&= TV_d\big(f, I_a^b\big) + TV_d\big(g, I_a^b\big).
\end{aligned}$$

(c) Suppose that $\xi = \{t_i\}_{i=0}^m$ and $\eta = \{s_j\}_{j=0}^n$ are partitions of $[a_1, b_1]$ and $[a_2, b_2]$ and I_{ij} are generated rectangles (6). It follows from the definition of W_2 that

$$\sum_{i=1}^m \sum_{j=1}^n \mathrm{md}_2(f_k, g_k, I_{ij}) \le W_2(f_k, g_k, I_a^b), \quad k \in \mathbb{N}.$$

Passing to the limit inferior as $k \to \infty$ and using the pointwise convergence of f_k to f and g_k to g together with (3), we find that

$$\sum_{i=1}^m \sum_{j=1}^n \mathrm{md}_2(f, g, I_{ij}) \le \liminf_{k\to\infty} W_2(f_k, g_k, I_a^b),$$

whence $W_2\big(f, g, I_a^b\big) \le \liminf_{k\to\infty} W_2\big(f_k, g_k, I_a^b\big)$. Recalling Lemma 1(e), we obtain (c): we should use the inequality

$$\liminf(\alpha_k + \beta_k) \ge \liminf \alpha_k + \liminf \beta_k$$

for all real sequences $\{\alpha_k\}$ and $\{\beta_k\}$ where the right-hand side is not of the form $\mp\infty$ or $\pm\infty$. (Moreover, observe that $W_2(\cdot, \cdot)$ possesses the additivity property of the form (8).) $\square$

Lemma 3. *If $(M, d, +)$ is a (complete) metric semigroup (abstract convex cone) then so is $\big(\mathrm{BV}_2\big(I_a^b; M\big), d_2, +\big)$.*

Proof. Let $f, g \in \mathrm{BV}_2\big(I_a^b; M\big)$. It is clear that if $f = g$ then $d_2(f, g) = 0$ and if $d_2(f, g) = 0$ then, by Lemma 2(a),

$$d(f(x), g(x)) = d(f(a), g(a)) = 0, \quad x \in I_a^b,\ x \ne a;$$

i.e., $f = g$. The symmetry of d_2, the triangle inequality for d_2, and the invariance of d_2 under translations follow from the corresponding properties of d.

Let us establish completeness. Suppose that $\{f_k\} \subset \mathrm{BV}_2\big(I_a^b; M\big)$ is a Cauchy sequence; i.e., $d_2(f_k, f_j) \to 0$ as $k, j \to \infty$. Then from Lemma 2(a) we find that $\{f_k(x)\}$ is a Cauchy sequence in M for all $x \in I_a^b$; therefore, there is a mapping $f : I_a^b \to M$ such that $d(f_k(x), f(x)) \to 0$ as $k \to \infty$ for all $x \in I_a^b$. By Lemma 2(c),

$$TW_d\big(f_k, f, I_a^b\big) \le \liminf_{j\to\infty} TW_d\big(f_k, f_j, I_a^b\big) \le \lim_{j\to\infty} d_2(f_k, f_j), \quad k \in \mathbb{N}.$$

Since $\{f_k\}$ is a Cauchy sequence in $\mathrm{BV}_2\big(I_a^b; M\big)$, we have

$$\limsup_{k\to\infty} TW_d\big(f_k, f, I_a^b\big) \le \lim_{k\to\infty} \lim_{j\to\infty} d_2(f_k, f_j) = 0;$$

whence $d_2(f_k, f) \to 0$ as $k \to \infty$. It remains to note that $f \in \mathrm{BV}_2\big(I_a^b; M\big)$: by Lemma 2(b), $\big\{TV_d\big(f_k, I_a^b\big)\big\}$ is a Cauchy sequence in $\mathbb{R}$; therefore, it is bounded and convergent and, by (9), for $TV_d(\cdot, \cdot)$ we find that

$$TV_d\big(f, I_a^b\big) \le \lim_{k\to\infty} TV_d\big(f_k, I_a^b\big) < \infty. \qquad \square$$

Let (N, ρ) be a metric space and let $(M, d, +)$ be a metric semigroup (abstract convex cone). As usual, an operator $T : N \to M$ is called *Lipschitzian* if its (least) *Lipschitz constant* is finite:

$$L(T) = \sup\{d(Tu, Tv)/\rho(u, v) \mid u, v \in N,\ u \neq v\},$$

and the set of all these operators is denoted by $\mathrm{Lip}(N; M)$. This set is closed with respect to the pointwise addition (multiplication by $\lambda \in \mathbb{R}^+$), since $L(T + S) \leq L(T) + L(S)$ ($L(\lambda T) = \lambda L(T)$) for $T, S \in \mathrm{Lip}(N; M)$ by (2). Given a fixed $u_0 \in N$, the translation-invariant metric d_L on $\mathrm{Lip}(N; M)$ is defined by the rule (for example, see [16])

$$d_L(T, S) = d(Tu_0, Su_0) + d_\ell(T, S), \quad T, S \in \mathrm{Lip}(N; M), \tag{17}$$

where

$$d_\ell(T, S) = \sup\{d(Tu + Sv, Su + Tv)/\rho(u, v) \mid u, v \in N,\ u \neq v\}.$$

In the following lemma we give the properties of the translation-invariant semimetric d_ℓ:

Lemma 4. *The following hold for $T, S \in \mathrm{Lip}(N; M)$:*

(a) $|d(Tu, Su) - d(Tv, Sv)| \leq d(Tu + Sv, Su + Tv) \leq d_\ell(T, S)\rho(u, v)$ *for $u, v \in N$;*

(b) $|L(T) - L(S)| \leq d_\ell(T, S) \leq L(T) + L(S)$;

(c) if $\{T_k, S_k\} \subset \mathrm{Lip}(N; M)$ and $d(T_k u, Tu) \to 0$, $d(S_k u, Su) \to 0$ as $k \to \infty$ for all $u \in N$ then $d_\ell(T, S) \leq \liminf_{k\to\infty} d_\ell(T_k, S_k)$.

Thus, $(\mathrm{Lip}(N; M), d_L, +)$ is a metric semigroup (abstract convex cone) which is complete if $(X, d, +)$ is complete.

Let $(N, \rho, +)$ and $(M, d, +)$ be two metric semigroups. An operator $T : N \to M$ is called *additive* if it satisfies the Cauchy equation: $T(u + v) = Tu + Tv$ for all $u, v \in N$. Denote by $\mathrm{L}(N; M)$ the set of all Lipschitzian additive operators from N to M. If, in addition, N and M contain zeros (denoted by the same symbol 0) and $T \in \mathrm{L}(N; M)$ then $T(0) = 0$, for $T(0) = T(0 + 0) = T(0) + T(0)$ and $d(0, T(0)) = d(T(0), T(0) + T(0)) = 0$. In this case $d_L = d_\ell$ (see (17) for $u_0 = 0$) is a metric on $\mathrm{L}(N; M)$ and the equality $L(T) = d_L(T, 0) = |T|_{d_L}$ is valid.

If $(N, \rho, +, \cdot)$ and $(M, d, +, \cdot)$ are two abstract convex cones then every additive continuous operator $T : N \to M$ also possesses the property $T(\lambda u) = \lambda Tu$ for all $\lambda \in \mathbb{R}^+$ and $u \in N$. Indeed, let $\{\lambda_k\}$ be a sequence of positive rational numbers converging to λ as $k \to \infty$. Additivity of T implies that $T(\lambda_k u) = \lambda_k Tu$ and continuity of T implies that $d\big(T(\lambda u), T(\lambda_k u)\big) \to 0$ as $k \to \infty$. By (4),

$$d(T(\lambda_k u), \lambda Tu) = d(\lambda_k Tu, \lambda Tu) = |\lambda_k - \lambda| d(Tu, 0)$$

and therefore

$$d(T(\lambda u), \lambda Tu) \leq d(T(\lambda u), T(\lambda_k u)) + d(T(\lambda_k u), \lambda Tu) \to 0 \quad \text{as } k \to \infty.$$

§ 3. Lipschitzian Superposition Operators. A Necessary Condition

The central result of this section is Theorem 1 which gives a necessary condition for Lipschitz continuity of a superposition operator $\mathscr{H}$ between abstract convex cones $\mathrm{BV}_2\big(I_a^b; M\big)$. To state it, we need the notion of left-left regularization of a mapping in $\mathrm{BV}_2\big(I_a^b; M\big)$ and two auxiliary lemmas (Lemmas 5 and 6).

If $(M, d, +)$ is a complete metric semigroup then we define the *left-left regularization* $f^- : I_a^b \to M$ of a mapping $f \in \mathrm{BV}_2\big(I_a^b; M\big)$ by the rule [25]

$$f^-(x_1, x_2) = \begin{cases} \lim\limits_{(y_1,y_2)\to(x_1-0,x_2-0)} f(y_1, y_2) & \text{if } a_1 < x_1 \leq b_1 \text{ and } a_2 < x_2 \leq b_2, \\ \lim\limits_{(y_1,y_2)\to(x_1-0,a_2+0)} f(y_1, y_2) & \text{if } a_1 < x_1 \leq b_1 \text{ and } x_2 = a_2, \\ \lim\limits_{(y_1,y_2)\to(a_1+0,x_2-0)} f(y_1, y_2) & \text{if } x_1 = a_1 \text{ and } a_2 < x_2 \leq b_2, \\ \lim\limits_{(y_1,y_2)\to(a_1+0,a_2+0)} f(y_1, y_2) & \text{if } x_1 = a_1 \text{ and } x_2 = a_2. \end{cases}$$

We should note that the condition $(y_1, y_2) \to (x_1 - 0, x_2 - 0)$ (shortly, $y \to x - 0$) is understood in the sense $(y_1, y_2) \in I_a^b$, $y_1 < x_1$, $y_2 < x_2$, and $(y_1, y_2) \to (x_1, x_2)$ in $\mathbb{R}^2$; the other three limits are understood similarly, and the limits themselves are calculated in the metric space M. Existence of all these limits will be proven below in Lemma 5.

A mapping $f : I_a^b \to M$ is called *left-left continuous* if

$$\lim_{(y_1,y_2)\to(x_1-0,x_2-0)} f(y_1, y_2) = f(x_1, x_2) \quad \text{for all } x_1 \in (a_1, b_1] \text{ and } x_2 \in (a_2, b_2].$$

Denote by $\mathrm{BV}_2^-\big(I_a^b; M\big)$ the subspace of $\mathrm{BV}_2\big(I_a^b; M\big)$ constituted by left-left continuous mappings on $(a_1, b_1] \times (a_2, b_2]$.

Lemma 5. *If $(M, d, +)$ is a complete metric semigroup and $f \in \mathrm{BV}_2\big(I_a^b; M\big)$ then $f^- \in \mathrm{BV}_2^-\big(I_a^b; M\big)$; moreover,*

$$V_2\big(f^-, I_a^b\big) \le V_2\big(f, I_a^b\big) \quad \text{and} \quad TV_d\big(f^-, I_a^b\big) \le 3TV_d\big(f, I_a^b\big).$$

Proof. 1. Let us show that the mapping f^- is defined correctly. Using (13), from [43, § III.5.3] we obtain existence of the left-left regularization $\nu_f^- : I_a^b \to \mathbb{R}$ of ν_f. Let us prove the existence of the limit $f^-(x) \in M$, for example, at a point $x = (x_1, x_2)$, where $x_i \in (a_i, b_i]$, $i = 1, 2$ (the other three possibilities are considered similarly). Take $y', y'' \in I_a^b$ such that $y' < x$ and $y'' < x$. If $y' \le y''$ or $y'' \le y'$ then, by (12),

$$d(f(y'), f(y'')) \le |\nu_f(y') - \nu_f(y'')| \to |\nu_f^-(x) - \nu_f^-(x)| = 0 \quad \text{as } y', y'' \to x.$$

If $y_1'' < y_1'$ and $y_2' < y_2''$ then, using again (12), we obtain

$$\begin{aligned} d(f(y'), f(y'')) &\le d(f(y_1', y_2'), f(y_1', y_2'')) + d(f(y_1', y_2''), f(y_1'', y_2'')) \\ &\le \nu_f(y_1', y_2'') - \nu_f(y_1', y_2') + \nu_f(y_1', y_2'') - \nu_f(y_1'', y_2'') \\ &\to \nu_f^-(x) - \nu_f^-(x) + \nu_f^-(x) - \nu_f^-(x) = 0 \quad \text{as } y', y'' \to x. \end{aligned}$$

Similarly, we examine the case when $y_1' < y_1''$ and $y_2'' < y_2'$. Thus, $d(f(y'), f(y'')) \to 0$ as $y', y'' \to x$, and we are left with applying Cauchy's criterion for existence of the limit of $f(y)$ as $y \to x - 0$ in a complete space M.

2. Let us show that f^- is left continuous at all points $x \in I_a^b$, $a < x \le b$. By [43, § III.5.4], all discontinuity points of a completely monotone function ν_f lie on at most countably many lines parallel to the coordinate axes. Then (12) implies that this property is enjoyed by the discontinuity points of f. Therefore, there is a sequence $\{y_k\} \subset I_a^b$ of continuity points of f such that $y_k < x$ for all $k \in \mathbb{N}$ and $y_k \to x$ in $\mathbb{R}^2$ as $k \to \infty$. Hence,

$$\lim_{y\to x-0} f^-(y) = \lim_{k\to\infty} f^-(y_k) = \lim_{k\to\infty} f(y_k) = \lim_{y\to x-0} f(y) = f^-(x) \quad \text{in } M.$$

3. Let us prove that f^- lies in $\mathrm{BV}_2\big(I_a^b; M\big)$. Suppose that $a_1 = t_0 < t_1 < \cdots < t_{m-1} < t_m = b_1$, $a_2 = s_0 < s_1 < \cdots < s_{n-1} < s_n = b_2$, and $\varepsilon > 0$ is given. By the definition of f^-, there exist points $t_i' \in (t_{i-1}, t_i)$, $i = 1, \ldots, m$, $s_j' \in (s_{j-1}, s_j)$, $j = 1, \ldots, n$, $t_0' \in (a_1, t_1')$, and $s_0' \in (a_2, s_1')$ such that

$$d(f^-(t_i, s_j), f(t_i', s_j')) \le \varepsilon/(4mn), \quad i = 0, 1, \ldots, m, \; j = 0, 1, \ldots, n.$$

Then, applying the triangle inequality and (2) together with (6), we obtain

$$\begin{aligned} \mathrm{md}(f^-, I_{ij}) &= d(f^-(t_{i-1}, s_{j-1}) + f^-(t_i, s_j), f^-(t_{i-1}, s_j) + f^-(t_i, s_{j-1})) \\ &\le d(f(t_{i-1}', s_{j-1}') + f(t_i', s_j'), f(t_{i-1}', s_j') + f(t_i', s_{j-1}')) \\ &+ d(f^-(t_{i-1}, s_{j-1}), f(t_{i-1}', s_{j-1}')) + d(f^-(t_i, s_j), f(t_i', s_j')) \\ &+ d(f^-(t_{i-1}, s_j), f(t_{i-1}', s_j')) + d(f^-(t_i, s_{j-1}), f(t_i', s_{j-1}')) \le \mathrm{md}(f, I_{ij}') + \varepsilon/(mn), \end{aligned}$$

where $I'_{ij} = [t'_{i-1}, t'_i] \times [s'_{j-1}, s'_j]$, $i = 1, \dots, m$, $j = 1, \dots, n$. Summing over these i and j, taking the supremum over all partitions of I_a^b, and using the arbitrariness of $\varepsilon > 0$, we find that $V_2\big(f^-, I_a^b\big) \le V_2\big(f, I_a^b\big)$.

To show that $V_{a_1}^{b_1}(f^-(\cdot, a_2)) < \infty$, take $a_1 = t_0 < t_1 < \dots < t_{m-1} < t_m = b_1$ and $\varepsilon > 0$. By the definition of f^-, we find $t'_i \in (t_{i-1}, t_i)$, $i = 1, \dots, m$, $t'_0 \in (a_1, t'_1)$, and $s_0 \in (a_2, b_2)$ such that

$$d(f^-(t_i, a_2), f(t'_i, s_0)) \le \varepsilon/(2m), \quad i = 0, 1, \dots, m.$$

From the triangle inequality for $i = 1, \dots, m$ we obtain

$$\begin{aligned} d(f^-(t_i, a_2), f^-(t_{i-1}, a_2)) &\le d(f(t'_i, s_0), f(t'_{i-1}, s_0)) + d(f^-(t_i, a_2), f(t'_i, s_0)) \\ +d(f^-(t_{i-1}, a_2), f(t'_{i-1}, s_0)) &\le d(f(t'_i, s_0), f(t'_{i-1}, s_0)) + (\varepsilon/m). \end{aligned}$$

Summing over i and applying (10), we find that

$$\sum_{i=1}^{m} d(f^-(t_i, a_2), f^-(t_{i-1}, a_2)) \le V_{a_1}^{b_1}(f(\cdot, s_0)) + \varepsilon \le V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2(f, I_{a_1, a_2}^{b_1, s_0}) + \varepsilon,$$

whence $V_{a_1}^{b_1}(f^-(\cdot, a_2)) \le V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2\big(f, I_a^b\big)$. Using (11), we similarly obtain the following estimate: $V_{a_2}^{b_2}(f^-(a_1, \cdot)) \le V_{a_2}^{b_2}(f(a_1, \cdot)) + V_2\big(f, I_a^b\big)$. $\square$

Some particular cases of the following lemma for operators T with compact convex values were also established in [46, Theorem 2; 47, Theorem 5.6]:

Lemma 6 [17, Theorem 1 and Corollary 2]. *Suppose that $(N, +)$ is a commutative semigroup with zero and division by 2 and $(M, d, +, \cdot)$ is a complete abstract convex cone. Then a mapping $T : N \to M$ satisfies the Jensen functional equation*

$$2T\left(\frac{u+v}{2}\right) = Tu + Tv \quad \text{in } M \text{ for all } u, v \in N$$

if and only if there exist a unique additive mapping $A : N \to M$ and a constant $h_0 \in M$ such that $Tu = Au + h_0$ for all $u \in N$.

The main result of the present section is the following

Theorem 1. *Suppose that $(N, \rho, +, \cdot)$ and $(M, d, +, \cdot)$ are two abstract convex cones such that M is complete and a mapping $h : I_a^b \times N \to M$ is the generator of a superposition operator $\mathscr{H}$ for $I = I_a^b$. If $\mathscr{H} \in \mathrm{Lip}\big(\mathrm{BV}_2\big(I_a^b; N\big); \mathrm{BV}_2\big(I_a^b; M\big)\big)$ then $h(x, \cdot) \in \mathrm{Lip}(N; M)$ for all $x \in I_a^b$ and there exist two mappings $f : I_a^b \to \mathrm{L}(N; M)$ and $h_0 : I_a^b \to M$ such that $f(\cdot)u$, $h_0 \in \mathrm{BV}_2^-\big(I_a^b; M\big)$ for all $u \in N$ and the representation $h^-(x, u) = f(x)u + h_0(x)$ holds for all $x \in I_a^b$ and $u \in N$, where $f(\cdot)u$ acts by the rule $x \mapsto f(x)u$ and $h^-(\cdot, u)$ is the left-left regularization of the mapping $h(\cdot, u)$ for each fixed $u \in N$.*

Proof. The Lipschitz continuity of $\mathscr{H}$ and Definition 5 of the metrics ρ_2 and d_2 on $\mathrm{BV}_2\big(I_a^b; N\big)$ and $\mathrm{BV}_2\big(I_a^b; M\big)$ yield the inequality $d_2(\mathscr{H} g_1, \mathscr{H} g_2) \le L(\mathscr{H}) \rho_2(g_1, g_2)$ for all $g_1, g_2 \in \mathrm{BV}_2\big(I_a^b; N\big)$ whose expanded form is

$$\begin{aligned} &d((\mathscr{H} g_1)(a), (\mathscr{H} g_2)(a)) + W_{a_1}^{b_1}((\mathscr{H} g_1)(\cdot, a_2), (\mathscr{H} g_2)(\cdot, a_2)) \\ &\quad + W_{a_2}^{b_2}((\mathscr{H} g_1)(a_1, \cdot), (\mathscr{H} g_2)(a_1, \cdot)) + W_2\big(\mathscr{H} g_1, \mathscr{H} g_2, I_a^b\big) \\ &\le L(\mathscr{H})\big(\rho(g_1(a), g_2(a)) + W_{a_1}^{b_1}(g_1(\cdot, a_2), g_2(\cdot, a_2)) \\ &\quad + W_{a_2}^{b_2}(g_1(a_1, \cdot), g_2(a_1, \cdot)) + W_2\big(g_1, g_2, I_a^b\big)\big). \end{aligned} \tag{18}$$

1. Show first that $h(x,\cdot) \in \mathrm{Lip}(N;M)$ for all $x \in I_a^b$. (Observe that the arguments of this step can be applied for every metric semigroup M.) For a point $x=(x_1,x_2)\in I_a^b$ we observe the following four possible cases:

(i) $a_1 < x_1 \le b_1$ and $a_2 < x_2 \le b_2$;
(ii) $a_1 < x_1 \le b_1$ and $x_2 = a_2$;
(iii) $x_1 = a_1$ and $a_2 < x_2 \le b_2$;
(iv) $x_1 = a_1$ and $x_2 = a_2$.

Define the functions $\zeta_{\alpha,\beta} \in \mathrm{Lip}(\mathbb{R};[0,1])$, where $\alpha,\beta\in\mathbb{R}$ and $\alpha<\beta$, by the rules

$$\zeta_{\alpha,\beta}(t) = \begin{cases} 0 & \text{if } t \le \alpha, \\ (t-\alpha)/(\beta-\alpha) & \text{if } \alpha \le t \le \beta, \\ 1 & \text{if } t \ge \beta. \end{cases} \tag{19}$$

Let $u_1,\,u_2 \in N$ be arbitrary.

Case (i). Define two mappings $g_1,\,g_2 \in \mathrm{BV}_2\big(I_a^b;N\big)$ by the rules

$$g_k(y_1,y_2) = \frac{1}{2}(\zeta_{a_1,x_1}(y_1) + \zeta_{a_2,x_2}(y_2))u_k, \quad y_k\in[a_k,b_k],\ k=1,2,$$

and note that $g_k(a)=0$, $k=1,2$. By (4),

$$W_{a_1}^{b_1}(g_1(\cdot,a_2),g_2(\cdot,a_2)) = \frac{1}{2}V_{a_1}^{b_1}(\zeta_{a_1,x_1})\rho(u_1,u_2) = \rho(u_1,u_2)/2$$

and similarly $W_{a_2}^{b_2}(g_1(a_1,\cdot),g_2(a_1,\cdot)) = \rho(u_1,u_2)/2$; moreover, $W_2\big(g_1,g_2,I_a^b\big)=0$; therefore, $\rho_2(g_1,g_2) = \rho(u_1,u_2)$ on the right-hand side of (18). Recalling that

$$(\mathscr{H}g_k)(a_1,a_2) = h(a_1,a_2,g_k(a_1,a_2)) = h(a_1,a_2,0), \quad k=1,2,$$

and using (1) and the invariance of d under translations, from (18) we find that

$$\begin{aligned} d(h(x,u_1),h(x,u_2)) &= d((\mathscr{H}g_1)(x_1,x_2),(\mathscr{H}g_2)(x_1,x_2)) \\ &\le d((\mathscr{H}g_1)(x_1,a_2)+(\mathscr{H}g_2)(a_1,a_2),(\mathscr{H}g_2)(x_1,a_2)+(\mathscr{H}g_1)(a_1,a_2)) \\ &\quad + d((\mathscr{H}g_1)(a_1,x_2)+(\mathscr{H}g_2)(a_1,a_2),(\mathscr{H}g_2)(a_1,x_2)+(\mathscr{H}g_1)(a_1,a_2)) \\ &\quad + d((\mathscr{H}g_1)(a_1,a_2)+(\mathscr{H}g_1)(x_1,x_2)+(\mathscr{H}g_2)(a_1,x_2)+(\mathscr{H}g_2)(x_1,a_2), \\ &\qquad (\mathscr{H}g_2)(a_1,a_2)+(\mathscr{H}g_2)(x_1,x_2)+(\mathscr{H}g_1)(a_1,x_2)+(\mathscr{H}g_1)(x_1,a_2)) \\ &\le W_{a_1}^{b_1}((\mathscr{H}g_1)(\cdot,a_2),(\mathscr{H}g_2)(\cdot,a_2)) + W_{a_2}^{b_2}((\mathscr{H}g_1)(a_1,\cdot),(\mathscr{H}g_2)(a_1,\cdot)) \\ &\quad + W_2(\mathscr{H}g_1,\mathscr{H}g_2,I_a^b) = d_2(\mathscr{H}g_1,\mathscr{H}g_2) \le L(\mathscr{H})\rho_2(g_1,g_2) = L(\mathscr{H})\rho(u_1,u_2) \end{aligned}$$

and thereby arrive at the claim in this case.

Cases (ii) and (iii). In Case (ii) we put $g_k(y_1,y_2)=\zeta_{a_1,x_1}(y_1)u_k$ for all $y_k\in[a_k,b_k]$, $k=1,2$. Then $g_k(a)=0$, $k=1,2$, $W_{a_1}^{b_1}(g_1(\cdot,a_2),g_2(\cdot,a_2)) = \rho(u_1,u_2)$, $W_{a_2}^{b_2}(g_1(a_1,\cdot),g_2(a_1,\cdot))=0$, and $W_2(g_1,g_2,I_a^b)=0$; therefore, $\rho_2(g_1,g_2)=\rho(u_1,u_2)$. Since $g_k(x_1,a_2)=u_k$, $k=1,2$, from (18) we find that

$$\begin{aligned} d(h(x_1,a_2,u_1),h(x_1,a_2,u_2)) &= d((\mathscr{H}g_1)(x_1,a_2),(\mathscr{H}g_2)(x_1,a_2)) \\ &= d((\mathscr{H}g_1)(x_1,a_2)+(\mathscr{H}g_2)(a_1,a_2),(\mathscr{H}g_2)(x_1,a_2)+(\mathscr{H}g_1)(a_1,a_2)) \\ &\le W_{a_1}^{b_1}((\mathscr{H}g_1)(\cdot,a_2),(\mathscr{H}g_2)(\cdot,a_2)) = d_2(\mathscr{H}g_1,\mathscr{H}g_2) \le L(\mathscr{H})\rho(u_1,u_2). \end{aligned}$$

In Case (iii) we put $g_k(y_1,y_2)=\zeta_{a_2,x_2}(y_2)u_k$ for all $y_k\in[a_k,b_k]$, $k=1,2$, and argue similarly.

Case (iv). Putting

$$g_k(y_1,y_2) = \frac{1}{2}(2-\zeta_{a_1,b_1}(y_1)-\zeta_{a_2,b_2}(y_2))u_k, \quad y_k\in[a_k,b_k],\ k=1,2,$$

we obtain $g_k(a) = u_k$, $k = 1, 2$,

$$W_{a_1}^{b_1}(g_1(\cdot, a_2), g_2(\cdot, a_2)) = W_{a_2}^{b_2}(g_1(a_1, \cdot), g_2(a_1, \cdot)) = \rho(u_1, u_2)/2,$$

and $W_2\big(g_1, g_2, I_a^b\big) = 0$; therefore, $\rho_2(g_1, g_2) = 2\rho(u_1, u_2)$. Recalling that $(\mathscr{H} g_k)(b_1, b_2) = h(b_1, b_2, 0)$, $k = 1, 2$, from (18) we find that

$$\begin{aligned} d(h(a_1, a_2, u_1), h(a_1, a_2, u_2)) &= d((\mathscr{H} g_1)(a_1, a_2), (\mathscr{H} g_2)(a_1, a_2)) \\ \le d((\mathscr{H} g_1)(b_1, a_2) &+ (\mathscr{H} g_2)(a_1, a_2), (\mathscr{H} g_2)(b_1, a_2) + (\mathscr{H} g_1)(a_1, a_2)) \\ +d((\mathscr{H} g_1)(a_1, b_2) &+ (\mathscr{H} g_2)(a_1, a_2), (\mathscr{H} g_2)(a_1, b_2) + (\mathscr{H} g_1)(a_1, a_2)) \\ +d((\mathscr{H} g_1)(a_1, a_2) &+ (\mathscr{H} g_1)(b_1, b_2) + (\mathscr{H} g_2)(a_1, b_2) + (\mathscr{H} g_2)(b_1, a_2), \\ (\mathscr{H} g_2)(a_1, a_2) &+ (\mathscr{H} g_2)(b_1, b_2) + (\mathscr{H} g_1)(a_1, b_2) + (\mathscr{H} g_1)(b_1, a_2)) \\ \le W_{a_1}^{b_1}((\mathscr{H} g_1)(\cdot, a_2), &(\mathscr{H} g_2)(\cdot, a_2)) + W_{a_2}^{b_2}((\mathscr{H} g_1)(a_1, \cdot), (\mathscr{H} g_2)(a_1, \cdot)) \\ +W_2(\mathscr{H} g_1, \mathscr{H} g_2, I_a^b) &\le d_2(\mathscr{H} g_1, \mathscr{H} g_2) \le 2L(\mathscr{H})\rho(u_1, u_2), \end{aligned}$$

which completes the proof of the first assertion.

2. Now, establish the representation for $h^-(x, u)$. We first take $x = (x_1, x_2) \in I_a^b$, where $x_1 \in (a_1, b_1]$ and $x_2 \in (a_2, b_2]$. Also, let $m \in \mathbb{N}$, $a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < x_1$ and $a_2 < \bar{\alpha}_1 < \bar{\beta}_1 < \bar{\alpha}_2 < \bar{\beta}_2 < \cdots < \bar{\alpha}_m < \bar{\beta}_m < x_2$. Inequality (18) and Definition 5 imply in particular that

$$\begin{aligned} &\sum_{i=1}^{m} d((\mathscr{H} g_1)(\beta_i, a_2) + (\mathscr{H} g_2)(\alpha_i, a_2), (\mathscr{H} g_2)(\beta_i, a_2) + (\mathscr{H} g_1)(\alpha_i, a_2)) \\ &+\sum_{i=1}^{m} d((\mathscr{H} g_1)(a_1, \bar{\beta}_i) + (\mathscr{H} g_2)(a_1, \bar{\alpha}_i), (\mathscr{H} g_2)(a_1, \bar{\beta}_i) + (\mathscr{H} g_1)(a_1, \bar{\alpha}_i)) \\ &\qquad +W_2\big(\mathscr{H} g_1, \mathscr{H} g_2, I_a^b\big) \le L(\mathscr{H})\rho_2(g_1, g_2). \end{aligned} \tag{20}$$

Let $\zeta_m : [a_1, b_1] \to [0, 1]$ and $\bar{\zeta}_m : [a_2, b_2] \to [0, 1]$ be the two Lipschitz continuous functions defined as follows:

$$\zeta_m(t) = \begin{cases} 0 & \text{if } a_1 \le t \le \alpha_1, \\ \zeta_{\alpha_i,\beta_i}(t) & \text{if } \alpha_i \le t \le \beta_i,\ i = 1, \ldots, m, \\ 1 - \zeta_{\beta_i,\alpha_{i+1}}(t) & \text{if } \beta_i \le t \le \alpha_{i+1},\ i = 1, \ldots, m-1, \\ 1 & \text{if } \beta_m \le t \le b_1, \end{cases} \tag{21}$$

$$\bar{\zeta}_m(s) = \begin{cases} 0 & \text{if } a_2 \le s \le \bar{\alpha}_1, \\ \zeta_{\bar{\alpha}_i,\bar{\beta}_i}(s) & \text{if } \bar{\alpha}_i \le s \le \bar{\beta}_i,\ i = 1, \ldots, m, \\ 1 - \zeta_{\bar{\beta}_i,\bar{\alpha}_{i+1}}(s) & \text{if } \bar{\beta}_i \le s \le \bar{\alpha}_{i+1},\ i = 1, \ldots, m-1, \\ 1 & \text{if } \bar{\beta}_m \le s \le b_2, \end{cases}$$

where $\zeta_{\alpha,\beta}$ are defined by (19). Given $u_1, u_2 \in N$, $y_1 \in [a_1, b_1]$, $y_2 \in [a_2, b_2]$, and $k = 1, 2$, we put

$$g_k(y_1, y_2) = \frac{1}{4}(\zeta_m(y_1) + \bar{\zeta}_m(y_2))u_1 + \frac{1}{4}(2 - \zeta_m(y_1) - \bar{\zeta}_m(y_2))u_2 + \frac{1}{2}u_k.$$

Since $\rho(g_1(y), g_2(y)) = \rho(u_1, u_2)/2$ for all $y \in I_a^b$, we have $\rho_2(g_1, g_2) = \rho(u_1, u_2)/2$. The following inequality holds for all $i = 1, \ldots, m$:

$$\begin{aligned} &d((\mathscr{H} g_1)(\beta_i, \bar{\beta}_i) + (\mathscr{H} g_2)(\alpha_i, \bar{\alpha}_i), (\mathscr{H} g_2)(\beta_i, \bar{\beta}_i) + (\mathscr{H} g_1)(\alpha_i, \bar{\alpha}_i)) \\ &\le d((\mathscr{H} g_1)(\beta_i, a_2) + (\mathscr{H} g_2)(\alpha_i, a_2), (\mathscr{H} g_2)(\beta_i, a_2) + (\mathscr{H} g_1)(\alpha_i, a_2)) \end{aligned}$$

$$+d((\mathscr{H}g_1)(a_1,\bar\beta_i)+(\mathscr{H}g_2)(a_1,\bar\alpha_i),(\mathscr{H}g_2)(a_1,\bar\beta_i)+(\mathscr{H}g_1)(a_1,\bar\alpha_i))$$
$$+d((\mathscr{H}g_1)(\alpha_i,a_2)+(\mathscr{H}g_1)(\beta_i,\bar\beta_i)+(\mathscr{H}g_2)(\alpha_i,\bar\beta_i)+(\mathscr{H}g_2)(\beta_i,a_2),$$
$$(\mathscr{H}g_2)(\alpha_i,a_2)+(\mathscr{H}g_2)(\beta_i,\bar\beta_i)+(\mathscr{H}g_1)(\alpha_i,\bar\beta_i)+(\mathscr{H}g_1)(\beta_i,a_2))$$
$$+d((\mathscr{H}g_1)(a_1,\bar\alpha_i)+(\mathscr{H}g_1)(\alpha_i,\bar\beta_i)+(\mathscr{H}g_2)(a_1,\bar\beta_i)+(\mathscr{H}g_2)(\alpha_i,\bar\alpha_i),$$
$$(\mathscr{H}g_2)(a_1,\bar\alpha_i)+(\mathscr{H}g_2)(\alpha_i,\bar\beta_i)+(\mathscr{H}g_1)(a_1,\bar\beta_i)+(\mathscr{H}g_1)(\alpha_i,\bar\alpha_i));$$

therefore, summing over $i=1,\dots,m$, by (20), we find that

$$\sum_{i=1}^{m} d((\mathscr{H}g_1)(\beta_i,\bar\beta_i)+(\mathscr{H}g_2)(\alpha_i,\bar\alpha_i),(\mathscr{H}g_2)(\beta_i,\bar\beta_i)+(\mathscr{H}g_1)(\alpha_i,\bar\alpha_i))$$
$$\le d_2(\mathscr{H}g_1,\mathscr{H}g_2)\le L(\mathscr{H})\rho(u_1,u_2)/2.$$

Since $g_1(\beta_i,\bar\beta_i)=u_1$, $g_2(\beta_i,\bar\beta_i)=(u_1+u_2)/2$, $g_1(\alpha_i,\bar\alpha_i)=(u_1+u_2)/2$, and $g_2(\alpha_i,\bar\alpha_i)=u_2$, we can rewrite the last inequality as

$$\sum_{i=1}^{m} d\left(h(\beta_i,\bar\beta_i,u_1)+h(\alpha_i,\bar\alpha_i,u_2),h\left(\beta_i,\bar\beta_i,\frac{u_1+u_2}{2}\right)+h\left(\alpha_i,\bar\alpha_i,\frac{u_1+u_2}{2}\right)\right)$$
$$\le L(\mathscr{H})\frac{\rho(u_1,u_2)}{2}. \qquad (22)$$

Using the fact that $\mathscr{H}$ maps $\mathrm{BV}_2(I_a^b;N)$ to $\mathrm{BV}_2(I_a^b;M)$ and that the constant mappings of two variables lie in $\mathrm{BV}_2(I_a^b;N)$, we find that $h(\cdot,u)=\mathscr{H}(u)\in\mathrm{BV}_2(I_a^b;M)$ for all $u\in N$. Then, by Lemma 5, the left-left regularization $h^-(\cdot,u)$ in the first two variables belongs to $\mathrm{BV}_2^-(I_a^b;M)$ for all $u\in N$. Passing to the limit in (22) as $(\alpha_1,\bar\alpha_1)\to(x_1-0,x_2-0)$ and using completeness of M, the definition of the left-left regularization $h^-(\cdot,u)$ of $x\mapsto h(x,u)$, and continuity of $+$ in M, we obtain

$$d\left(h^-(x_1,x_2,u_1)+h^-(x_1,x_2,u_2),h^-\left(x_1,x_2,\frac{u_1+u_2}{2}\right)+h^-\left(x_1,x_2,\frac{u_1+u_2}{2}\right)\right)$$
$$\le L(\mathscr{H})\frac{\rho(u_1,u_2)}{2m}.$$

Hence, letting $m\to\infty$, we come to the following equality valid for all $u_1,u_2\in N$:

$$d\left(h^-(x,u_1)+h^-(x,u_2),h^-\left(x,\frac{u_1+u_2}{2}\right)+h^-\left(x,\frac{u_1+u_2}{2}\right)\right)=0. \qquad (23)$$

Since d is a metric on M and M is a convex cone, we conclude now that

$$h^-(x,u_1)+h^-(x,u_2)=h^-\left(x,\frac{u_1+u_2}{2}\right)+h^-\left(x,\frac{u_1+u_2}{2}\right)=2h^-\left(x,\frac{u_1+u_2}{2}\right).$$

Thus, the operator $h^-(x,\cdot):N\to M$ satisfies the following Jensen functional equation:

$$2h^-\left(x,\frac{u_1+u_2}{2}\right)=h^-(x,u_1)+h^-(x,u_2),\quad u_1,u_2\in N. \qquad (24)$$

Now, let $a_1<x_1\le b_1$ and $x_2=a_2$. If $m\in\mathbb{N}$, $a_1<\alpha_1<\beta_1<\dots<\alpha_m<\beta_m<x_1$, and $a_2<\bar\alpha_1<\bar\beta_1<\dots<\bar\alpha_m<\bar\beta_m<b_2$ then the above arguments yield estimate (22). Passing to the limit in this estimate as $(\alpha_1,\bar\beta_m)\to(x_1-0,a_2+0)$, we obtain (23) and hence (24). Similarly, we consider the cases in which $x_1=a_1$ and $a_2<x_2\le b_2$ or $x_1=a_1$ and $x_2=a_2$.

Consequently, the Jensen equation (24) holds for all $x\in I_a^b$.

By Lemma 6, for every $x \in I_a^b$, there exist an additive operator $f(x)(\cdot) : N \to M$ and a constant $h_0(x) \in M$ such that

$$h^-(x,u) = f(x)u + h_0(x), \quad u \in N. \tag{25}$$

Since $f(x)(0) = 0$, it follows from (25) that $h^-(x,0) = h_0(x)$ for all $x \in I_a^b$, however, as observed above, $h(\cdot,0) \in \mathrm{BV}_2\big(I_a^b; M\big)$; therefore, by Lemma 5, we find that $h_0 = h^-(\cdot,0) \in \mathrm{BV}_2^-\big(I_a^b; M\big)$. At step 1 we demonstrated that

$$d(h(x,u_1),h(x,u_2)) \le 2L(\mathscr{H})\rho(u_1,u_2), \quad x \in I_a^b,\ u_1,u_2 \in N;$$

therefore, taking the left-left regularization, we find that this inequality is valid with h^- instead of h. By (25), we find that

$$d(f(x)u_1, f(x)u_2) = d(f(x)u_1 + h_0(x), f(x)u_2 + h_0(x))$$
$$= d(h^-(x,u_1),h^-(x,u_2)) \le 2L(\mathscr{H})\rho(u_1,u_2), \quad u_1,u_2 \in N,$$

so that $f(x) \in \mathrm{L}(N;M)$ and hence $f : I_a^b \to \mathrm{L}(N;M)$.

We are left with showing that if $u \in N$ then $f(\cdot)u \in \mathrm{BV}_2^-\big(I_a^b; M\big)$. Since the mapping $h(\cdot,u)$ lies in $\mathrm{BV}_2\big(I_a^b; M\big)$, by Lemma 5, the mappings h_0 and $h^-(\cdot,u)$ lie in $\mathrm{BV}_2^-\big(I_a^b; M\big)$. In the inequalities below we use (1) and (25) several times. If $x,y \in I_a^b$ and $x \le y$ then

$$\mathrm{md}(f(\cdot)u, I_x^y) = d(f(x_1,x_2)u + f(y_1,y_2)u, f(x_1,y_2)u + f(y_1,x_2)u)$$
$$\le d(h^-(x_1,x_2,u) + h^-(y_1,y_2,u), h^-(x_1,y_2,u) + h^-(y_1,x_2,u))$$
$$+d(h_0(x_1,x_2) + h_0(y_1,y_2), h_0(x_1,y_2) + h_0(y_1,x_2)) = \mathrm{md}\big(h^-(\cdot,u), I_x^y\big) + \mathrm{md}\big(h_0, I_x^y\big),$$

whence

$$V_2\big(f(\cdot)u, I_a^b\big) \le V_2\big(h^-(\cdot,u), I_a^b\big) + V_2\big(h_0, I_a^b\big).$$

By analogy, if $t,s \in [a_1,b_1]$ then

$$d(f(t,a_2)u, f(s,a_2)u) \le d(h^-(t,a_2,u),h^-(s,a_2,u)) + d(h_0(t,a_2),h_0(s,a_2)),$$

whence

$$V_{a_1}^{b_1}(f(\cdot,a_2)u) \le V_{a_1}^{b_1}(h^-(\cdot,a_2,u)) + V_{a_1}^{b_1}(h_0(\cdot,a_2)),$$

and a similar estimate holds for $V_{a_2}^{b_2}(f(a_1,\cdot)u)$. Thus,

$$TV_d\big(f(\cdot)u, I_a^b\big) \le TV_d\big(h^-(\cdot,u), I_a^b\big) + TV_d\big(h_0, I_a^b\big).$$

The left-left continuity of $f(\cdot)u$ follows from the fact that, for every point $x = (x_1,x_2)$, where $x_k \in (a_k,b_k]$, $k = 1,2$, letting $I_a^b \ni y \to x - 0$, we obtain

$$d(f(y)u, f(x)u) \le d(h^-(y,u),h^-(x,u)) + d(h_0(y),h_0(x)) \to 0.$$

Theorem 1 is proven completely. □

Closing this section and the first part of the article, we make some remarks and complements to the main result.

Remark 1. An assertion similar to Theorem 1 is valid for the right-right, right-left, or left-right regularization of the mapping $h(\cdot,u)$, $u \in N$. However, in the representation $h^-(x,u) = f(x)u + h_0(x)$, we cannot replace h^- with h in general: an appropriate example is constructed in [25, Theorem 3].

Remark 2. Suppose that $I_k = [a_k,b_k]$ and $\mathrm{P}_k(I_k;N) \subset N^{I_k}$ is a family of mappings possessing the property: for all $u_1,u_2 \in N$, $m \in \mathbb{N}$, and $a_k < \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m < b_k$, the mapping

$I_k \ni t \mapsto \zeta_m(t)u_1 + u_2 \in N$ belongs to $\mathrm{P}_k(I_k; N)$, where $k = 1, 2$ and the function ζ_m has the form (21). Put $\mathrm{P}\big(I_a^b; N\big) = \mathrm{P}_1(I_1; N) + \mathrm{P}_2(I_2; N)$ and endow this set with the metric ρ_2 of $\mathrm{BV}_2\big(I_a^b; N\big)$. Then the conclusion of Theorem 1 remains valid, if we replace the assumption of Lipschitz continuity of $\mathscr{H}$ with the condition $\mathscr{H} \in \mathrm{Lip}\big(\mathrm{P}\big(I_a^b; N\big); \mathrm{BV}_2\big(I_a^b; M\big)\big)$.

Remark 3. Suppose that the conditions of Theorem 1 are satisfied. Denote by $\mathrm{B}(N; M)$ the set of all bounded additive operators from N to M. From the proof of Theorem 1 we see that the following result is valid: if the superposition operator $\mathscr{H}$ acts from $\mathrm{BV}_2\big(I_a^b; N\big)$ to $\mathrm{BV}_2\big(I_a^b; M\big)$ and is (*globally*) *bounded*; i.e., there is a constant $C \geq 0$ such that $d_2(\mathscr{H}g_1, \mathscr{H}g_2) \leq C$ for all $g_1, g_2 \in \mathrm{BV}_2\big(I_a^b; N\big)$ (see also Remark 2), then $h(x, \cdot) \in \mathrm{B}(N; M)$ for all $x \in I_a^b$ and there is a mapping $h_0 \in \mathrm{BV}_2^-\big(I_a^b; M\big)$ such that $h^-(x, u) = h_0(x)$ for all $x \in I_a^b$ and $u \in N$. Indeed, there exist mappings $f : I_a^b \to \mathrm{B}(N; M)$ and $h_0 \in \mathrm{BV}_2^-\big(I_a^b; M\big)$ for which $h^-(x, u) = f(x)u + h_0(x)$, $x \in I_a^b$, $u \in N$. Since $d(h(x, u_1), h(x, u_2)) \leq C$ for $x \in I_a^b$ and $u_1, u_2 \in N$, we have $d(f(x)u_1, f(x)u_2) = d(h^-(x, u_1), h^-(x, u_2)) \leq C$. Consequently, for an arbitrary rational $\lambda > 0$ and every $u \in N$ we have

$$\lambda d(f(x)u, 0) = d(\lambda f(x)u, 0) = d(f(x)(\lambda u), f(x)(0)) \leq C,$$

so that $f(x)u = 0$ and hence $f(x) = 0$ for all $x \in I_a^b$.

Remark 4. Assume that $h : N \to M$ in Theorem 1 (i.e., h is independent of the first argument $x \in I_a^b$). The following assertion is valid: The superposition operator $\mathscr{H}$ generated by h maps $\mathrm{BV}_2\big(I_a^b; N\big)$ to $\mathrm{BV}_2\big(I_a^b; M\big)$ and satisfies the Lipschitz condition if and only if there exist $f \in \mathrm{L}(N; M)$ and $h_0 \in M$ such that $hu = fu + h_0$ in M for all $u \in N$. Indeed, it follows from Theorem 1 that $h(u) = f(x)u + h_0(x)$, whence $h(0) = h_0(x)$ for all $x \in I_a^b$. Moreover, if $x, y \in I_a^b$ then

$$d(f(x)u, f(y)u) = d(f(x)u + h(0), f(y)u + h(0)) = d(h(u), h(u)) = 0;$$

therefore, $f(x)u = f(y)u$ for all $u \in N$ and hence $f(x) = f(y)$ in $\mathrm{L}(N; M)$. Sufficiency follows from Theorem 2 of part II of this article.

Remark 5. Let $(Y, |\cdot|)$ be a real normed vector space. A set-valued operator $T : N \to \mathrm{cbc}(Y)$ from an abstract convex cone $(N, \rho, +, \cdot)$ to $\mathrm{cbc}(Y)$ is called *linear* if it is $\overset{*}{+}$-*additive* (i.e., $T(u+v) = Tu \overset{*}{+} Tv$ for all $u, v \in N$) and *nonnegatively homogeneous* (i.e., $T(\lambda u) = \lambda Tu$ for all $\lambda \in \mathbb{R}^+$ and $u \in N$). Note that if T is linear then $T(0) = \{0\}$. Denote by $\mathbb{L}(N; \mathrm{cbc}(Y))$ the abstract convex cone of all linear Lipschitzian set-valued operators from N to $\mathrm{cbc}(Y)$ endowed with the pointwise operations (for which we keep the notations of the operations on $\mathrm{cbc}(Y)$) and the metric $D_L = D_\ell$:

$$D_L(T, S) = \sup_{u, v \in N, u \neq v} D(Tu \overset{*}{+} Sv, Su \overset{*}{+} Tv)/\rho(u, v).$$

Hence, we see that Theorem 1 remains valid if $(M, d, +) = (\mathrm{cbc}(Y), D, \overset{*}{+})$, $(Y, |\cdot|)$ is a Banach space, and $\mathrm{L}(N; M)$ is replaced by $\mathbb{L}(N; \mathrm{cbc}(Y))$. Moreover, if N is a vector space then the operator $f(x)(\cdot)$ for every $x \in I_a^b$ is univalent (so that $f : I_a^b \to \mathrm{L}(N; Y)$); this is a consequence of the fact that if $u \in N$ then $(-u) \in N$; therefore, by $\overset{*}{+}$-additivity of the operator $f(x)(\cdot)$, we find that

$$f(x)(u) \overset{*}{+} f(x)(-u) = f(x)(u + (-u)) = f(x)(0) = \{0\}.$$

Moreover, if N is real then we can consider $\mathrm{L}(N; Y)$ as the usual space of all bounded linear operators from N to Y.

Remark 6. An analog of Theorem 1 is also valid for mappings and superposition operators of a single variable if we put $I_a^b = [a, b] \subset \mathbb{R}$, replace BV_2 everywhere with BV_1 (see also Remark 2), and assume that $h^-(x, u) = \lim_{y \to x-0} h(y, u)$ for $a < x \leq b$, $h^-(a, u) = \lim_{x \to a+0} h^-(x, u)$ in M for all $u \in N$, and $\mathrm{BV}_1^-\big(I_a^b; M\big)$ is the subset of $\mathrm{BV}_1\big(I_a^b; M\big)$ constituted by the left continuous mappings on $(a, b]$. Following the proof of Theorem 1, we only sketch the main steps of the proof in this case. For $g_1, g_2 \in \mathrm{BV}_1\big(I_a^b; N\big)$ the Lipschitz condition for $\mathscr{H}$ has the form

$$d((\mathscr{H}g_1)(a), (\mathscr{H}g_2)(a)) + W_a^b(\mathscr{H}g_1, \mathscr{H}g_2) \le L(\mathscr{H})\big(\rho(g_1(a), g_2(a)) + W_a^b(g_1, g_2)\big).$$

In particular, if $m \in \mathbb{N}$ and $a \le \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_m < \beta_m \le b$ then

$$\sum_{i=1}^{m} d(h(\beta_i, g_1(\beta_i)) + h(\alpha_i, g_2(\alpha_i)), h(\beta_i, g_2(\beta_i)) + h(\alpha_i, g_1(\alpha_i))) \le L(\mathscr{H})\rho_1(g_1, g_2).$$

Hence, for $m = 1$ and $\alpha_1 = a$ we see that $d(h(x, u_1), h(x, u_2)) \le 2L(\mathscr{H})\rho(u_1, u_2)$, $x \in [a, b]$, $u_1, u_2 \in N$, if we put $\beta_1 = x$ and $g_k(y) = \zeta_{a,x}(y)u_k$ for $a < x \le b$ and $\beta_1 = b$ and $g_k(y) = (1 - \zeta_{a,b}(y))u_k$ for $x = a$, $y \in [a, b]$, $k = 1, 2$. If $a < x \le b$, $a < \alpha_1$, and $\beta_m < x$ then, putting

$$g_k(y) = \frac{1}{2}\zeta_m(y)u_1 + \frac{1}{2}(1 - \zeta_m(y))u_2 + \frac{1}{2}u_k, \quad y \in [a, b],\ k = 1, 2,$$

we find that

$$\sum_{i=1}^{m} d\left(h(\beta_i, u_1) + h(\alpha_i, u_2), h\left(\beta_i, \frac{u_1 + u_2}{2}\right) + h\left(\alpha_i, \frac{u_1 + u_2}{2}\right)\right) \le L(\mathscr{H})\frac{\rho(u_1, u_2)}{2},$$

whence, letting $\alpha_1 \to x - 0$, we standardly find that

$$d\left(h^-(x, u_1) + h^-(x, u_2), h^-\left(x, \frac{u_1 + u_2}{2}\right) + h^-\left(x, \frac{u_1 + u_2}{2}\right)\right) \le L(\mathscr{H})\frac{\rho(u_1, u_2)}{2m}.$$

The remaining part of the proof is the same as in Theorem 1.

The result of the above remark generalizes the results of [8, 9] (in the set-valued case we should put $(M, d, +) = (\mathrm{cbc}(Y), D, \overset{*}{+})$).

Remarks 1–5 with due changes can be translated to the case of the functions in $\mathrm{BV}_1([a, b]; M)$ and superposition operators of a single variable.

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