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To the blessed memory of Vitaly Lazarevich Ginzburg

Global in Time Solutions to Kolmogorov–Feller Pseudodifferential Equations with Small Parameter

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Abstract. The goal in this paper is to demonstrate a new method for constructing global-in-time approximate (asymptotic) solutions of (pseudodifferential) parabolic equations with a small parameter. We show that, in the leading term, such a solution can be constructed by using characteristics, more precisely, by using solutions of the corresponding Hamiltonian system and without using any integral representation. For completeness, we also briefly describe the well-known scheme developed by V. P. Maslov for constructing global-in-time solutions.

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INTRODUCTION

The goal of the present paper is to present a new approach to the construction of singular (i.e., containing the Dirac δ-function as a summand) solutions to the continuity equation and to show how these solutions can be used to construct the global in time solution of the Cauchy problem for Kolmogorov–Feller-type equations with diffusion, potential, and jump terms. As is well known, the asymptotic solutions of the Cauchy problem for linear equations with a small parameter $\varepsilon > 0$ can be constructed by the WKB method [27]. In the framework of this method, the initial partial differential equation is reduced to a system of equations consisting of the Hamilton–Jacobi equation and several transport equations. All these equations can be solved under the assumption that the Hamilton system has smooth solutions corresponding to the above-mentioned Hamilton–Jacobi equation (the trajectories of the Hamilton system fiber the phase space). In general, this ensures only the existence of the classical solution in small with respect to time. In this case, if, for example, the Hamilton function is time-independent, then, on the time intervals where the Hamilton–Jacobi equation has a smooth solution, the Cauchy problem for this equation (as well as for the corresponding transport equations) is invertible in time.

As is well known, the above system of equations (the Hamilton–Jacobi equation and several transport equations) arises when constructing WKB asymptotic (approximating) solutions of the form $u_{\rm as} = \exp\{\frac{i}{h}S(x,t)\}\varphi(x,t)$ for wave equations and Schrödinger-type equations [27] and in the construction of approximate solutions of the form $u_\varepsilon = \exp\{-\frac{1}{\varepsilon}S(x,t)\}\varphi(x,t)$ for parabolic equations [23].

We note that, in the first case (for stationary symbols), the invertibility in time is typical not only for solutions of the above type, but also for more general solutions because of the properties of the very equation; however, in the other case, the general solutions of the Cauchy problem fail to have this property, and the fact that smooth "WKB-type" solutions $\exp\{-\frac{1}{\varepsilon}S(x,t)\}\varphi(x,t)$ are invertible in time distinguishes this class of solutions from the other solutions.

Now we explain the time-invertibility condition in more detail and describe (below in the paper) a method for constructing such solutions.

First, note that the function $S(x,t)$ can be defined as the pointwise limit

$$S(x,t) = -\lim_{\varepsilon\to 0}\varepsilon\log u_\varepsilon(x,t),$$

where u_ε is a solution of the equation of parabolic type with a small parameter appearing in [24].

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If the function $S(x,t)$ thus defined exists and is smooth, then it is a (classical) solution of the Hamilton–Jacobi equation [24, 25]

$$S_t + H(S_x, x) = 0. \tag{0.1}$$

There is a well-known exact formula expressing the solutions of this equation in terms of the trajectories of the corresponding Hamilton system

$$\dot{x} = H_p(p,x), \qquad \dot{p} = -H_x(p,x). \tag{0.2}$$

The fact that the solution of Eq. (0.1) is smooth for $t = t_0$ means that the Lagrangian manifold Λ^n_t obtained by a shift of the initial manifold $\Lambda^n_0 = \{x, \frac{\partial S}{\partial x}|_{t=0}\}$ along the trajectories of (0.2) can be uniquely projected on R^n_x for all $t \in [0, t_0]$.

Since the shift along the trajectories of (0.2) is invertible in time, this implies that the resolving operator of Eq. (0.1) is also invertible in time. Under the same conditions of unique projection, the solution of the transport equation for the function $\varphi(x,t)$ (the amplitude) is given by the formula

$$\varphi(x,t) = C/\sqrt{J(x,t)}, \tag{0.3}$$

where C is a constant along the projection of the trajectories of (0.2) on R^n_x and $J(x,t)$ is the Jacobian of the shift mapping along these projections (the uniqueness of the projection mapping implies the uniqueness and the invertibility of the shift mapping).

Under the above conditions, formulas (0.3) are also invertible. All that was said above can be illustrated by the following simple example.

Consider the heat conduction equation

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} = 0, \qquad u\Big|_{t=0} = e^{-S_0(x)/\varepsilon}\varphi_0, \tag{0.4}$$

where the function $S_0(x) \geqslant 0$ is assumed to be smooth and bounded (together with its derivatives) and $\varphi \in C^\infty_0$; for example,

$$S_0(x) = \int_{-\infty}^{x} (1 + \tanh z)dz = x + \log\cosh x + \log 2.$$

Here the following additional condition is satisfied:

$$d^2 S_0/dx^2 \geqslant 0. \tag{0.5}$$

The Hamilton function corresponding to (0.3) is $H(p,x) = p^2$ and, respectively, the trajectories of system (0.2) are of the form

$$x(x_0,t) = x_0 + 2tp(x_0,t), \qquad p(x_0,t) \equiv p(x_0,0) = \frac{dS_0}{dx_0} = 1 + \tanh x_0. \tag{0.6}$$

It is obvious that condition (0.5) implies the unique globally-in-t solvability of the equation $x = x_0 + 2t\,\partial S_0/\partial x_0$ in x_0, and hence the global solvability of the corresponding Hamilton–Jacobi equation and the transport equation in the class of smooth functions.

Further, for Eq. (0.3), it is possible to write out the Green function, which is a solution of Eq. (0.4),

$$G(x,\xi,t) = \frac{1}{\sqrt{2\pi t\varepsilon}} e^{-(x-\xi)^2/4t\varepsilon},$$

such that $G(x,\xi,0) = \delta(x-\xi)$. As is well known, one can write

$$u = \int G(x,\xi,t) u(\xi,0)\, d\xi.$$

After the change $t \to -t$, we obtain the Green function for the inverse heat conduction equation, and it can readily be seen that

$$u|_{t=0} + O(\varepsilon^N) = \int G(x, \xi, -t) e^{-\hat{S}(\xi,t)/\varepsilon} \hat{\varphi}(\xi, t)\, d\xi, \tag{0.7}$$

where $\hat{S}(x,t)$ and $\hat{\varphi}(x,t)$ are solutions of the Hamilton–Jacobi equations and of the transport equation

$$\frac{\partial \hat{\varphi}}{\partial t} + 2\frac{\partial S}{\partial x}\frac{\partial \hat{\varphi}}{\partial c} + \frac{\partial^2 S}{\partial x^2}\hat{\varphi} = 0,$$

at time t, satisfying the initial conditions $\hat{S}|_{t=0} = S_0(x)$, $\hat{\varphi}|_{t=0} = \varphi_0(x)$, where $N > 0$ is an arbitrary positive number and $O(\varepsilon^N)$ is such that $e^{S_0(x)/\varepsilon} O(\varepsilon^N) = O(\varepsilon^N)$.

Relation (0.7) can be verified by using the Laplace method. The last relation means that the estimate $O(\varepsilon^N)$ is not quite suitable here; see [26] for details.

The situation is absolutely different if (0.5) is replaced by the inequality

$$\frac{d^2 S_0}{dx^2} < 0 \tag{0.8}$$

at least for some value of x_0, for example,

$$S_0 = \int_{-\infty}^{x} \cosh^{-1} z dz = 2 \arctan \exp x + \pi.$$

In this case, the Jacobian of the shift mapping along the trajectories $x = x_0 + 2t\frac{dS_0}{dx_0}$ vanishes at $t^* = (\min |2\frac{d^2 S_0}{dx_0^2}|)^{-1}$ (in the example, we have $d^2 S_0/dx_0^2 = \tan x \cosh^{-1} x$ and $t^* = 9/2\sqrt{2}$), where the maximum is taken over all points at which the inequality (0.8) is satisfied. In our example, this is $x_0 = 0$. For $t > t^*$, the Lagrangian manifold shifted along the trajectories (0.6),

$$\Lambda_t^1 = \{x = x_0 - 2t \cosh^{-1} x_0, p = \cosh^{-1} x_0\},$$

forms an S-shaped curve, and there are three points of this curve above some point x in the plane (x, p). These three points are associated with three (local) “WKB-type” solutions

$$u_j = e^{-S_j(x,t)/\varepsilon} \varphi_j(x, t).$$

It is clear that their linear combination

$$u = \sum_{j=1}^{3} c_j u_j \tag{0.9}$$

is also a solution, i.e., it satisfies the equation with the same accuracy as each of the functions u_j, $j = 1, 2, 3$. However, the functions by themselves are not equivalent.

For example, it is clear that, if the inequality $S_1(\bar{x}, t) > S_2(\bar{x}, t)$ holds at some point $\bar{x}$, then the “WKB” solutions u_1 and u_2 at the point $\bar{x}$ satisfy the relation

$$u_1|_{x=\bar{x}} = e^{-S_1(\bar{x},t)/\varepsilon} \varphi_1(\bar{x}, t) = e^{-S_1(\bar{x},t)/\varepsilon} \varphi_2(\bar{x}, t) \big(e^{-(S_1 - S_2)/\varepsilon} \varphi_1/\varphi_2\big)|_{x=\bar{x}} = u_2 O(\varepsilon^N), \tag{0.10}$$

where $N > 0$ is an arbitrary number. This follows from the fact that the difference $(S_1 - S_2)|_{\bar{x}}$ in the parentheses (in the exponent) is positive.

Thus, in formula (0.9), at each point, it is necessary to choose the term where the function S_j is minimal. Such a choice leads to an expression of the form

$$u = e^{-\phi(x,t)/\varepsilon}\varphi(x,t), \quad \text{where} \quad \phi = \phi(x,t) = \min_x \{S_j(x,t)\}. \tag{0.11}$$

It is clear that the expression (0.11) is the leading term of the approximate solution. However, its substitution into the equation (to verify that it is a solution in a sense) is not a trivial problem, because the function $\phi(x,t)$ thus defined is not smooth; it is only continuous, with bounded first derivative.

Certainly, we can avoid this difficulty if we first calculate the terms in (0.9) and then proceed with (0.11). Such a construction, which takes into account the fact that the functions S_j and φ_j can loose smoothness at the points where the Jacobian of the projection mapping of the Lagrangian manifold on R^n_x vanishes, was proposed by V. P. Maslov [26] and is ideologically similar to the construction of the Maslov canonical operator (the Fourier integral operators); see [8, 10]. Another version of Maslov's construction was suggested in [28, 9].

The above procedure is quite appropriate for constructing an asymptotic solution of the Cauchy problem; some estimates for the discrepancy between the asymptotic and exact solutions have also been obtained.

However, attempts to apply these constructions to solve the inverse problem face insurmountable difficulties, and the details of the construction using integral representations play no role here.

The problem is related to the information we need from the solution at $t = T$ to recover the solution for $0 \leqslant t < T$.

Indeed, as was shown above, if there are singularities in the projection of the Lagrangian manifold on R^n_x, then the solution can have the form (0.9) at some points, whereas, as was shown above, some terms in this sum are "infinitesimal" as compared with others. We can "measure" only expressions of the form (0.11) which contain no information about the parts of the Lagrangian manifold corresponding to the "infinitesimal" terms of the solution. However, if we move backwards in time, then these (unknown) parts of the Lagrangian manifold can enter domains that are uniquely projected to R^n_x, and then, in the projection, they are responsible for the principal part of the solution.

Thus, the following assertions hold.

(i) If the function $\phi(x,t) = \lim_{\varepsilon\to 0}(-\varepsilon \log u_\varepsilon)$ is smooth at $t = T$, then the principal part of the solution u_ε of the Cauchy problem can be recovered for $0 \leqslant t < T$ on the entire space R^n_x.

(ii) If the derivatives of the function $\phi(x,t)$ have singularities (discontinuities) for $t = T$, then the principal part of u_ε (u_ε is the solution of the Cauchy problem) cannot be recovered for $0 < t < T$ on the entire domain, because there are domains on which is impossible to reconstruct the solution.

In our future considerations for the nonsmooth case, we use the relationship between the transport and continuity equations. The relation between the solutions of the continuity equation and of the system consisting of the Hamilton–Jacobi equation together with the transport equation has been well studied earlier in the case of a smooth action functional.

Let the velocity field u be determined as the family of velocities of points on the projections of the trajectories of the Hamiltonian system corresponding to the Hamilton–Jacobi equation. In this velocity field, as was pointed out by Madelung in [13], the squared solution of the transport equation satisfies the continuity equation

$$\rho_t + (\nabla, u\rho) + a\rho = 0 \tag{0.12}$$

with the additional term $a\rho$, which is defined below (a is equal to 0 if the Hamiltonian is formally self-adjoint). The main obstacle to the extension of this correspondence globally in time is that, in general, the solutions of the Hamilton–Jacobi equation are smooth only locally in time. The loss of smoothness is equivalent to the appearance of singularities of the velocity field, as was mentioned above. Until recently, there was no approach to constructing formulas for solutions of the continuity equation for a discontinuous velocity field. In Madelung's approach, the divergent form of the continuity equation (in contrast to the transport equation) is very important and enables us to introduce a concept of global solution, despite the singularities of the velocity field.

In the present work, we generalize Madelung's approach to the case in which the singular support of the velocity field is a stratified manifold transversal to the velocity field trajectories. This holds, for example, in the one-dimensional case under the assumption that, for any $t \in [0, T]$, the singular support is discrete and has no limit points (which holds, for example, for real-analytic data and Hamiltonians).

The class of solutions constructed in this way admits both forward and backward motion in time. Here we discuss only the construction by itself, and the invertibility problem will be discussed in another paper.

1. GENERALIZED SOLUTIONS OF THE CONTINUITY EQUATION

Here we follow the approach developed in [1], where the solution is understood in the sense of an integral identity, which, in turn, follows from the fact that relation (0.12) can be understood in the sense of the distributional space $\mathcal{D}(\mathbb{R}^{n+1}_{x,t})$. The first step in this way has been done in [14], see also [15, 16], where the approach based on smooth approximations of the solutions was used.

We specially note that the integral identities in [1] can be derived without using the construction of nonconservative products [2, 4] of nonsmooth and generalized functions (or measure solutions [5]), and the value of the velocity on the discontinuity lines (surfaces) is not given a priori but is calculated. For the case treated in [1], the integral identities exactly coincide with the identities derived by using the construction of nonconservative product (measure solutions) in the situation described above at the end of the introduction, which we shall now make more precise.

First, consider an $(n-1)$-dimensional surface γ_t moving in $\mathbb{R}^n_x$, determined by the equation $\gamma_t = \{x; t = \psi(x)\}$, where $\psi \in C^1(\mathbb{R}^n)$, and $\nabla\psi \neq 0$ in the domain in $\mathbb{R}^n_x$ on which we work.

This is equivalent to finding a surface by an equation of the form $S(x,t) = 0$ ($S \in C^1$ in both variables, $S(x,t) = 0$, $\nabla_{x,t}S|_{S=0} \neq 0$) under the condition that $\partial S/\partial t \neq 0$.

Recall that the situation with $\partial S/\partial t = 0$ can also be covered by making the change of variables $x_i' = x_i - c_i t$ with an appropriately chosen c_i, $i = 1, \dots, n$, by solving the problem with the moving surface and then returning to the original variables. Possible generalizations are considered below in this section.

Next, assume that ζ belongs to $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^1_+)$. Then, by definition,

$$\langle \delta(t - \psi(x)), \zeta(x,t) \rangle = \int_{\mathbb{R}^n} \zeta(x, \psi(x))\, dx,$$

where δ is the Dirac delta function and $\langle , \rangle$ is the distributional pairing (with respect to the variables $t \in \mathbb{R}^1_+$ and $x \in \mathbb{R}^n$).

Let $\delta(t - \psi(x))$ be applied to the test function $\eta \in C_0^\infty(\mathbb{R}^n)$. Then

$$\langle \delta(t - \psi(x)), \eta(x) \rangle = \int_{\gamma_t} \eta\, d\omega_\psi,$$

where $d\omega$ is the Leray form [6] on the surface $\{t = \psi(x)\}$ such that $-d\psi d\omega_\psi = dx_1 \dots dx_n$.

One can show that (see [1, 6])

$$\langle \delta(t - \psi(x)), \eta(x) \rangle = \int_{\gamma_t} \frac{\eta(x)}{|\nabla\psi|}\, d\sigma.$$

First, we assume that the solution ρ to Eq. (0.12) has the form

$$\rho(x,t) = R(x,t) + e(x)\delta(t - \psi(x)), \tag{1.1}$$

where $R(x,t)$ is a piecewise smooth function with possible discontinuity at $\{t = \psi(x)\}$:

$$R = R_0(x,t) + H(t - \psi(x))R_1(x,t),$$

$e \in C(\mathbb{R}^n)$ and e is compactly supported, $\psi \in C^2$ and $\nabla\psi \neq 0$ for $x \in \operatorname{supp} e$, and $H(z)$ stands for the Heaviside function.

It is clear that the term $e(x)\delta'(t-\psi(x))$ appears in (0.12) if we differentiate the distribution $\delta(t-\psi(x))$ with respect to t. Hence, it is necessary to have in (0.12)

$$(\nabla, \rho u) = -e(x)\delta'(t-\psi) + \text{smoother summands}$$

since $\nabla\delta(t-\psi) = -\nabla\psi\delta'(t-\psi)$. Then we must have

$$\rho u = \frac{e\nabla\psi}{|\nabla\psi|^2}\delta(t-\psi) + \text{ smoother summands.}$$

Now we formulate an integral identity, thus defining a generalized solution to the continuity equation. Write $\Gamma = \{(x,t); t = \psi(x)\}$; this is an n-dimensional surface in $\mathbb{R}^n \times \mathbb{R}^1_+$. Let

$$u(x,t) = u_0(x,t) + H(t-\psi)u_1(x,t),$$

where ψ is the same function as above, and $u_0, u_1 \in C(\mathbb{R}^n \times \mathbb{R}^1_+)$.

Consider Eq. (0.12) in the sense of distributions. For all $\zeta(x,t) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^1_+)$, $\zeta(x,0) = 0$, we have

$$\Big\langle \frac{\partial\rho}{\partial t} + (\nabla, \rho u), \zeta \Big\rangle = -\langle \rho, \zeta_t\rangle - \langle \rho u, \nabla\zeta\rangle.$$

Substituting the singular terms for ρ and ρu calculated above, we arrive at the following definition.

Definition 1.1. A function $\rho(x,t)$ determined by relation (1.1) is called a *generalized δ-shock-wave-type solution to* (0.12) *on the surface* $\{t = \psi(x)\}$ if the following integral identity holds:

$$\int_0^\infty\int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta)\,dx\,dt + \int_\Gamma \frac{e}{|\nabla\psi|}\frac{d}{dn_\perp}\zeta(x,t)\,dx = 0 \tag{1.2}$$

for all test functions $\zeta(x,t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^1_+)$, $\zeta(x,0) = 0$, $d/dn_\perp = \big(\nabla\psi/|\nabla\psi|, \nabla\big) + |\nabla\psi|\,\partial/\partial t$.

We have also the relation

$$\int_{\mathbb{R}^n} \frac{e}{|\nabla\psi|}\frac{d}{dn}\zeta(x,\psi)\,dx = \int_\Gamma \frac{e}{|\nabla\psi|}\frac{d}{dn_\perp}\zeta(x,t)\,dx.$$

Note that the vector $n_\perp$ is orthogonal to the vector $(\nabla\psi, -1)$, which is normal to the surface Γ, i.e., $\frac{d}{dn_\perp}$ lies in the plane tangent to Γ.

We can give a geometric definition of the field $\frac{d}{dn_\perp}$. The trajectories of this vector field are curves lying on the surface Γ, and they are orthogonal to all sections of this surface produced by the planes $t = \text{const}$. Furthermore, it is clear that the expression $\frac{1}{|\nabla\psi|}$ is an absolute value of the normal velocity of a point on γ_t, i.e., on the cross-section of Γ by the plane $t = \text{const}$, and the expression $\frac{1}{|\nabla\psi|}\cdot\frac{\nabla\psi}{|\nabla\psi|} \overset{\text{def}}{=} \vec{V}_n$ is the vector of normal velocity of a point on γ_t. Thus, we have another representation:

$$\int_\Gamma \frac{e}{|\nabla\psi|}\frac{d}{dn_\perp}\zeta(x,t)\,dx = \int_\Gamma e\Big((\vec{V}_n, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\,dx,$$

where $V_n = \pi^*(v_n)$, v_n is the normal velocity of a point on γ_t, and π^* is induced by the projection mapping $\pi\colon \Gamma \to R^n_x$.

It results from the latter definition that the following relations must hold:

$$R_t + (\nabla, Ru) + aR\zeta = 0 \qquad \text{for all points} \quad (x,t) \notin \Gamma,$$

$$([R]-|\nabla\psi|[Ru_n])+\Big(\frac{d}{dn}\Big)^*\frac{e}{|\nabla\psi|}=0 \qquad \text{for all points} \quad (x,t)\in\Gamma. \tag{1.3}$$

The last relation can be represented as

$$\mathcal{K}E+(d/dn)E=[Ru_n]|\nabla\psi|-[R], \quad \text{where} \quad E=e/|\nabla\psi|, \quad K=\left(\nabla,\frac{\nabla\psi}{|\nabla\psi|}\right)=\operatorname{div}\nu \tag{1.4}$$

(ν is the normal on the surface $\{t=\psi(x)\}$), and, as is known, $\mathcal{K}$ is the mean curvature of the cross-section of the surface Γ by the plane $t=\text{const}$; finally, $\frac{d}{dn}=(\frac{\nabla\psi}{|\nabla\psi|},\nabla)$.

Assume that there are two surfaces $\Gamma_i=\{(x,t);t=\psi_i(x)\}$ in $\mathbb{R}^n\times\mathbb{R}^1_+$, $i=1,2$, whose intersection is a smooth surface $\hat\gamma=\{(x,t);(t=\psi_1)\cap(t=\psi_2)\}$ belonging to the third surface $\Gamma_{(3)}=\{(x,t);t=\psi_3(x)\}$. Further, assume that the surface $\Gamma_{(3)}$ is a continuation of the surfaces $\Gamma^{(i)}$ in the following sense. Denote curves on the surfaces Γ_i by $n_\perp^{(i)}$ and assign, to every point $(\hat x,\hat t\,)$ on the surface $\hat\gamma$, the graph consisting of the trajectories $n_\perp^{(1)}$ and $n_\perp^{(2)}$ entering $(\hat x,\hat t\,)$ and of the trajectory $n_\perp^{(3)}$ leaving this point (i.e., the trajectories $n_\perp^{(i)}$ fiber the surface $\Gamma^{(i)}$). We also assume that the surface (stratified manifold) $\Gamma_\cup=\Gamma_{(1)}\cup\Gamma_{(2)}\cup\Gamma_{(3)}$ consists of points belonging to these graphs. Next, assume that $u(x,t)$ is a piecewise smooth vector field whose trajectories enter $\Gamma_\cup$.

Definition 1.2. Let

$$u(x,t)=u_0(x,t)+\sum_{i=1}^{3}H(t-\psi_i)u_{1i}(x,t),$$

where ψ is the same function as above, and $u_0,u_{1i}\in C(\mathbb{R}^n\times\mathbb{R}^1_+)$. The function $\rho(x,t)$ determined by the relation

$$\rho(x,t)=R(x,t)+\sum_{i=1}^{3}e_i(x)\delta(t-\psi_i(x)), \quad \text{where} \quad R(x,t)\in C^1(\mathbb{R}^n\times\mathbb{R}^1_+)\setminus\{\bigcup\Gamma_t^{(i)}\},$$

is referred to as a *generalized δ-shock wave type solution to* (1.2) *corresponding to the stratified manifold* $\Gamma_\cup$ provided that the integral identity

$$\int_0^\infty\!\!\int_{\mathbb{R}^n}(R\zeta_t+(uR,\nabla\zeta)+aR\zeta)\,dx\,dt+\sum_{i=1}^{3}\int_{\Gamma_{(i)}}\frac{e_i}{|\nabla\psi_i|}\frac{d}{dn_\perp^{(i)}}\zeta(x,t)\,dx=0 \tag{1.5}$$

holds for all test functions $\zeta(x,t)\in\mathcal{D}(\mathbb{R}^n\times\mathbb{R}^1_+)$, $\zeta(x,0)=0$, $\frac{d}{dn_\perp^{(i)}}=\big(\frac{\nabla\psi_i}{|\nabla\psi_i|},\nabla\big)+|\nabla\psi_i|\frac{\partial}{\partial t}$.

As above, this relation implies the first equation in (1.3) outside $\Gamma_\cup$, equations of the type of the second equation in (1.3) on strata $\Gamma_{(i)}$, and the Kirchhoff-type relation on $\hat\gamma$,

$$(e_1+e_2)\big|_{\hat\gamma}=e_3\big|_{\hat\gamma}. \tag{1.6}$$

Let us now consider the case $\operatorname{codim}\Gamma>1$. First, we note that the second integral in (1.2) can be represented as

$$\int_\Gamma\frac{e}{|\nabla\psi|}\frac{d}{dn_\perp}\zeta(x,t)\,dx=\int_\Gamma e\bigg(\Big(\frac{\nabla\psi}{|\nabla\psi|^2},\nabla\Big)+\frac{\partial}{\partial t}\bigg)\zeta(x,t)\,dx.$$

We note further that, if the surface Γ is defined by the equation $S(x,t)=0$ rather than by the simpler equation $\{t=\psi(x)\}$ presented at the beginning of this section, then

$$\vec V_n=-\frac{S_t}{|\nabla S|}\cdot\frac{\nabla S}{|\nabla S|}=-\frac{S_t}{|\nabla S|^2}\nabla S$$

and the new vector field $d/dn_\perp = (\vec{V}_n, \nabla) + \partial/\partial t$ certainly remains tangent to Γ.

Therefore, in this more general case, using the new vector $\vec{V}_n$, we can again represent the integral identity of Definition 1.1 as

$$\int_0^\infty\int_{\mathbb{R}^n} \big(R\zeta_t + (uR, \nabla\zeta) + aR\zeta\big)\,dxdt + \int_\Gamma e\Big((\vec{V}_n, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\,dx = 0. \tag{1.7}$$

This form of the integral identity can easily be generalized to the case in which Γ is a smooth surface in $\mathbb{R}^{n+1}$ of codimension > 1.

In this case, instead of $\vec{V}_n$, we can use a vector $\vec{v}$ transversal to Γ for which the field $(\vec{v}, \nabla) + \frac{\partial}{\partial t}$ is tangent to Γ. Note that the vector $\vec{v}$ is uniquely determined by this condition, which can be treated as "the calculation of the velocity value on the discontinuity" from the viewpoint of [5, 7].

Moreover, in this case, the expression for ρ does not contain the Heaviside function, and it is assumed that the trajectories of the field u are smooth, nonsingular outside Γ, and transversal to Γ at each point of Γ. In this case, the function ρ has the form $\rho(x,t) = R(x,t) + e(x)\delta(\Gamma)$, where $R \in C^1(\mathbb{R}^{n+1} \setminus \Gamma)$, $e \in C^1(\Gamma)$, and the function $\delta(\Gamma)$ is given by

$$\langle \delta(\Gamma), \zeta(x,t)\rangle = \int_\Gamma \zeta\omega,$$

where ω is the Leray form on Γ. If $\Gamma = \{S_1(x,t) = 0 \cap \dots \cap S_k(x,t) = 0\}$, $k \in [1, n]$, then ω is defined by the relation $dt\,dx_1 \dots d\,x_n = dS_1 \dots dS_k\omega$, see [6, p. 274].

In this case, assume that the functions S_k are sufficiently smooth (for example, $C^2(\mathbb{R}^n \times \mathbb{R}^1_+)$) and that their differentials on Γ are linearly independent.

Moreover, we can assume that the inequality

$$J = \frac{\mathcal{D}(S_1, \dots, S_n)}{\mathcal{D}(t, x_1, \dots, x_{n-1})} \neq 0$$

holds. This inequality is an analog of the condition $S_t \neq 0$ (see the beginning of the section) and enables us to represent ω in the form $\omega = J^{-1}dx_k \dots dx_n$.

The integral identity, which is an analog of (1.7), has the form

$$\int_0^\infty\int_{\mathbb{R}^n} \big(R\zeta_t + (uR, \nabla\zeta) + aR\zeta\big)\,dxdt + \int_\Gamma e\Big((v, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\omega = 0.$$

Integrating the last relation by parts, we obtain equations for the functions e and R, similarly to (1.4).

Assume now that the singular support of the velocity field is the stratified manifold $\bigcup\Gamma_i$ with smooth strata Γ_t of codimensions $n_i \geqslant 1$ and that the velocity field trajectories are transversal to $\bigcup\Gamma$ and are entering trajectories. Then the general solution of Eq. (0.12) is of the form

$$\rho(x,t) = R(x,t) + \sum e_i\delta(\Gamma_i), \tag{1.8}$$

where $R(x,t)$ is smooth outside $\bigcup\Gamma_i$, the e_i are functions defined on the strata Γ_i, and the sum is taken over all strata Γ_i.

The integral identities determining a generalized solution of this kind are

$$\int_0^\infty\int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta)\,dxdt + \sum_i \int_{\Gamma_i} e_i\Big[\Big((v_i, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\Big]\omega_i = 0. \tag{1.9}$$

This implies that, outside $\bigcup\Gamma_i$, the function R satisfies the continuity equation $R_t + (\nabla, uR) + aR = 0$, and, on the strata Γ_j, for $n_j = 1$, equations of the form (1.4) hold, which contain the values of R brought to Γ_t along the trajectories. For $n_l = n-k$, $k>1$, on the strata Γ_l, we have the equations

$$\frac{\partial}{\partial t} e_l \mu_l + (\nabla, v_l e_l \mu_l) = F_l \mu_l, \tag{1.10}$$

where μ_l stands for the density of the measure ω_l with respect to the measure on Γ_l which is left-invariant with respect to the field $\frac{\partial}{\partial t} + \langle v_l, \nabla \rangle$, and F_l is defined by the following construction. Denote the ε-neighborhood of Γ_l by Γ_l^ε and denote its boundary by $\partial\Gamma_l^\varepsilon$. Consider the integral arising after integrating by parts,

$$\int_{\partial\Gamma_l^\varepsilon} \zeta \rho u_{nl} \omega_l^\varepsilon,$$

where u_{nl} is the normal component of the velocity u on $\partial\Gamma_l^\varepsilon$, while ω_l^ε stands for the Leray measure on $\partial\Gamma_l^\varepsilon$, and ζ is a test function. Passing to the limit as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0} \int_{\partial\Gamma_l^\varepsilon} \zeta \rho u_{nl} \omega_l^\varepsilon = \int_{\Gamma_l} \zeta \; F_l \omega_l.$$

As is well known, outside $\bigcup \Gamma_i$, the function $R(x,t)$ can be calculated by using the famous Cauchy formula

$$R(x,t) = \rho_0(x,t) \left| \frac{Dx}{Dx_0} \right|^{-1} \exp\left(-\int_0^t a \, dt' \right), \tag{1.11}$$

where ρ_0 is constant along the trajectories of the field and outside $\bigcup \Gamma_i$ and $\left|\frac{Dx}{Dx_0}\right|$ is the Jacobian of the mapping corresponding to the shift along the trajectories of u, while the integral in the exponent is calculated along the trajectories of the field u. This formula implies that the limit as $\varepsilon \to 0$ of the above integral exists.

It follows from what was said above that the function R is defined independently of the values of v_i on the strata under the condition that the field trajectories enter $\bigcup \Gamma_i$.

In conclusion, consider the case in which the coefficient a has a singular support on $\bigcup \Gamma_i$, i.e., $a = f(u)$. In this case, write $a\rho = \check{a}\rho + \sum f(v_i) e_i \delta(\Gamma_i)$, where $\check{a} = f(u)$ outside $\bigcup \Gamma_i$. Note that such a choice of the term $a\rho$ is not unique in this case. However, first, it is consistent with the common concept of measure solution (see [3, 5]) and, second, it is of no importance when constructing the solution outside $\bigcup \Gamma_i$ for the case in which the trajectories u enter $\bigcup \Gamma_i$.

In this case, identity (1.9) becomes

$$\int_0^\infty \int_{\mathbb{R}^n} \big(R\zeta_t + (uR, \nabla\zeta) + f(u) R\zeta \big) \, dx dt + \sum_i \int_{\Gamma_i} e_i \Big[\Big((v_i, \nabla) + \frac{\partial}{\partial t} + f(v_i) \Big) \zeta(x,t) \Big] \omega_i = 0, \tag{1.12}$$

and Eq. (1.7) can be represented as

$$\frac{\partial}{\partial t}(e_l \mu_l) + (\nabla, v_l e_l \mu_l) + f(v_l) = F_l \mu_l. \tag{1.13}$$

The above considerations imply the following statement.

Theorem 1.1. *Let the following conditions be satisfied for* $t \in [0,T]$, $T>0$:

(1) $\bigcup \Gamma_i$ *is a stratified manifold with smooth strata* Γ_i;

(2) *the trajectories of the field* u *are smooth outside* $\bigcup \Gamma_i$, *enter* $\bigcup \Gamma_i$, *and do not intersect* $\bigcup \Gamma_i$ *from outside*;

(3) *equations* (1.13) *are solvable on the strata* Γ_i;

(4) *the Kirchhoff laws are satisfied on the intersections of strata* Γ_i.

Then there is a general solution to the continuity equation (0.1) *with* $a = f(u)$ *in the sense of the integral identity* (1.12).

2. MASLOV'S TUNNEL ASYMPTOTICS

Recall that the asymptotic solutions of a general Cauchy problem for an equation with pure imaginary characteristics was first constructed by Maslov [8]. In the present paper, we consider only the following Cauchy problem:

$$-h\,\partial u/\partial t + P\big(\overset{2}{x}, -h\,\partial\overset{1}{/}\partial x\big)u = 0, \qquad u(x,t,h)|_{t=0} = e^{-S_0(x)/h}\varphi^0(x), \tag{2.1}$$

where $P(x,\xi)$ is the (smooth) symbol of the Kolmogorov–Feller operator [9], $S_0 \geqslant 0$ is a smooth function, $\varphi^0 \in C_0^\infty$, and $h \to +0$ is a small parameter characterizing the frequency and the amplitude of jumps of the Markov stochastic process whose transition probability is given by $P(x,\xi)$. To be more precise, we can keep in mind the following form of $P(x,\xi)$:

$$P(x,\xi) = (A(x)\xi,\xi) + V(x) + \int_{\mathbb{R}^n} \big(e^{i(\xi,\nu)} - 1\big)\mu(x,d\nu),$$

where $A(x)$ is a positive-definite smooth matrix function and $\mu(x,d\nu)$ is a family of bounded measures which is smooth with respect to x. The symbol $P(x,\xi)$ can also depend on t, we shall go into details below.

Locally in t, an asymptotic solution of problem (2.1) can be constructed according to the scheme of the WKB method, see [8], namely, a solution is constructed in the form

$$u = e^{-S(x,t)/h}\sum_{i=0}^{\infty}\varphi_i(x,t)h^i$$

in the sense of asymptotic series. In this case, for the functions $S(x,t)$ and $\varphi_0(x,t)$, we obtain the following problems:

$$\begin{gathered}\partial S/\partial t + P\big(x, \partial S/\partial x\big) = 0,\\ S(x,t)|_{t=0} = S_0(x),\end{gathered} \tag{2.2}$$

$$\begin{gathered}\frac{\partial\varphi_0}{\partial t} + \left(\nabla_\xi P\Big(x,\frac{\partial S}{\partial x}\Big), \nabla\varphi_0\right) + \frac{1}{2}\sum_{ij}\frac{\partial^2 P}{\partial\xi_i\partial\xi_j}\frac{\partial^2 S}{\partial x_i\partial x_j}\varphi_0 = 0,\\ \varphi_0(x,t)|_{t=0} = \varphi^0(x).\end{gathered} \tag{2.3}$$

As is known, the solution of problem (2.2) is constructed by using solutions of the Hamiltonian system that are assumed to exist and to be smooth,

$$\begin{gathered}\dot{x} = \nabla_\xi P(x,p), \qquad x|_{t=0} = x_0,\\ \dot{p} = -\nabla_x P(x,p), \qquad p|_{t=0} = \nabla S_0(x_0).\end{gathered} \tag{2.4}$$

This solution is smooth on the support of $\varphi_0(x,t)$ for all t such that the Jacobian $\big|Dx/Dx_0\big|$ does not vanish for $x_0 \in \operatorname{supp}\varphi^0(x)$. Denote by g_H^t the translation mapping along the trajectories of the Hamiltonian system (2.4).

Recall that the plot

$$\Lambda_0^n = \{x = x_0, p = \nabla S_0(x_0)\}$$

is the initial Lagrangian manifold corresponding to Eq. (2.2) and $\Lambda_t^n = g_h^t\Lambda_0^n$ is the Lagrangian manifold corresponding to Eq. (2.2) at time t. Let $\pi\colon \Lambda_t^n \to \mathbb{R}_x^n$ be the projection of Λ_t^n to $\mathbb{R}_x^n$, which is assumed to be proper. The point $\alpha \in \Lambda_t^n$ is said to be *essential* if

$$\hat{S}(\alpha,t) = \min_{\beta\in\pi^{-1}(\alpha)}\hat{S}(\beta,t)$$

and *nonessential* otherwise. Here $\hat{S}$ stands for the action on Λ_t^n given by the formula

$$\hat{S}(\beta,t)=\int_0^t p\,dx-H\,dt,$$

where the integral is taken along the trajectories of (2.4) whose origin is projected to $x_0=\beta$. As is known, $S(x,t)=\hat{S}(\pi^{-1}x,t)$ at the regular points at which the projection π is bijective.

The global-in-time asymptotic solution of problem (2.1) is given by the Maslov tunnel canonical operator.

To define this operator, following [8, 10], we introduce the set of essential points $\bigcup\gamma_{it}\subset\Lambda_t^n$. This set is closed because the projection π is proper, that is, for all x, the set of p such that $(x,p)\in\Lambda_t^n,\pi(x,p)=x$, is finite.

Suppose that the open domains $U_j\subset\Lambda_t^n$ form a locally finite covering of the set $\bigcup\gamma_{it}$. If the set U_j consists of regular points, then we write

$$u_j=e^{-S_j(x,t)/h}\varphi_{0j}(x,t),\quad\text{where}\quad\varphi_{0j}(x,t)=\psi_{0j}(x,t)\left|\frac{Dx_0}{Dx}\right|^{1/2}\tag{2.5}$$

and $\psi_{0j}(x,t)$ stands for the solution of the equation

$$\frac{\partial\psi_{0j}}{\partial t}+(P_\xi(x,\nabla S_j),\nabla\psi_{0j})-\frac{1}{2}\operatorname{tr}\frac{\partial^2 P}{\partial x\partial\xi}(x,\nabla S_j)\psi_{0j}=0,\tag{2.6}$$

which exists and is smooth whenever $\left|Dx/Dx_0\right|\neq 0$. The solution u_j in the domain containing essential (nonregular) points (at which $d\pi$ is degenerate) is given as follows: the canonical change of variables is carried out in such a way that the nonregular points become regular, after which we determine a fragment of the solution in the new coordinates by formula (2.5) and return to the old variables, applying the "quantum" inverse canonical transformation to the solution obtained in the new coordinates.

The Hamiltonian determining this canonical transformation is of the form

$$H_\sigma=\frac{1}{2}\sum_{i=1}^n\sigma_k p_k^2,$$

where $\sigma_1,\dots,\sigma_n=\text{const}>0$.

The canonical transformation to the new variables is given by the translation by the time -1 along the trajectories of the Hamiltonian H_σ. One can prove (see [8, 10]) that the family of sets σ for which the change of variables takes a regular point into a nonregular is not empty.

Next, the solution near the essential point is determined by the relation

$$u_j=e^{\frac{1}{h}\hat{H}_\sigma}\tilde{u}_j,\tag{2.7}$$

where $\tilde{u}_j$ is given by formula (2.5) in the new variables and

$$\hat{H}_\sigma=\frac{1}{2}\sum_{k=1}^n\sigma_k\big(-h\,\partial/\partial x_k\big)^2.$$

On the intersections of singular (containing singular points) and nonsingular charts (without singular points), we must match S_j and ψ_{0j}. This can be done by applying the Laplace method to the integral whose kernel is a fundamental solution for the operator $-h\frac{\partial}{\partial t}+\hat{H}_\sigma$. The integral appears when we write out the right-hand side of (2.7) in detail. In this case, since the solution is real, the Maslov index which occurs in hyperbolic problems, as is well known [8], does not occur here. The complete representation of the solution of (2.1) is obtained by summing functions of the type (2.5) and (2.7) over all the domains U_j; for more detail, see [8, 10].

The asymptotics thus constructed is justified, i.e., the proximity between the exact and asymptotic solutions of the Cauchy problem (2.1) is proved [8, 9]. More precisely, it is proved that the estimate $u(x,t,h) - u_j = O(h)$ holds at the points of the set $\pi(\bigcup \gamma_{it})$ at which the projection π is bijective.

In the preceding case, we noted that values of the solution of the continuity equation at nonregular points are independent of the values of the solution on the support of the singularity (certainly, the inverse influence takes place) by the condition that the velocity field trajectories enter the singular support.

In the case of the canonical operator construction briefly described above, the relation between the solutions at essential and nonessential points is also unilateral, namely, the essential points are "bypassed" (using (2.7)), whereas the values of the functions $\tilde{\psi}_{oj}$ contained in $\tilde{u}_j$ on the singularity support do not determine the values at the regular points (however, the converse fails to hold).

Note that the function $S(x,t)$ given by $S(x,t)|_{U_j} = S_j(\pi^{-1}(\alpha),t)$ is globally determined and continuous at the points of the domain $\pi(\bigcup \gamma_{it}) \subset \mathbb{R}^n_x$. Denote this set by $\bigcup \Gamma_i$ and assume that this is a stratified manifold with smooth strata Γ_{it} of different codimensions. Note that, for example, if the inequality $\nabla(S_i(x,t) - S_j(x,t)) \neq 0$ holds while we pass from one branch $\Lambda^n_t \cap \bigcup \gamma_{it}$ to another one, then the set $\pi\{(\tilde{S}_i - \tilde{S}_j) = 0\}$ generates a smooth stratum of codimension 1. In the one-dimensional case, all strata are points or curves on the (x,t)-plane (under the above assumptions about the singularities that are discrete).

Consider now the equation for ψ^2_{0j}. Denote this function by ρ. We then obtain

$$\partial\rho/\partial t + (\nabla, u\rho) + a\rho = 0, \tag{2.8}$$

where $u(x,t) = \nabla_\xi P(x, \nabla S)$ and $a = -\operatorname{tr} \frac{\partial^2 P}{\partial x \partial \xi}(x, \nabla S)$.

If the condition $\operatorname{Hess}_\xi P(x,\xi) > 0$ is satisfied, then it follows from the implicit function theorem that $\nabla S(x,t) = F(x, u(x,t))$, where $F(x,u)$ is a smooth function and $a = f(x,u)$, where $f(x,z)$ is a smooth function again.

Let us return to formula (1.8) and denote the regular part of ρ (in the sense of distributions) by ρ_{reg}. This leads to

Theorem 2.1. *Suppose that the following conditions are satisfied for $t \in [0,T]$, $T > 0$.*

(1) *There exists a smooth solution of the Hamiltonian system* (2.4)*.*

(2) *The singularities of the velocity field $u = \nabla_\xi P(x, \nabla S)$ form a stratified manifold with smooth strata and with $\operatorname{Hess}_\xi P(x,\xi) > 0$.*

(3) *There is a generalized solution ρ of the Cauchy problem for Eq.* (2.8) *in the sense of the integral identity* (1.10)*.*

Then, at the points of $\pi(\bigcup \gamma_{it})$, where the projection π is bijective, the asymptotic solution of the Cauchy problem (2.1) *is of the form*

$$u = \exp(-S(x,t)/h)(\sqrt{\rho_{\mathrm{reg}}} + O(h)).$$

This theorem is a global-in-time analog of the corresponding Madelung observation concerning local solutions of Schrödinger-type equations.

Let us now show a relationship between the solutions of the continuity and transport equations. It is easy to see that, by construction, a solution to the transport equation is equal to ρ_{reg} outside the points of singular support. Moreover, one can define $\sqrt{\rho}$ globally in the distributional sense, and the square root thus defined is equal to ρ_{reg}. To prove this statement, it is sufficient to note that the δ-shock type solution to the continuity equation in the form (1.8) can be obtained ([12, 11]) as a weak limit of a weak asymptotic solution of the form

$$\rho(x,t) = \rho_{\mathrm{reg}} + \sum e_i \delta_\varepsilon(\Gamma_i), \tag{2.9}$$

where $x \in R^n$, ε is an auxiliary small parameter and δ_ε is a regularization of the δ-function of the form $\delta_\varepsilon = \varepsilon^{-(n-n_i)}\omega(S_i/\varepsilon)$. Here S_i are smooth functions such that $\nabla S_i|_{S_i=0} \neq 0$, the equation $S_i = 0$ defines the strata Γ_i, $n_i = \dim \Gamma_i$, and $\omega = \omega(\eta)$ belongs to the Schwartz space of test functions. Without loss of generality, one can consider the case $n_i = 0$ in an (x,t) half-plane for each fixed t. In this case, the function S_i can be chosen in the form $S = x - \phi(t)$, and our statement becomes

$$\lim_{\varepsilon\to 0}\int_{R^1}\left(\sqrt{(\rho_{\rm reg} + e_i\varepsilon^{-1}\omega((x-\phi)/\varepsilon))} - \sqrt{\rho_{\rm reg}}\right)\varphi(x)dx = 0, \tag{2.10}$$

for an arbitrary test function $\varphi(x)$ in C_0^∞.

After the change of variables $\eta = (x-\phi)/\varepsilon$, the (2.10) obtains the following equivalent form

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{R^1}\left(\sqrt{\varepsilon\rho_{\rm reg} + e_i\omega(\eta)} - \sqrt{\varepsilon\rho_{\rm reg}}\right)d\eta = 0, \tag{2.11}$$

which is obviously true under the trivial inequality $\sqrt{1+z} \leqslant 1 + z/2$. Indeed, consider the domain in R^1_η, where $\omega^\alpha < \rho_{\rm reg}\varepsilon$ for some α, $0 < \alpha < 1$. Then the above inequality gives the estimate $O(\varepsilon^{1/2})$ for the integral under the sign of the limit in (2.11). In the case of $\omega^\alpha \geqslant \rho_{\rm reg}\varepsilon$, because the function ω is decreasing (belongs to the Schwartz space), the measure of the corresponding domain can be estimated as $(\rho_{\rm reg}\varepsilon)^{-1/N}$, where N is an arbitrary positive number. This completes the proof.

3. PARTICULAR CASES

The theorem stated in the previous section needs some assumptions. The most restrictive is item 3 of the above theorem, i.e., the existence of a global generalized solution to the continuity equation. Under the above assumptions, it is possible to construct this solution by using characteristics: however, this is possible only if the structure of singular support of u is preserved in the course of time, and all sections of the above stratified manifold by planes of the form $t = \text{const}$ are diffeomorphic. A more complicated situation arises if the singularities of the velocity field change their structure. In this case, the problem of constructing a global-in-time generalized solution to the continuity equation has not been solved yet. The obstacle is that, in this case, as a rule, one has no global-in-time expression for the velocity field u. In turn, this gives no possibility to apply formula (1.11) to construct a global solution to the continuity equation. In the multidimensional case, as far as we know, there is only one result concerning shock wave generation [14] which enables one to construct global-in-time approximation of the shock wave formation process. However, this construction is slightly different from what we need here. In the one-dimensional case, the situation is better, and we have all required formulas.

Let us begin with the spatially homogeneous case. Here the problem is equivalent to that of constructing the formula for a global solution to the conservation law equation

$$\partial v/\partial t + \partial P(v,t)/\partial x = 0. \tag{3.1}$$

Here $P(-h\,\partial/\partial x, t)$ is the same operator as in (2.1), but it is assumed to be independent of x, has the symbol $P(\xi,t)$, and $v = \partial S/\partial x$. The velocity field u, in this case, is $P_\xi(v,t)$. In [11], there is a construction of a global solution to the continuity equation, where the velocity field is given by the solution of equation (3.1). Because the set of singular points is discrete by our assumptions, without loss of generality, one can consider the case in which there is only one point of singularity. Denote the corresponding (smooth) initial condition by u_0, the moment at which the singularity happens by t^*, and the point of singularity by x^*.

The first step of the construction suggested in [11, 12] (see also [18]) is that we change u_0 in a small neighborhood of the origin x_0^* of the trajectory coming to x^* when $t = t^*$. Denote this new part of initial data by $u_1(x_0)$ for $x_0 \in (x_0^* - \beta, x_0^* + \beta)$, $\beta \to 0$, and assume that

$$\varepsilon\beta^{-1} \to 0, \qquad \varepsilon \to 0. \tag{3.2}$$

Introduce the function $u_1 = u_1(x_0, t)$ as a solution of the implicit equation

$$P'_\xi(u_1, t) = -K(t)x_0 + b(t). \tag{3.3}$$

The last equation is solvable provided that $\mathrm{Hess}_\xi P(x,\xi) > 0$, as was assumed above.

The functions $K(t)$ and $b(t)$ are defined from the condition of continuity of the characteristics flow, i.e.,

$$u_1(x_0^* - \beta, t) = u_0(x_0^* - \beta, t), \qquad u_1(x_0^* + \beta, t) = u_0(x_0^* + \beta, t).$$

It is easy to see that this choice of u_1 insures that the Jacobian $\big|Dx/Dx_0\big|$ is identically equal to 0 for $t = t^*$ and $x_0 \in (x_0^* - \beta, x_0^* + \beta)$. Here we remove the situation of identical equality that can be destroyed by a small perturbation based on the usual topological concept of general position. However, this construction follows from the algebraic concept and enables us to present the solution of (3.1) in the form of a linear combination of Heaviside functions (see [11]).

The second step of our construction for an approximation is a modification of the definition of characteristics. Write

$$\dot{x} = (1 - B)P'_\xi(u_1(x_0, t), t) + Bc, \qquad x_0 \in (x_0^* - \beta, x_0^* + \beta) \tag{3.4}$$

and $\dot{x} = P'_\xi(u_0, t)$, where x_0 does not belong to $(x_0^* - \beta, x_0^* + \beta)$ and

$$c = \frac{P(v(x(x_0^* + \beta, t), t)) - P(v(x(x_0^* - \beta, t), t))}{v(x(x_0^* + \beta, t), t) - v(x(x_0^* - \beta, t), t)}.$$

The initial data for (3.4) are

$$x\big|_{t=0} = x_0 + A\varepsilon, \qquad \varepsilon > 0.$$

The function B in (3.4) is of the form $B = B((t - t^*)/\varepsilon)$ and $B(z)$ is smooth, monotone, and increasing from 0 to 1 for $z \in (-\infty, \infty)$. Just as in [11, 12], one can prove that there is an $A = \mathrm{const}$ such that the Jacobian $\big|Dx/Dx_0\big|$ calculated by using the above characteristics is nonzero, but it is of order $O(\varepsilon)$ as $t \geqslant t^* + O(\beta)$ for $x_0 \in (x_0^* - \beta, x_0^* + \beta)$. Using the velocity field generated by $\dot{x}$, we can construct a global-in-time (smooth) solution of the continuity equation in the form (1.11). After that, passing to the limit as $\varepsilon \to 0$, we obtain a generalized solution of the continuity equation in the sense of the definition in Section 1, just as in [12].

Spatially inhomogeneous one-dimensional case. We follow the scheme introduced above. The case under consideration can be treated in the same way as the previous one, with some modifications. Firstly, assume that the symbol $P = P(x, \xi)$ does not depend on t. In the present case, this assumption (which means that the mapping g_P^t is invertible) will be used to construct an insertion in the initial data. In the previous case, we have done this by using the implicit function theorem, see (3.3).

Let Λ_0^1 be a smooth nonsingular curve (with respect to the projection π) in the (x, p) space, which is a Lagrangian manifold corresponding to the initial data of our problem. Consider the Lagrangian manifold $\Lambda_{t^*}^1 = g_P^{t^*}\Lambda_0^1$ and assume that there is only one point singular with respect to the projection to the x axis, and that this projection is x^*. Let β be as above. Write $t_1^* = t^* + \beta$. By the assumption that $P''_{\xi\xi}$ is positive, for $t = t_1^*$, the Lagrangian manifold $\Lambda_{t_1^*}^1$ has two parts which contain essential points, and these parts form a shock-wave-type curve with the jump at the point x_1^*, where $S_{\mathrm{left}}(x_1^*, t_1^*) = S_{\mathrm{right}}(x_1^*, t_1^*)$. Let us connect these parts by a vertical line, and thus obtain a new Lagrangian manifold, which is a piecewise smooth continuous curve with two angular points (the ends of the vertical part; the distance between them being of order β). Denote this manifold by $\hat{\Lambda}_{t_1^*}^1$ and apply the mapping $g_P^{-t_1}$ for a sufficiently small t_1 to this manifold. This mapping obviously exists and is a diffeomorphism, because our Hamiltonian P does not depend on t. Consider the manifold $g_P^{-t_1}\hat{\Lambda}_{t_1^*}^1$ thus obtained as a new Lagrangian manifold corresponding to our problem for $t = t_1^* - t_1$, replacing the manifold $\Lambda_{t_1^* - t_1}^1$ by $g_P^{-t_1}\hat{\Lambda}_{t_1^*}^1$. As was said above, the latter manifold is a piecewise smooth curve with two angular points, and all points of the curve outside the part between these angular points are regular. Moreover, there is a sufficiently small t_1 such that the part of the curve between these angular points contains only regular points; these statements hold because P'' is positive and stationarity and t_1 can be chosen to be small enough (and independent of ε).

Denote the projections of the above angular points on the manifold $g_P^{-t_1}\hat{\Lambda}^1_{t_1^*}$ to the x axis by a_1 and a_2, $a_1 < a_2$, and note that $\left|a_1 - a_2\right|$ is of order β.

As in the previous example, introduce the new characteristics system

$$\dot{x} = (1-B)P'_\xi(x(x_0,t), p(x_0,t)) + Bc, \tag{3.5}$$

$$\dot{p} = -(1-B)P'_x(x(x_0,t), p(x_0,t)), \qquad x_0 \in (a_1, a_2),$$

and

$$\dot{x} = P'_\xi(x,p), \qquad \dot{p} = -P'_x(x,p), \tag{3.6}$$

if x_0 does not belong to (a_1, a_2). We have chosen

$$c = \frac{P(v(x(a_2,t), p(a_2,t))) - P(x(a_1,t), p(a_1,t))}{p(x(a_2,t),t) - p(x(a_1,t),t)}. \tag{3.7}$$

The initial data for (3.5), (3.6) are

$$x\big|_{t=0} = x_0 + A\varepsilon \quad \text{and} \quad p\big|_{t=0} = p_0(x_0),$$

where

$$(x_0, p_0(x_0)) = g_P^{-t_1}\hat{\Lambda}^1_{t_1^*}.$$

The expression on the right-hand side of (3.7) is the direct analog of the well-known Rankine–Hugoniot expression for the velocity of the shock propagation. In the case under consideration, this is the velocity of the point $\check{x}$ on the x axis, where

$$S_{\text{left}}(\check{x}, t) = S_{\text{right}}(\check{x}, t).$$

By assumption, we have only one singular point if we are considering the family of manifolds Λ^1_t, $t \in [0, t^*]$. Moreover, by construction, the Jacobian $J = Dx/Dx_0$ (evaluated by using the solutions of system (3.5)) is nonzero. More precisely,

$$\lim_{\varepsilon \to 0} J = H(t^* - t)J_0,$$

where J_0 is the Jacobian calculated using the solutions of (3.5) for $B = 0$ ($J_0 = 0$ for $t = t^*$ by construction) and

$$J \geqslant H(t^* - t)J_0 + C\varepsilon,$$

where $C = \text{const} > 0$. This statement directly follows from (3.5) if we take the properties of the function B into account. This means that the velocity field generated by the projections of the solution of system (3.5), (3.6) to the x axis has disjoint trajectories for $\varepsilon > 0$. Thus, we can use this field to construct solutions of the continuity equation. It remains to note that, just as in [12], it can readily be seen that the limits of these solutions satisfy the integral identities introduced in Section 1 as the definition of generalized solutions to the continuity equation.

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