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ADVANCES IN Mathematics
Advances in Mathematics 230 (2012) 414–432
www.elsevier.com/locate/aim

Pursuing the double affine Grassmannian II: Convolution

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Received 13 January 2011; accepted 13 December 2011

Available online 15 December 2011

Communicated by Roman Bezrukavnikov

Abstract

This is the second paper of a series (started by Braverman and Finkelberg, 2010 [2]) which describes a conjectural analog of the affine Grassmannian for affine Kac–Moody groups (also known as the double affine Grassmannian). The current paper is dedicated to describing a conjectural analog of the convolution diagram for the double affine Grassmannian. In the case when $G = \mathrm{SL}(n)$ our conjectures can be derived from Nakajima (2009) [12].

Keywords: Affine Grassmannian; Convolution; Quiver varieties

1. Introduction

1.1. The usual affine Grassmannian

Let G be a connected complex reductive group and let $\mathcal{K} = \mathbb{C}((s))$, $\mathcal{O} = \mathbb{C}[[s]]$. By the *affine Grassmannian* of G we shall mean the quotient $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$. It is known (cf. [1,10]) that Gr_G is the set of $\mathbb{C}$-points of an ind-scheme over $\mathbb{C}$, which we will denote by the same symbol. Note that Gr_G is defined for any (not necessarily reductive) group G.

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.12.007

Let $\Lambda = \Lambda_G$ denote the coweight lattice of G and let $\Lambda^\vee$ denote the dual lattice (this is the weight lattice of G). We let $2\rho_G^\vee$ denote the sum of the positive roots of G.

The group-scheme $G(\mathcal{O})$ acts on Gr_G on the left and its orbits can be described as follows. One can identify the lattice Λ_G with the quotient $T(\mathcal{K})/T(\mathcal{O})$. Fix $\lambda \in \Lambda_G$ and let s^λ denote any lift of λ to $T(\mathcal{K})$. Let Gr_G^λ denote the $G(\mathcal{O})$-orbit of s^λ (this is clearly independent of the choice of s^λ). The following result is well known:

Lemma 1.2.

(1)

$$\mathrm{Gr}_G = \bigcup_{\lambda \in \Lambda_G} \mathrm{Gr}_G^\lambda .$$

(2) *We have* $\mathrm{Gr}_G^\lambda = \mathrm{Gr}_G^\mu$ *if an only if* λ *and* μ *belong to the same* W*-orbit on* Λ_G (*here* W *is the Weyl group of* G). *In particular,*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda_G^+} \mathrm{Gr}_G^\lambda .$$

(3) *For every* $\lambda \in \Lambda^+$ *the orbit* Gr_G^λ *is finite-dimensional and its dimension is equal to* $\langle \lambda, 2\rho_G^\vee \rangle$.

Let $\overline{\mathrm{Gr}_G}^\lambda$ denote the closure of Gr_G^λ in Gr_G; this is an irreducible projective algebraic variety; one has $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G}^\lambda$ if and only if $\lambda - \mu$ is a sum of positive roots of $G^\vee$. We will denote by IC^λ the intersection cohomology complex on $\overline{\mathrm{Gr}_G}^\lambda$. Let $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ denote the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G. It is known that every object of this category is a direct sum of the IC^λ's.

1.3. Transversal slices

Consider the group $G[s^{-1}] \subset G((s))$; let us denote by $G[s^{-1}]_1$ the kernel of the natural ("evaluation at ∞") homomorphism $G[s^{-1}] \to G$. For any $\lambda \in \Lambda$ let $\mathrm{Gr}_{G,\lambda} = G[s^{-1}] \cdot s^\lambda$. Then it is easy to see that one has

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_{G,\lambda} .$$

Let also $\mathcal{W}_{G,\lambda}$ denote the $G[s^{-1}]_1$-orbit of s^λ. For any $\lambda, \mu \in \Lambda^+$, $\lambda \geqslant \mu$ set

$$\mathrm{Gr}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_{G,\mu}, \qquad \overline{\mathrm{Gr}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathrm{Gr}_{G,\mu}$$

and

$$\mathcal{W}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathcal{W}_{G,\mu}, \qquad \overline{\mathcal{W}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathcal{W}_{G,\mu}.$$

Note that $\overline{\mathcal{W}}^{\lambda}_{G,\mu}$ contains the point s^{μ} in it. The variety $\overline{\mathcal{W}}^{\lambda}_{G,\mu}$ can be thought of as a transversal slice to Gr^{μ}_G inside $\overline{\mathrm{Gr}}^{\lambda}_G$ at the point s^{μ} (cf. [2, Lemma 2.9]).

1.4. The convolution

We can regard $G(\mathcal{K})$ as a total space of a $G(\mathcal{O})$-torsor over Gr_G. In particular, by viewing another copy of Gr_G as a $G(\mathcal{O})$-scheme, we can form the associated fibration

$$\mathrm{Gr}_G \star \mathrm{Gr}_G := G(\mathcal{K}) \underset{G(\mathcal{O})}{\times} \mathrm{Gr}_G .$$

One has the natural maps $p, m : \mathrm{Gr}_G \star \mathrm{Gr}_G \to \mathrm{Gr}_G$ defined as follows. Let $g \in G(\mathcal{K})$, $x \in \mathrm{Gr}_G$. Then

$$p(g \times x) = g \mod G(\mathcal{O}); \qquad m(g \times x) = g \cdot x.$$

For any $\lambda_1, \lambda_2 \in \Lambda^+_G$ let us set $\mathrm{Gr}^{\lambda_1}_G \star \mathrm{Gr}^{\lambda_2}_G$ to be the corresponding subscheme of $\mathrm{Gr}_G \star \mathrm{Gr}_G$; this is a fibration over $\mathrm{Gr}^{\lambda_1}_G$ with the typical fiber $\mathrm{Gr}^{\lambda_2}_G$. Its closure is $\overline{\mathrm{Gr}}^{\lambda_1} \star \overline{\mathrm{Gr}}^{\lambda_2}$. In addition, we define

$$\left(\mathrm{Gr}^{\lambda_1}_G \star \mathrm{Gr}^{\lambda_2}_G\right)^{\lambda_3} = m^{-1}\left(\mathrm{Gr}^{\lambda_3}_G\right) \cap \left(\mathrm{Gr}^{\lambda_1}_G \star \mathrm{Gr}^{\lambda_2}_G\right).$$

It is known (cf. [9]) that

$$\dim\left(\left(\mathrm{Gr}^{\lambda_1}_G \star \mathrm{Gr}^{\lambda_2}_G\right)^{\lambda_3}\right) = \langle \lambda_1 + \lambda_2 + \lambda_3, \rho^{\vee}_G \rangle. \tag{1.1}$$

(It is easy to see that although $\rho^{\vee}_G \in \frac{1}{2}\Lambda^{\vee}_G$, the RHS of (1.1) is an integer whenever the above intersection is non-empty.)

Starting from any perverse sheaf $\mathcal{T}$ on Gr_G and a $G(\mathcal{O})$-equivariant perverse sheaf $\mathcal{S}$ on Gr_G, we can form their twisted external product $\mathcal{T} \tilde{\boxtimes} \mathcal{S}$, which will be a perverse sheaf on $\mathrm{Gr}_G \star \mathrm{Gr}_G$. For two objects $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ we define their convolution

$$\mathcal{S}_1 \star \mathcal{S}_2 = m_!(\mathcal{S}_1 \tilde{\boxtimes} \mathcal{S}_2).$$

The following theorem, which is a categorical version of the Satake equivalence, is a starting point for this paper, cf. [9,6,10]. The best reference so far is [1, Section 5.3].

Theorem 1.5.

(1) *Let* $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$. *Then* $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$.
(2) *The convolution* $\star$ *extends to a structure of a tensor category on* $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$.
(3) *As a tensor category,* $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ *is equivalent to the category* $\mathrm{Rep}(G^{\vee})$. *Under this equivalence, the object* IC^{λ} *goes over to the irreducible representation* $L(\lambda)$ *of* $G^{\vee}$ *with highest weight* λ.

1.6. n-fold convolution

Similarly to the above, we can define the n-fold convolution diagram

$$m_n : \underbrace{\mathrm{Gr}_G \star \cdots \star \mathrm{Gr}_G}_{n} \to \mathrm{Gr}_G .$$

Here

$$\underbrace{\mathrm{Gr}_G \star \cdots \star \mathrm{Gr}_G}_{n} = \underbrace{G(\mathcal{K}) \underset{G(\mathcal{O})}{\times} \cdots \underset{G(\mathcal{O})}{\times} G(\mathcal{K})}_{n-1} \underset{G(\mathcal{O})}{\times} \mathrm{Gr}_G$$

and m_n is the multiplication map. Thus, given n objects $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ we may consider the convolution $\mathcal{S}_1 \star \cdots \star \mathcal{S}_n$; this will be again an object of $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ which under the equivalence of Theorem 1.5 corresponds to n-fold tensor product in $\mathrm{Rep}(G^\vee)$. In particular, let $\lambda_1, \ldots, \lambda_n$ be elements of Λ^+. One can consider the corresponding subvariety $\overline{\mathrm{Gr}}_G^{\lambda_1} \star \cdots \star \overline{\mathrm{Gr}}_G^{\lambda_n}$ in $\underbrace{\mathrm{Gr}_G \star \cdots \star \mathrm{Gr}_G}_{n}$. Then the convolution $\mathrm{IC}^{\lambda_1} \star \cdots \star \mathrm{IC}^{\lambda_n}$ is just the direct image $(m_n)_!(\mathrm{IC}(\overline{\mathrm{Gr}}_G^{\lambda_1} \star \cdots \star \overline{\mathrm{Gr}}_G^{\lambda_n}))$. In particular, we have an isomorphism

$$(m_n)_!\big(\mathrm{IC}\big(\overline{\mathrm{Gr}}_G^{\lambda_1} \star \cdots \star \overline{\mathrm{Gr}}_G^{\lambda_n}\big)\big) \simeq \bigoplus_{\nu \in \Lambda^+} \mathrm{IC}^\nu \otimes \mathrm{Hom}\big(L(\nu), L(\lambda_1) \otimes \cdots \otimes L(\lambda_n)\big). \tag{1.2}$$

1.7. The group G_{aff}

From now on we assume that G is almost simple and simply connected. To a connected reductive group G as above one can associate the corresponding affine Kac–Moody group G_{aff} in the following way. One can consider the polynomial loop group $G[t, t^{-1}]$ (this is an infinite-dimensional group ind-scheme)

It is well known that $G[t, t^{-1}]$ possesses a canonical central extension $\widetilde{G}$ of $G[t, t^{-1}]$:

$$1 \to \mathbb{G}_m \to \widetilde{G} \to G\big[t, t^{-1}\big] \to 1.$$

Moreover, $\widetilde{G}$ has again a natural structure of a group ind-scheme.

The multiplicative group $\mathbb{G}_m$ acts naturally on $G[t, t^{-1}]$ and this action lifts to $\widetilde{G}$. We denote the corresponding semi-direct product by G_{aff}; we also let $\mathfrak{g}_{\mathrm{aff}}$ denote its Lie algebra.

The Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ is an untwisted affine Kac–Moody Lie algebra. In particular, it can be described by the corresponding affine root system. We denote by $\mathfrak{g}_{\mathrm{aff}}^\vee$ the *Langlands dual affine Lie algebra* (which corresponds to the dual affine root system) and by $G_{\mathrm{aff}}^\vee$ the corresponding dual affine Kac–Moody group, normalized by the property that it contains $G^\vee$ as a subgroup (cf. [2, Section 3.1] for more details).

We denote by $\Lambda_{\mathrm{aff}} = \mathbb{Z} \times \Lambda \times \mathbb{Z}$ the coweight lattice of G_{aff}; this is the same as the weight lattice of $G_{\mathrm{aff}}^\vee$. Here the first $\mathbb{Z}$-factor is responsible for the center of $G_{\mathrm{aff}}^\vee$ (or $\widehat{G}^\vee$); it can also be thought of as coming from the loop rotation in G_{aff}. The second $\mathbb{Z}$-factor is responsible for the loop rotation in $G_{\mathrm{aff}}^\vee$ (it may also be thought of as coming from the center of G_{aff}). We denote by Λ^+_{aff} the set of dominant weights of $G^\vee_{\mathrm{aff}}$ (which is the same as the set of dominant coweights of G_{aff}). We also denote by $\Lambda_{\mathrm{aff},k}$ the set of weights of $G^\vee_{\mathrm{aff}}$ of level k, i.e. all the weights of the form $(k, \overline{\lambda}, n)$. We put $\Lambda^+_{\mathrm{aff},k} = \Lambda^+_{\mathrm{aff}} \cap \Lambda_{\mathrm{aff},k}$.

Important notational convention. From now on we shall denote elements of Λ by $\overline{\lambda}, \overline{\mu}, \ldots$ (instead of just writing $\lambda, \mu, \ldots$) in order to distinguish them from the coweights of G_{aff} (= weights of $G^\vee_{\mathrm{aff}}$), which we shall just denote by $\lambda, \mu, \ldots$.

Let $\Lambda^+_k \subset \Lambda$ denote the set of dominant coweights of G such that $\langle \overline{\lambda}, \alpha \rangle \leqslant k$ when α is the highest root of $\mathfrak{g}$. Then it is well known that a weight $(k, \overline{\lambda}, n)$ of $G^\vee_{\mathrm{aff}}$ lies in $\Lambda^+_{\mathrm{aff},k}$ if and only if $\overline{\lambda} \in \Lambda^+_k$ (thus $\Lambda_{\mathrm{aff},k} = \Lambda^+_k \times \mathbb{Z}$).

Let also W_{aff} denote affine Weyl group of G which is the semi-direct product of W and Λ. It acts on the lattice Λ_{aff} (resp. $\widehat{\Lambda}$) preserving each $\Lambda_{\mathrm{aff},k}$ (resp. each $\widehat{\Lambda}_k$). In order to describe this action explicitly it is convenient to set $W_{\mathrm{aff},k} = W \ltimes k\Lambda$ which naturally acts on Λ. Of course the groups $W_{\mathrm{aff},k}$ are canonically isomorphic to W_{aff} for all k. Then the restriction of the W_{aff}-action to $\Lambda_{\mathrm{aff},k} \simeq \Lambda \times \mathbb{Z}$ comes from the natural $W_{\mathrm{aff},k}$-action on the first multiple.

It is well known that every W_{aff}-orbit on $\Lambda_{\mathrm{aff},k}$ contains unique element of $\Lambda^+_{\mathrm{aff},k}$. This is equivalent to saying that $\Lambda^+_k \simeq \Lambda / W_{\mathrm{aff},k}$.

1.8. Transversal slices for $\mathrm{Gr}_{G_{\mathrm{aff}}}$

Our main dream is to create an analog of the affine Grassmannian Gr_G and the above results about it in the case when G is replaced by the (infinite-dimensional) group G_{aff}. The first attempt to do so was made in [2]: namely, in [2] we have constructed analogs of the varieties $\overline{\mathcal{W}}^\lambda_{G,\mu}$ in the case when G is replaced by G_{aff}. In the current paper, we are going to construct analogs of the varieties $m_n^{-1}(\overline{\mathcal{W}}^\lambda_{G,\mu}) \cap (\mathrm{Gr}^{\lambda_1}_G \star \cdots \star \mathrm{Gr}^{\lambda_n}_G)$ and $m_n^{-1}(\overline{\mathcal{W}}^\lambda_{G,\mu}) \cap (\overline{\mathrm{Gr}}^{\lambda_1}_G \star \cdots \star \overline{\mathrm{Gr}}^{\lambda_n}_G)$ (here $\lambda = \lambda_1 + \cdots + \lambda_n$) when G is replaced by G_{aff}. We shall also construct (cf. Section 3.11) analogs of the corresponding pieces in the *Beilinson–Drinfeld Grassmannian* for G_{aff} (cf. Section 3.10 for a short digression on the Beilinson–Drinfeld Grassmannian for G).

To formulate the idea of our construction, let us first recall the construction of the affine analogs of the varieties $\overline{\mathcal{W}}^\lambda_{G,\mu}$. Let $\mathrm{Bun}_G(\mathbb{A}^2)$ denote the moduli space of principal G-bundles on $\mathbb{P}^2$ trivialized at the "infinite" line $\mathbb{P}^1_\infty \subset \mathbb{P}^2$. This is an algebraic variety which has connected components parametrized by non-negative integers, corresponding to different values of the second Chern class of the corresponding bundles; we denote the corresponding connected component by $\mathrm{Bun}^a_G(\mathbb{A}^2)$ (here $a \geqslant 0$). According to [3] one can embed $\mathrm{Bun}^a_G(\mathbb{A}^2)$ (as an open dense subset) into a larger variety $\mathcal{U}^a_G(\mathbb{A}^2)$ which is called *the Uhlenbeck moduli space of G-bundles on $\mathbb{A}^2$ of second Chern class a*.[1] Furthermore, for any $k \geqslant 0$, let $\Gamma_k \subset \mathrm{SL}(2)$ be the group of k-th roots of unity. This group acts naturally on $\mathbb{A}^2$ and $\mathbb{P}^2$ and this action can be lifted to an action of Γ_k on $\mathrm{Bun}_G(\mathbb{A}^2)$ and $\mathcal{U}_G(\mathbb{A}^2)$. This lift depends on a choice of a homomorphism $\Gamma_k \to G$ which is responsible for the action of Γ_k on the trivialization of our G-bundles on $\mathbb{P}^1_\infty$; it is explained in [2] that to such a homomorphism one can associate a dominant weight μ of $G^\vee_{\mathrm{aff}}$ of level k; in the future we shall denote the set of all such weights by Λ^+_k. We denote by $\mathrm{Bun}_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ the set of fixed points of Γ_k on $\mathrm{Bun}_G(\mathbb{A}^2)$. In [2] we construct a bijection

[1] This space is an algebraic analog of the Uhlenbeck compactification of the moduli space of instantons on a Riemannian 4-manifold.

between connected components of $\mathrm{Bun}_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ and dominant weights λ of G_{aff} such that $\lambda \geqslant \mu$. We denote the corresponding connected component by $\mathrm{Bun}^{\lambda}_{G,\mu}(\mathbb{A}^2/\Gamma_k)$; we also denote by $\mathcal{U}^{\lambda}_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ its closure in $\mathcal{U}_G(\mathbb{A}^2)$.[2] In [2] we explain in what sense the variety $\mathcal{U}^{\lambda}_{G,\mu}$ should be thought of as the correct version of $\overline{\mathcal{W}}^{\lambda}_{G_{\mathrm{aff}},\mu}$.

1.9. Bundles on mixed Kleinian stacks

For a scheme X endowed with an action of a finite group Γ we shall denote by $X//\Gamma$ the scheme-theoretic (categorical) quotient of X by Γ; similarly, we denote by X/Γ the corresponding quotient stack.[3]

Given positive integers $k_1, \dots, k_n$ such that $\sum_{i=1}^n k_i = k$, we set

$$\mathbb{A}^2//\Gamma_k = \underline{S}_k \subset \overline{\underline{S}}_k := \mathbb{P}^2//\Gamma_k, \qquad \mathbb{A}^2/\Gamma_k = S_k \subset \overline{S}_k = \mathbb{P}^2/\Gamma_k.$$

We define $\widetilde{S}_k$ (resp. $\widetilde{\overline{S}}_k$) as the minimal resolution of $\underline{S}_k$ (resp. $\overline{\underline{S}}_k$) at the point 0. The exceptional divisor $E \subset \widetilde{S}_k$ is an A_{k-1}-diagram of projective lines $E_1, \dots, E_{k-1}$. Since any E_i is a -2-curve, it is possible to blow down an arbitrary subset of $\{E_1, \dots, E_{k-1}\}$. We set $\vec{k} = (k_1, \dots, k_n)$, and we define $\underline{S}_{\vec{k}}$ (resp. $\overline{\underline{S}}_{\vec{k}}$) as the result of blowing down all the lines except for $E_{k_1}, E_{k_1+k_2}, \dots, E_{k_1+\dots+k_{n-1}}$ in $\widetilde{S}_k$ (resp. $\widetilde{\overline{S}}_k$). The surface $\underline{S}_{\vec{k}}$ (resp. $\overline{\underline{S}}_{\vec{k}}$) possesses canonical stacky resolution $S_{\vec{k}}$ (resp. $\overline{S}_{\vec{k}}$). We will denote by $s_1, \dots, s_n \in S_{\vec{k}}$ the torus fixed points with the automorphism groups $\Gamma_{k_1}, \dots, \Gamma_{k_n}$.

We denote by $\mathrm{Bun}_G(S_{\vec{k}})$ the moduli space of G-bundles on $\overline{S}_{\vec{k}}$ trivialized on the boundary divisor $\overline{S}_{\vec{k}} \setminus S_{\vec{k}}$. For a bundle $\mathcal{F} \in \mathrm{Bun}_G(S_{\vec{k}})$, the group Γ_{k_i} acts on its fiber $\mathcal{F}_{s_{k_i}}$ at the point s_{k_i}, and hence defines a conjugacy class of maps $\Gamma_{k_i} \to G$, i.e. an element of $\Lambda^+_{k_i}$. Similarly, the action of Γ_k at the fiber of $\mathcal{F}$ at infinity defines an element of Λ^+_k. We denote by $\mathrm{Bun}^{\overline{\lambda}^{(1)},\dots,\overline{\lambda}^{(n)}}_{G,\overline{\mu}}(S_{\vec{k}})$ the subset of $\mathrm{Bun}_G(S_{\vec{k}})$ formed by all $\mathcal{F} \in \mathrm{Bun}_G(S_{\vec{k}})$ such that $\mathcal{F}_{s_{k_i}}$ is of class $\overline{\lambda}^{(i)}$, and $\mathcal{F}_\infty$ is of class $\overline{\mu}$. To unburden the notations, we will write $\overline{\boldsymbol{\lambda}}$ for $(\overline{\lambda}^{(1)}, \dots, \overline{\lambda}^{(n)})$, and $\mathrm{Bun}^{\overline{\boldsymbol{\lambda}}}_{G,\overline{\mu}}(S_{\vec{k}})$ for $\mathrm{Bun}^{\overline{\lambda}^{(1)},\dots,\overline{\lambda}^{(n)}}_{G,\overline{\mu}}(S_{\vec{k}})$. Clearly, it is a union of connected components of $\mathrm{Bun}_G(S_{\vec{k}})$. We denote by $\mathrm{Bun}^{\overline{\boldsymbol{\lambda}},d/k}_{G,\overline{\mu}}(S_{\vec{k}})$ the intersection of $\mathrm{Bun}^{\overline{\boldsymbol{\lambda}}}_{G,\overline{\mu}}(S_{\vec{k}})$ with $\mathrm{Bun}^{d/k}_G(S_{\vec{k}})$. (Here $\mathrm{Bun}^{d/k}_G(S_{\vec{k}})$ denotes the moduli space of G-bundles of second Chern class d/k. Here d/k is the second Chern class on the stack; it is a rational number with denominator k.)

1.10. Uhlenbeck spaces and convolution

Our first goal in this paper is to define a certain partial Uhlenbeck compactification $\mathcal{U}^{\overline{\boldsymbol{\lambda}},d/k}_{G,\overline{\mu}}(S_{\vec{k}}) \supset \mathrm{Bun}^{\overline{\boldsymbol{\lambda}},d/k}_{G,\overline{\mu}}(S_{\vec{k}})$. The definition is given in Section 2 for $G = \mathrm{SL}(N)$ using Nakajima's quiver varieties and in Section 3 for general G (using all possible embeddings of G into

[2] More precisely, in [2] we construct an open and closed subvariety $\mathrm{Bun}^{\lambda}_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ inside $\mathrm{Bun}_G(\mathbb{A}^2/\Gamma_k)$ and formulate a conjecture, saying that it is connected (and thus it is a connected component of $\mathrm{Bun}_G(\mathbb{A}^2/\Gamma_k)$). This conjecture is proved in [2] for $G = \mathrm{SL}(n)$ and it is still open in general.

[3] The categorical quotient $X//\Gamma$ may not exist in general, but we will only deal with the case X is affine or projective, when the categorical quotient does exist.

$\mathrm{SL}(N)$). Choosing certain lifts $\lambda^{(i)}$ of $\bar{\lambda}^{(i)}$ and μ of $\bar{\mu}$ to level k dominant weights of $G^\vee_{\mathrm{aff}}$ we will redenote $\mathcal{U}^{\bar{\boldsymbol\lambda},d/k}_{G,\bar{\mu}}(S_{\vec{k}})$ by $\mathcal{U}^{\boldsymbol\lambda}_{G,\mu}(S_{\vec{k}})$. We will also construct a proper birational morphism $\varpi : \mathcal{U}^{\boldsymbol\lambda}_{G,\mu}(S_{\vec{k}}) \to \mathcal{U}^{\lambda}_{G,\mu}(S_k)$ for $\lambda = \lambda^{(1)} + \cdots + \lambda^{(n)}$. We believe that ϖ is the correct analog of the convolution morphism

$$m_n^{-1}\big(\overline{\mathcal{W}}^{\lambda}_{G,\mu}\big) \cap \big(\overline{\mathrm{Gr}}^{\lambda_1}_G \star \cdots \star \overline{\mathrm{Gr}}^{\lambda_n}_G\big) \to \overline{\mathcal{W}}^{\lambda}_{G,\mu}.$$

In particular, in the case $G = \mathrm{SL}(N)$ we prove an analog of (1.2) for the morphism ϖ (the proof follows from the results of [12] by a fairly easy combinatorial argument). We conjecture that a similar decomposition holds for general G.

Let us note that the above conjecture is somewhat reminiscent of the results of [4] where similar moduli spaces have been used in order to prove the existence of convolution for the *spherical Hecke algebra* of G_{aff} (recall that the tensor category $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is a categorification of the spherical Hecke algebra of G).

1.11. Axiomatic approach to Uhlenbeck spaces

We note that the above constructions of the relevant Uhlenbeck spaces and morphisms between them are rather ad hoc; to give the reader certain perspective, let us formulate what we would expect from Uhlenbeck spaces for general smooth 2-dimensional Deligne–Mumford stacks. The constructions of Section 2 and Section 3 may be viewed as a partial verification of these expectations in the case of mixed Kleinian stacks.

1.11.1. Spaces

Let S be a smooth 2-dimensional Deligne–Mumford stack. Let $\overline{S}$ be its smooth compactification, and let $D \subset \overline{S}$ be the divisor at infinity. Let $\underline{S} \subset \underline{\overline{S}}$ be the coarse moduli spaces. In our applications we only consider the stacks with cyclic automorphism groups of points; more restrictively, only toric stacks.

Let G as before be an almost simple simply connected complex algebraic group. We assume that there are no G-bundles on $\overline{S}$ equipped with a trivialization on D with nontrivial automorphisms (preserving the trivialization). In this case there is a fine moduli space $\mathrm{Bun}_G(S)$ of the pairs (a G-bundle on $\overline{S}$; its trivialization on D). We believe that $\mathrm{Bun}_G(S)$ is open dense in the *Uhlenbeck completion* $\mathcal{U}_G(S)$. We believe that $\mathcal{U}_{\mathrm{SL}(N)}(S)$ is a certain quotient of the moduli stack of perverse coherent sheaves on S which are n-dimensional vector bundles off finitely many points.

A nontrivial homomorphism $\varrho : G \to \mathrm{SL}(N)$ gives rise to the closed embedding $\varrho_* : \mathrm{Bun}_G(S) \hookrightarrow \mathrm{Bun}_{\mathrm{SL}(N)}(S)$ which we expect to extend to a morphism $\mathcal{U}_G(S) \hookrightarrow \mathcal{U}_{\mathrm{SL}(N)}(S)$.

1.11.2. Morphisms

Assume that we have a proper morphism $\pi : \underline{\overline{S}} \to \underline{\overline{S}}'$ which is an isomorphism in the neighbourhoods of D, D'. We believe π gives rise to a birational proper morphism $\varpi : \mathcal{U}_G(S) \to \mathcal{U}_G(S')$. If $\varrho : G \to \mathrm{SL}(N)$ is a nontrivial representation of G, and $\phi \in \mathcal{U}_G(S)$, we choose a perverse coherent sheaf F on S representing $\varrho_*(\phi)$. According to Theorem 4.2 of [8], there is an equivalence of derived coherent categories on S and S' (it is here that we need the assumption that S and S' are toric). This equivalence takes F to a perverse coherent sheaf F' on S'. We believe that the class of F' in $\mathcal{U}_{\mathrm{SL}(N)}(S')$ equals $\varrho_*(\varpi(\phi))$.

1.11.3. Families

Assume we have a morphism $\overline{\mathcal{S}} \to \mathcal{X}$ where $\mathcal{X}$ is a variety, and for every $x \in \mathcal{X}$ the fiber $\overline{S}_x$ over x is of type considered in Section 1.11.1. Then there should exist a morphism of varieties $\mathcal{U}_G(\mathcal{S}) \to \mathcal{X}$ such that for every $x \in \mathcal{X}$ the fiber $\mathcal{U}_G(\mathcal{S})_x$ is isomorphic to $\mathcal{U}_G(S_x)$ where $S_x \subset \overline{S}_x$ is the canonical stacky resolution of $\underline{S}_x \subset \overline{\underline{S}}_x$, cf. Section 2.1 (note that we *do not require* the existence of a family of stacks over $\mathcal{X}$ with fibers S_x).

2. The case of $G = \mathrm{SL}(N)$

2.1. Stacky resolutions and derived equivalences

In this subsection we would like to implement the constructions announced in Section 1.10 in the case $G = \mathrm{SL}(N)$. To do that let us first discuss some preparatory material.

Let $\underline{S}$ be an algebraic surface and let $s_1, \dots, s_n$ be distinct points on $\underline{S}$ such that the formal neighbourhood of s_i is isomorphic to the formal neighbourhood of 0 in the surface $\mathbb{A}^2//\Gamma_{k_i}$ for some $k_i \geqslant 1$; note that Artin's algebraization theorem implies that such an isomorphism exists also étale-locally. Let us also assume that $\underline{S}$ is smooth away from $s_1, \dots, s_n$. Recall that for any $k \geqslant 1$ the surface $\mathbb{A}^2//\Gamma_k$ possesses canonical minimal resolution $\pi : \widetilde{\mathbb{A}^2//\Gamma_k} \to \mathbb{A}^2//\Gamma_k$ whose special fiber is a tree of type A_{k-1} of $\mathbb{P}^1$'s having self-intersection -2. Similarly, we have a stacky resolution $\mathbb{A}^2/\Gamma_k \to \mathbb{A}^2//\Gamma_k$. The existence of the above resolutions implies the existence of a resolution $\widetilde{S} \to \underline{S}$ and a stacky resolution[4] $S \to \underline{S}$ which near every s_i are étale locally isomorphic to respectively $\widetilde{\mathbb{A}^2//\Gamma_{k_i}}$ and $\mathbb{A}^2/\Gamma_{k_i}$.

For any scheme Y let us denote by $D^b\mathsf{Coh}(Y)$ the bounded derived category of coherent sheaves on Y. Recall (cf. [7] and [5]) that we have an equivalence of derived categories

$$\Psi : D^b\mathsf{Coh}\big(\widetilde{\mathbb{A}^2//\Gamma_k}\big) \to D^b\mathsf{Coh}\big(\mathbb{A}^2/\Gamma_k\big).$$

This equivalence is given by a kernel which is a sheaf on $\widetilde{\mathbb{A}^2//\Gamma_k} \times \mathbb{A}^2/\Gamma_k$ (and not a complex of sheaves). Thus (by gluing in étale topology) a similar kernel can also be defined on the product $\widetilde{S} \times S$ and it will define an equivalence $D^b\mathsf{Coh}(\widetilde{S}) \to D^b\mathsf{Coh}(S)$ which we shall again denote by Ψ.

2.2. Recall the setup of [2, 7.1]. Following [12] we denote by $I = \{1, \dots, k\}$ the set of vertices of the affine cyclic quiver; k stands for the affine vertex, and $I_0 = I \setminus \{k\}$. Given $\vec{a} = (a_1, \dots, a_n) \in \mathbb{A}^n$ such that

$$k_1 a_1 + \dots + k_n a_n = 0, \tag{2.1}$$

we consider a k-tuple $(b_1 = a_1, \dots, b_{k_1} = a_1, b_{k_1+1} = a_2, \dots, b_{k_1+k_2} = a_2, \dots, b_{k_1+\dots+k_{n-1}} = a_{n-1}, b_{k_1+\dots+k_{n-1}+1} = a_n, \dots, b_k = a_n)$. We consider another k-tuple of complex numbers $\zeta^\circ_{\mathbb{C}}$ such that $\zeta^\circ_{\mathbb{C},i} := b_i - b_{i+1}$ for $i = 1, \dots, k$ (for $i = k$ it is understood that $i + 1 = 1$).

[4] The existence follows from the fact that every automorphism of $\mathbb{A}^2/\Gamma_k$ which is trivial over $\mathbb{A}^2//\Gamma_k \setminus \{0\}$, is trivial and the same is true over any étale neighbourhood of 0 in $\mathbb{A}^2//\Gamma_k$.

Furthermore, we set $I_0 \supset I_0^+ := \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_{n-1}\}$, and $I_0^0 := I_0 \setminus I_0^+$. We consider a vector $\zeta_{\mathbb{R}}^\bullet \in \mathbb{R}^I$ with coordinates $\zeta_{\mathbb{R},i}^\bullet = 0$ for $i \in I_0^0$, and $\zeta_{\mathbb{R},j}^\bullet = 1$ for $j \in I_0^+$, and $\zeta_{\mathbb{R},k}^\bullet = 1-n$.

Recall the setup of Section 1 of [11]. In this note we are concerned with the cyclic A_{k-1}-quiver only, so in particular, $\delta = (1,\ldots,1)$. We consider the GIT quotient $X_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)} := \{\xi \in M(\delta,0)\colon \mu(\xi) = -\zeta_{\mathbb{C}}^\circ\}//_{-\zeta_{\mathbb{R}}^\bullet}(G_\delta/\mathbb{C}^*)$ (see (1.5) of [11]). It is a partial resolution of the categorical quotient $X_{(\zeta_{\mathbb{C}}^\circ,0)} := \{\xi \in M(\delta,0)\colon \mu(\xi) = -\zeta_{\mathbb{C}}^\circ\}//(G_\delta/\mathbb{C}^*)$. The above surfaces admit the following explicit description: The surface $X_{(\zeta_{\mathbb{C}}^\circ,0)}$ is isomorphic to the affine surface given by the equation

$$xy = (z-a_1)^{k_1}\cdots(z-a_n)^{k_n}.$$

Note that when all a_i are equal to 0 we just get the equation $xy = z^n$ which defines a surface isomorphic to $\mathbb{A}^2//\Gamma_k$. The surface $X_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)}$ has the following properties: if all the points a_i are distinct, then $X_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)} = X_{(\zeta_{\mathbb{C}}^\circ,0)}$. On the other hand, if all a_i are equal (and thus they have to be equal to zero by (2.1)) then $X_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)}$ is obtained from $\widetilde{\mathbb{A}^2//\Gamma_k}$ by blowing down all the exceptional $\mathbb{P}^1$'s except those whose numbers are $k_1, k_1+k_2, \ldots, k_1+\cdots+k_{n-1}$. We leave the general case (i.e. the case of general a_i's) to the reader.

The surface $X_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)}$ (resp. $X_{(\zeta_{\mathbb{C}}^\circ,0)}$) is of the type discussed in Section 2.1 and thus it has canonical minimal stacky resolution, which we shall denote by $S_{\vec{k}}^{\vec{a}}$ (resp. $S'^{\vec{a}}_{\vec{k}}$).

If we choose a generic stability condition $\zeta_{\mathbb{R}}^\circ$ in the hyperplane $\zeta_{\mathbb{R}}\cdot\delta = 0$, then the corresponding GIT quotient $X_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\circ)}$ is smooth; moreover, it is the minimal resolution of singularities of $X_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)}$. Recall the compactification $\overline{X}_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\circ)}$ introduced in Section 3 of [11]. According to Section 2.1 we have the equivalence $\Psi : D^b\mathsf{Coh}(\overline{X}_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\circ)}) \xrightarrow{\sim} D^b\mathsf{Coh}(\overline{S}_{\vec{k}}^{\vec{a}})$. Recall the line bundles $\mathcal{R}_i$, $i \in I$, and their homomorphisms ξ on $\overline{X}_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\circ)}$, introduced in Sections 1(iii) and 3(i) of [11]. We will denote $\Psi(\mathcal{R}_i)$ by $\mathcal{R}_i^\bullet$. This is a line bundle on $\overline{S}_{\vec{k}}^{\vec{a}}$ (this follows from the fact that a similar statement is true for the equivalence $D^b\mathsf{Coh}(\widetilde{\mathbb{A}^2//\Gamma_k}) \to D^b\mathsf{Coh}(\mathbb{A}^2/\Gamma_k)$ under which the bundle $\mathcal{R}_i$ goes to the Γ_k-equivariant sheaf on $\mathbb{A}^2$ corresponding to the structure sheaf of $\mathbb{A}^2$ on which Γ_k acts by its i-th character).

2.3. We consider the quiver variety $\mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)}(V,W)$ for the stability condition $\zeta_{\mathbb{R}}^\bullet$, see Section 2 of [12]. We consider a vector $\zeta_{\mathbb{R}}^\pm := \zeta_{\mathbb{R}}^\bullet \pm (\varepsilon,\ldots,\varepsilon) \in \mathbb{R}^I$ for $0 < \varepsilon \ll 1$. Note that it lies in an (open) chamber of the stability conditions, so the corresponding quiver varieties $\mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\pm)}(V,W)$ are smooth. Moreover, since $\zeta_{\mathbb{R}}^\bullet$ lies in a face adjacent to the chamber of $\zeta_{\mathbb{R}}^\pm$, we have the proper morphism $\pi_{\zeta_{\mathbb{R}}^\bullet,\zeta_{\mathbb{R}}^\pm} : \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\pm)}(V,W) \to \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ,\zeta_{\mathbb{R}}^\bullet)}(V,W)$.

The construction of [11, (1.7), 3(ii)] associates to any ADHM data $(B,a,b) \in \mathbf{M}(V,W)$ satisfying $\mu(B,a,b) = \zeta_{\mathbb{C}}^\circ$ a complex of vector bundles

$$\mathrm{L}(\mathcal{R}^{\bullet *}, V)(-\ell_\infty) \xrightarrow{\sigma} \mathrm{E}(\mathcal{R}^{\bullet *}, V) \oplus \mathrm{L}(\mathcal{R}^{\bullet *}, W) \xrightarrow{\tau} \mathrm{L}(\mathcal{R}^{\bullet *}, V)(\ell_\infty) \tag{2.2}$$

on $\overline{S}_{\vec{k}}^{\vec{a}}$.

The following proposition is a slight generalization of Proposition 4.1 of [11].

Proposition 2.4. *Let $(B,a,b)\in\mu^{-1}(\zeta_{\mathbb{C}}^{\circ})$ and consider the complex* (2.2). *We consider σ,τ as linear maps on the fiber at a point in $\overline{S}_{\vec{k}}^{\vec{a}}$. Then*:

(1) *(B,a,b) is $\zeta_{\mathbb{R}}^{-}$-stable if and only if σ is injective possibly except finitely many points, and τ is surjective at any point.*
(2) *(B,a,b) is $\zeta_{\mathbb{R}}^{\bullet}$-semistable if and only if σ is injective and τ is surjective possibly except finitely many points.*

Proof. The proof is parallel to that of Proposition 4.1 of [11], with the use of Lemma 3.2 of [12] in place of Corollary 4.3 of [11]. □

2.5. We consider the Levi subalgebra $\mathfrak{l}\subset\mathfrak{sl}(k)\subset\mathfrak{sl}(k)_{\mathrm{aff}}$ whose set of simple roots is I_0^0, i.e. $\{\alpha_1,\dots,\alpha_{k_1-1},\alpha_{k_1+1},\dots,\alpha_{k_1+k_2-1},\dots,\alpha_{k-1}\}$. We will denote by $\mathbb{Z}[I_0^0]$ the root lattice of $\mathfrak{l}$. The multiplication by the affine Cartan matrix $A_{k-1}^{(1)}$ *embeds* $\mathbb{Z}[I_0^0]$ into the weight lattice P_{aff} of $\mathfrak{sl}(k)_{\mathrm{aff}}$ spanned by the fundamental weights $\omega_0,\dots,\omega_{k-1}$, so we will identify $\mathbb{Z}[I_0^0]$ with a sublattice of P_{aff}. The inclusion $\mathfrak{l}\subset\mathfrak{sl}(k)$ also gives rise to the embedding $\mathbb{Z}[I_0^0]\subset P$ into the weight lattice of $\mathfrak{sl}(k)$.

We have $\mathbf{w}=\underline{\dim}W=(w_1,\dots,w_k)$, $\mathbf{v}=\underline{\dim}V=(v_1,\dots,v_k)$. We set $N:=w_1+\dots+w_k$. Recall the setup of [2, 7.3]. We associate to the pair $(\mathbf{v},\mathbf{w})$ the $\mathfrak{sl}(k)_{\mathrm{aff}}$-weight $\mathbf{w}'=\sum_{i=1}^k w_i'\omega_i:=\sum_{i=1}^k w_i\omega_i-\sum_{i=1}^k v_i\alpha_i$. In this note we restrict ourselves to the pairs $(\mathbf{v},\mathbf{w})$ satisfying the condition

$$\mathbf{w}'\in N\omega_0+\mathbb{Z}\big[I_0^0\big]. \tag{2.3}$$

The geometric meaning of this condition is as follows. Proposition 2.4 implies that $\mathfrak{M}^{\mathrm{reg}}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}(V,W)$ is the moduli space of vector bundles on the stack $\overline{S}_{\vec{k}}^{\vec{a}}$ trivialized at infinity. The condition (2.3) guaranties that these vector bundles have trivial determinant, i.e. reduce to $\mathrm{SL}(N)$.

In effect, the determinant in question is a line bundle on $\overline{S}_{\vec{k}}^{\vec{a}}$ trivialized at infinity. So the determinant is trivial iff its restriction to the open substack $S_{\vec{k}}^{\vec{a}}$ is trivial, i.e. is a zero element of $\mathrm{Pic}(S_{\vec{k}}^{\vec{a}})$. Recall that $K(S_{\vec{k}}^{\vec{a}})\simeq P_{\mathrm{aff}}$, $\mathcal{R}_i^{\bullet}\mapsto\omega_i$, and we have the homomorphism $\det:K(S_{\vec{k}}^{\vec{a}})\to\mathrm{Pic}(S_{\vec{k}}^{\vec{a}})$. The class in $K(S_{\vec{k}}^{\vec{a}})\simeq P_{\mathrm{aff}}$ of any vector bundle in $\mathfrak{M}^{\mathrm{reg}}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}(V,W)$ is given by $\mathbf{w}'\in P_{\mathrm{aff}}$. So the triviality of its determinant is a consequence of the following lemma.

Lemma 2.6. *There is a canonical isomorphism* $\mathrm{Pic}(S_{\vec{k}}^{\vec{a}})\simeq P/\mathbb{Z}[I_0^0]$ *such that the homomorphism* $\det:K(S_{\vec{k}}^{\vec{a}})\to\mathrm{Pic}(S_{\vec{k}}^{\vec{a}})$ *identifies with the composition of the projection $P_{\mathrm{aff}}\to P$, $\omega_i\mapsto\omega_i-\delta_{i0}\omega_0$, and the projection $P\to P/\mathbb{Z}[I_0^0]$.*

Proof. Let $\widetilde{X}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$ stand for the minimal resolution of the surface $X_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$. Let $X^{\circ}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$ stand for the open subset of $X_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$ obtained by removing all the singular points. The projection $\widetilde{X}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}\to X_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$ identifies $X^{\circ}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$ with the open subset of $\widetilde{X}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$ obtained by removing the components $\{E_i,\ i\in I_0^0\}$ of the exceptional divisor. Since any line bundle on $X^{\circ}_{(\zeta_{\mathbb{C}}^{\circ},\zeta_{\mathbb{R}}^{\bullet})}$ extends uniquely to a line bundle on $S_{\vec{k}}^{\vec{a}}$, we obtain the restriction to the open subset homomorphism

$\mathrm{Pic}(\widetilde{X}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})}) \twoheadrightarrow \mathrm{Pic}(X^\circ_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})}) = \mathrm{Pic}(S^{\vec{a}}_{\vec{k}})$. Clearly, the kernel of this restriction homomorphism is spanned by the classes of the line bundles $\langle[\mathcal{O}(E_i)],\ i\in I_0^0\rangle$ in $\mathrm{Pic}(\widetilde{X}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})})$.

Now recall that we have a canonical isomorphism $\mathrm{Pic}(\widetilde{X}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})}) \simeq P$ such that the composition $\det : P_{\mathrm{aff}} = K(\widetilde{X}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})}) \to \mathrm{Pic}(\widetilde{X}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})}) \simeq P$ identifies with the projection $p : P_{\mathrm{aff}} \to P$, $\omega_i \mapsto \omega_i - \delta_{i0}\omega_0$. Moreover, the class $[\mathcal{O}(E_i)] \in \mathrm{Pic}(\widetilde{X}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})})$ gets identified with $p(\alpha_i)$. This follows by embedding $\widetilde{X}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})}$ as a slice into Grothendieck simultaneous resolution $\widetilde{\mathfrak{sl}}_k$.

This completes the proof of the lemma. □

2.7. Our next goal is to encode the quiver data $(\mathbf{v},\mathbf{w})$ by the weight data of $\mathfrak{sl}(N)_{\mathrm{aff}}$. From now on we assume that $\mathbf{w}$ corresponds to an N-dimensional representation of Γ_k with *trivial determinant*, i.e. a homomorphism $\Gamma_k \to \mathrm{SL}(N)$. Then the dominant weight $w_1\omega_1 + \cdots + w_{k-1}\omega_{k-1}$ of $\mathfrak{sl}(k)$ is actually a weight of $\mathrm{PSL}(k)$, and can be written uniquely as a generalized Young diagram $\tau = (\tau_1 \geqslant \cdots \geqslant \tau_k)$ such that $\tau_i - \tau_{i+1} = w_i$ for any $1 \leqslant i \leqslant k-1$, and $\tau_1 - \tau_k \leqslant N$, and $\tau_1 + \cdots + \tau_k = 0$, cf. [2, 7.3]. Under the bijection $\Psi_{N,k}$ of [2] $\mathbf{w}$ corresponds to a level k dominant weight $\overline{\mu} \in \Lambda_k^+$ of $\widehat{\mathfrak{sl}(N)}$ which can also be written as a generalized Young diagram $(\mu_1 \geqslant \cdots \geqslant \mu_N)$ such that $\mu_1 - \mu_N \leqslant k$, and $\mu_1 + \cdots + \mu_N = 0$. We write $\tau = {}^t\overline{\mu}$, and $\overline{\mu} = {}^t\tau$.

Here is an explicit construction of the transposition operation on the generalized Young diagrams. If $\overline{\mu}$ consists of all zeroes, then so does τ. Otherwise we assume $\mu_r > 0 \geqslant \mu_{r+1}$ for some $0 < r < N$. Then we have an *ordinary* Young diagram $\overline{\mu}' := (k + \mu_{r+1} \geqslant k + \mu_{r+2} \geqslant \cdots \geqslant k + \mu_N \geqslant \mu_1 \geqslant \cdots \geqslant \mu_r)$ formed by *positive* integers. We denote the *ordinary* transposition ${}^t\overline{\mu}'$ by $\tau' = (\tau_1' \geqslant \cdots \geqslant \tau_k')$, and finally we set $\tau = {}^t\overline{\mu} := (\tau_1' + r - N \geqslant \cdots \geqslant \tau_k' + r - N)$. In other words,

$$\begin{aligned}\tau &= {}^t\overline{\mu}\\ &= \left(r^{\mu_r}, (r-1)^{\mu_{r-1}-\mu_r}, \ldots, 1^{\mu_1-\mu_2}, 0^{k+\mu_N-\mu_1}, (-1)^{\mu_{N-1}-\mu_N}, \ldots, (r-N)^{-\mu_{r+1}}\right).\end{aligned} \quad (2.4)$$

Furthermore, we write down the weight $\mathbf{w}' = \sum_{i=1}^k w_i\omega_i - \sum_{i=1}^k v_i\alpha_i$ as a sequence of integers $(\sigma_1,\ldots,\sigma_k)$. The condition $\mathfrak{M}^{\mathrm{reg}}_{(\zeta^\circ_\mathbb{C},\zeta^\bullet_\mathbb{R})}(V,W) \neq \emptyset$ implies $\sigma_i \geqslant \sigma_{i+1}$ for $i \in I_0^0$, and $\sigma_{k_0+\cdots+k_{p-1}+1} - \sigma_{k_0+\cdots+k_p} \leqslant N$ for any $0 < p \leqslant n$, where we put for convenience $k_0 = 0$. The condition (2.3) implies that $\sigma_{k_1+\cdots+k_{p-1}+1} + \cdots + \sigma_{k_1+\cdots+k_p} = 0$ for any $0 < p \leqslant n$. Thus the sequence $(\sigma_{k_1+\cdots+k_{p-1}+1},\ldots,\sigma_{k_1+\cdots+k_p})$ is a generalized Young diagram to be denoted by $\sigma^{(p)}$. The transposed generalized Young diagram $\overline{\lambda}^{(p)} := {}^t\sigma^{(p)}$ corresponds to the same named level k_p dominant $\widehat{\mathfrak{sl}(N)}$-weight $\overline{\lambda}^{(p)} \in \Lambda^+_{k_p}(\widehat{\mathfrak{sl}(N)})$.

Recall that the affine Weyl group W_{aff} acts on the set of level N weights of $\widehat{\mathfrak{sl}(k)}$. If we write down these weights as the sequences $(\chi_1,\ldots,\chi_k)$ then the action of W_{aff} is generated by permutations of χ_i's and the operations which only change χ_i, χ_j for some pair $i, j \in I$; namely, $\chi_i \mapsto \chi_i + N$, $\chi_j \mapsto \chi_j - N$.

Lemma 2.8. *The sequence $(\sigma_1,\ldots,\sigma_k)$ is W_{aff}-conjugate to ${}^t(\overline{\lambda}^{(1)} + \cdots + \overline{\lambda}^{(n)})$.*

Proof. To simplify the notation we assume that $n = 2$; the general case is not much different. Let $\overline{\lambda}^{(1)} = (\lambda_1^{(1)} \geqslant \cdots \geqslant \lambda_N^{(1)})$, and $\overline{\lambda}^{(2)} = (\lambda_1^{(2)} \geqslant \cdots \geqslant \lambda_N^{(2)})$. We set $\overline{\lambda} = (\lambda_1 \geqslant \cdots \geqslant \lambda_N)$ where $\lambda_i := \lambda_i^{(1)} + \lambda_i^{(2)}$. We assume $\lambda_{r_1}^{(1)} > 0 \geqslant \lambda_{r_1+1}^{(1)}$ for some $0 < r_1 < N$, and $\lambda_{r_2}^{(2)} > 0 \geqslant \lambda_{r_2+1}^{(2)}$ for some $0 < r_2 < N$. If $r_1 = r_2$, then the formula (2.4) makes it clear that the sequence $(\sigma_1, \dots, \sigma_k)$ being a concatenation of the sequences $(\sigma_1, \dots, \sigma_{k_1}) = {}^t\overline{\lambda}^{(1)}$ and $(\sigma_{k_1+1}, \dots, \sigma_k) = {}^t\overline{\lambda}^{(2)}$ differs by a permutation from the sequence ${}^t(\overline{\lambda}^{(1)} + \overline{\lambda}^{(2)})$.

Otherwise we assume $r_1 > r_2$, and $\lambda_r > 0 \geqslant \lambda_{r+1}$ for some $r_1 \geqslant r \geqslant r_2$. Once again, to simplify the exposition, let us assume that $r_1 > r > r_2$. According to the formula (2.4), if we reorder the concatenation of ${}^t\overline{\lambda}^{(1)}$ and ${}^t\overline{\lambda}^{(2)}$ to obtain a nonincreasing sequence, we get

$$\big(r_1^{\lambda_1^{(1)}}, \dots, r^{\lambda_r^{(1)} - \lambda_{r+1}^{(1)}}, \dots, r_2^{\lambda_{r_2}^{(1)} - \lambda_{r_2+1}^{(1)} + \lambda_{r_2}^{(2)}}, \dots, (r_1 - N)^{-\lambda_{r_1+1}^{(1)} + \lambda_{r_1}^{(2)} - \lambda_{r_1+1}^{(2)}}, \dots,$$
$$(r - N)^{\lambda_r^{(2)} - \lambda_{r+1}^{(2)}}, \dots, (r_2 - N)^{-\lambda_{r_2+1}^{(2)}}\big).$$

On the other hand, the sequence ${}^t(\overline{\lambda}^{(1)} + \overline{\lambda}^{(2)})$ reads

$$\big(r^{\lambda_r^{(1)} + \lambda_r^{(2)}}, \dots, r_2^{\lambda_{r_2}^{(1)} + \lambda_{r_2}^{(2)} - \lambda_{r_2+1}^{(1)} - \lambda_{r_2+1}^{(2)}}, \dots,$$
$$(r_1 - N)^{\lambda_{r_1}^{(1)} + \lambda_{r_1}^{(2)} - \lambda_{r_1+1}^{(1)} - \lambda_{r_1+1}^{(2)}}, \dots, (r - N)^{-\lambda_{r+1}^{(1)} - \lambda_{r+1}^{(2)}}\big).$$

Now it is immediate to check that for any residue h modulo N its multiplicity in the latter sequence is the sum of multiplicities of the same residues in the former sequence. This means that the former sequence is W_{aff}-conjugate to the latter one. The lemma is proved. □

2.9. *Birational convolution morphism*

Recall that we have a proper morphism $\pi_{0,\zeta^\bullet} : \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, \zeta_{\mathbb{R}}^\bullet)}(V, W) \to \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V, W)$ introduced in [12, 3.2]. Since $\mathbf{w}'$ is not necessarily dominant weight of $\widehat{\mathfrak{sl}(k)}$, the open stratum $\mathfrak{M}^{\mathrm{reg}}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V, W)$ may be empty. However, replacing $\mathbf{v}$ by $\mathbf{v}' = (v_1', \dots, v_k')$ so that $\mathbf{w}'' = \sum_{i=1}^k w_i'' \omega_i := \sum_{i=1}^k w_i \omega_i - \sum_{i=1}^k v_i' \alpha_i$ is *dominant* and W_{aff}-conjugate to $\mathbf{w}'$, we can identify $\mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V, W)$ with $\mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V', W)$. Moreover, in this case the open subset $\mathfrak{M}^{\mathrm{reg}}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V', W)$ is not empty, and the morphism $\pi_{0,\zeta^\bullet} : \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, \zeta_{\mathbb{R}}^\bullet)}(V, W) \to \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V', W)$ is birational. Recall that for $\zeta_{\mathbb{C}}^\circ = 0$, in Section 7 of [2] we identified $\mathfrak{M}_{(0,0)}(V', W)$ with the Uhlenbeck space $\mathcal{U}^\lambda_{\mathrm{SL}(N),\mu}(\mathbb{A}^2/\Gamma_k)$ for certain level k dominant $\mathfrak{sl}(N)_{\mathrm{aff}}$-weights λ, μ. In the notations of current Section 2.7 we have $\mu = (k, \overline{\mu}, -\frac{1}{2k}(2d + (\overline{\mu}, \overline{\mu}) - (\overline{\lambda}, \overline{\lambda})))$, $\lambda = (k, \overline{\lambda}, 0)$. Here $d = \sum_{i=1}^k v_i'$, and $\overline{\lambda} = \sum_{p=1}^n \overline{\lambda}^{(p)}$ according to Lemma 2.8.

For $1 \leqslant p \leqslant n$ we introduce a level k_p dominant $\mathfrak{sl}(N)_{\mathrm{aff}}$-weight $\lambda^{(p)} := (k_p, \overline{\lambda}^{(p)}, 0)$. We set $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$. For arbitrary $\zeta_{\mathbb{C}}^\circ$ we *define* $\mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S^{\vec{a}}_{\vec{k}})$ as $\mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, \zeta_{\mathbb{R}}^\bullet)}(V, W)$. We *define* the convolution morphism $\varpi : \mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\lambda}_{\mathrm{SL}(N),\mu}(S'^{\vec{a}}_{\vec{k}})$ as $\pi_{0,\zeta^\bullet} : \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, \zeta_{\mathbb{R}}^\bullet)}(V, W) \to \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V', W)$. We will mostly use the particular case $\varpi : \mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S_{\vec{k}}) \to \mathcal{U}^{\lambda}_{\mathrm{SL}(N),\mu}(S_k) = \mathcal{U}^{\lambda}_{\mathrm{SL}(N),\mu}(\mathbb{A}^2/\Gamma_k)$ defined as $\pi_{0,\zeta^\bullet} : \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, \zeta_{\mathbb{R}}^\bullet)}(V, W) \to \mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, 0)}(V', W)$ for $\zeta_{\mathbb{C}}^\circ = (0, \dots, 0)$.

2.10. *Tensor product*

Recall the notations of Section 2.3. Now the construction of Section 5(i) of [11] gives rise to a morphism $\eta^\pm$ from $\mathfrak{M}_{(\zeta_{\mathbb{C}}^\circ, \zeta_{\mathbb{R}}^\pm)}(V, W)$ to the moduli stack of certain perverse coherent sheaves on $\overline{S}^{\vec{a}}_{\vec{k}}$ trivialized at ℓ_∞. It follows from Proposition 2.4(1) (“only if” part) that the image of η^- consists of torsion free sheaves, which implies that the image of η^+ consists of the perverse sheaves which are Serre-dual to the torsion free sheaves. We will denote the connected component of the moduli stack of torsion free sheaves (resp. of Serre-dual of torsion free sheaves) on $\overline{S}^{\vec{a}}_{\vec{k}}$ birationally mapping to $\mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S^{\vec{a}}_{\vec{k}})$ by $\mathcal{G}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$ (resp. by $\mathcal{SG}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$).

Lemma 2.11. *The morphisms* $\eta^- : \mathfrak{M}_{(\zeta^\circ_{\mathbb{C}},\zeta^-_{\mathbb{R}})}(V,W) \to \mathcal{G}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$, $\eta^+ : \mathfrak{M}_{(\zeta^\circ_{\mathbb{C}},\zeta^+_{\mathbb{R}})}(V,W) \to \mathcal{SG}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$ *are isomorphisms.*

Proof. Follows from Proposition 2.4(1) by the argument of Section 5 of [11]. □

We consider the locally closed subvariety $\mathbf{M}_{(\zeta^\circ_{\mathbb{C}},\zeta^\bullet_{\mathbb{R}})}(V,W) \subset \mu^{-1}(\zeta^\circ_{\mathbb{C}}) \subset \mathbf{M}(V,W)$ formed by all the $\zeta^\bullet_{\mathbb{R}}$-semistable modules. Let us denote by $\mathrm{Perv}^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$ the moduli stack of perverse coherent sheaves on $\overline{S}^{\vec{a}}_{\vec{k}}$ trivialized at ℓ_∞ and having the same numerical invariants as the torsion free sheaves in $\mathcal{G}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$. The construction of Section 5(i) of [11] gives rise to a morphism $\eta^\bullet$ from the stack $\mathbf{M}_{(\zeta^\circ_{\mathbb{C}},\zeta^\bullet_{\mathbb{R}})}(V,W)/GL_V$ to $\mathrm{Perv}^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$.

Lemma 2.12. $\eta^\bullet : \mathbf{M}_{(\zeta^\circ_{\mathbb{C}},\zeta^\bullet_{\mathbb{R}})}(V,W)/GL_V \to \mathrm{Perv}^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}})$ *is an isomorphism.*

Proof. Follows from Proposition 2.4(2) by the argument of Section 5 of [11]. □

It follows from Lemma 2.11 that we have a projective morphism $\pi_{\zeta^\bullet,\zeta^-} : \mathcal{G}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S^{\vec{a}}_{\vec{k}})$ (resp. $\pi_{\zeta^\bullet,\zeta^+} : \mathcal{SG}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S^{\vec{a}}_{\vec{k}})$).

Lemma 2.13. *If E is a torsion free coherent sheaf on $S^{\vec{a}}_{\vec{k}}$, and E' is the Serre dual of a torsion free coherent sheaf on $S^{\vec{a}}_{\vec{k}}$, then $E \otimes E'$ is a perverse coherent sheaf on $S^{\vec{a}}_{\vec{k}}$.*

Proof. Clearly, $\underline{H}^{>1}(E\otimes E')$ vanishes, $\underline{H}^1(E\otimes E')$ is a torsion sheaf supported at finitely many points, and $\underline{H}^0(E\otimes E')$ is torsion free. The same is true for the Serre dual sheaf of $E\otimes E'$ (being a tensor product of the same type). □

Thus we obtain a morphism $\mathcal{G}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}}) \times \mathcal{SG}ies'^{\boldsymbol{\lambda}}_{\mu'}(S^{\vec{a}}_{\vec{k}}) \to \mathrm{Perv}^{\boldsymbol{\lambda}\otimes'\boldsymbol{\lambda}}_{\mu\otimes\mu'}(S^{\vec{a}}_{\vec{k}})$. Here we understand $\overline{\mu}$ (resp. $\overline{\mu}'$) as a homomorphism $\Gamma_k \to \mathrm{SL}(N)$ (resp. $\Gamma_k \to \mathrm{SL}(N')$), and $\overline{\mu}\otimes\overline{\mu}'$ as the tensor product homomorphism $\Gamma_k \to \mathrm{SL}(NN')$; similarly for $\overline{\lambda}$'s. Furthermore, we set $\lambda^{(p)} \otimes {}'\lambda^{(p)} := (k_i, \overline{\lambda}^{(p)} \otimes {}'\overline{\lambda}^{(p)}, 0)$, and $\boldsymbol{\lambda}\otimes{}'\boldsymbol{\lambda} = (\lambda^{(1)}\otimes{}'\lambda^{(1)},\dots,\lambda^{(n)}\otimes{}'\lambda^{(n)})$. Finally, for $\mu = (k,\overline{\mu},m)$, $\mu' = (k,\overline{\mu}',m')$ we set $\mu\otimes\mu' := (k,\overline{\mu}\otimes\overline{\mu}',\mathsf{m})$ where

$$\mathsf{m} := mN' + m'N + \frac{1}{2k}\Bigg[N'(\overline{\mu},\overline{\mu}) - N'\Bigg(\sum_{p=1}^n \overline{\lambda}^{(p)}, \sum_{p=1}^n \overline{\lambda}^{(p)}\Bigg) + N\big(\overline{\mu}',\overline{\mu}'\big) - N\Bigg(\sum_{p=1}^n {}'\overline{\lambda}^{(p)}, \sum_{p=1}^n {}'\overline{\lambda}^{(p)}\Bigg) - \big(\overline{\mu}\otimes\overline{\mu}',\overline{\mu}\otimes\overline{\mu}'\big) + \Bigg(\sum_{p=1}^n \overline{\lambda}^{(p)}\otimes{}'\overline{\lambda}^{(p)}, \sum_{p=1}^n \overline{\lambda}^{(p)}\otimes{}'\overline{\lambda}^{(p)}\Bigg)\Bigg].$$

Composing this morphism with the further projection (due to Lemma 2.12) $\mathrm{Perv}^{\boldsymbol{\lambda}\otimes'\boldsymbol{\lambda}}_{\mu\otimes\mu'}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\boldsymbol{\lambda}\otimes'\boldsymbol{\lambda}}_{\mathrm{SL}(NN'),\mu\otimes\mu'}(S^{\vec{a}}_{\vec{k}})$ we obtain the morphism $\tau : \mathcal{G}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}}) \times \mathcal{S}\mathcal{G}ies'^{\boldsymbol{\lambda}}_{\mu'}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\boldsymbol{\lambda}\otimes'\boldsymbol{\lambda}}_{\mathrm{SL}(NN'),\mu\otimes\mu'}(S^{\vec{a}}_{\vec{k}})$.

Proposition 2.14. *The morphism τ factors through $\bar{\tau} : \mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S^{\vec{a}}_{\vec{k}}) \times \mathcal{U}'^{\boldsymbol{\lambda}}_{\mathrm{SL}(N'),\mu'}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\boldsymbol{\lambda}\otimes'\boldsymbol{\lambda}}_{\mathrm{SL}(NN'),\mu\otimes\mu'}(S^{\vec{a}}_{\vec{k}})$.*

Proof. Let us denote $\mathcal{G}ies^{\boldsymbol{\lambda}}_{\mu}(S^{\vec{a}}_{\vec{k}}) \times \mathcal{S}\mathcal{G}ies'^{\boldsymbol{\lambda}}_{\mu'}(S^{\vec{a}}_{\vec{k}})$ by X, and $\mathcal{U}^{\boldsymbol{\lambda}}_{\mathrm{SL}(N),\mu}(S^{\vec{a}}_{\vec{k}}) \times \mathcal{U}'^{\boldsymbol{\lambda}}_{\mathrm{SL}(N'),\mu'}(S^{\vec{a}}_{\vec{k}})$ by Y, and $\mathcal{U}^{\boldsymbol{\lambda}\otimes'\boldsymbol{\lambda}}_{\mathrm{SL}(NN'),\mu\otimes\mu'}(S^{\vec{a}}_{\vec{k}})$ by Z for short. We have to prove that the morphism $\tau : X \to Z$ factors through the morphism $\pi := \pi_{\zeta^{\bullet}_{\mathbb{R}},\zeta^{-}_{\mathbb{R}}} \times \pi_{\zeta^{\bullet}_{\mathbb{R}},\zeta^{+}_{\mathbb{R}}} : X \to Y$, and a morphism $\bar{\tau} : Y \to Z$. It is easy to see that τ contracts the fibers of π, that is for any $y \in Y$ we have $\tau(\pi^{-1}(y)) = z$ for a certain point $z \in Z$. It means that the image T of $\pi \times \tau : X \to Y \times Z$ projects onto Y bijectively. Furthermore, T is a closed subvariety of $Y \times Z$ since both π and τ are proper. Finally, Y is normal by a theorem of Crawley–Boevey. This implies that the projection $T \to Y$ is an isomorphism of algebraic varieties. Hence T is the graph of a morphism $Y \to Z$. This is the desired morphism $\bar{\tau}$.

This argument was explained to us by A. Kuznetsov. □

3. Tannakian approach

3.1. Given an almost simple simply connected group G, and the weights $\overline{\mu} \in \Lambda^{+}_{k}$, $\overline{\lambda}^{(i)} \in \Lambda^{+}_{k_i}$, $1 \leqslant i \leqslant n$, and a positive integer d, we consider the moduli space $\mathrm{Bun}^{\overline{\boldsymbol{\lambda}},d/k}_{G,\overline{\mu}}(S^{\vec{a}}_{\vec{k}})$ introduced in Section 1.9. It classifies the G-bundles on the stack $\overline{S}^{\vec{a}}_{\vec{k}}$ of second Chern class d/k, trivialized at infinity such that the class of the fiber at infinity is given by $\overline{\mu}$, while the class of the fiber at s_i is given by $\overline{\lambda}^{(i)}$.

Conjecture 3.2. $\mathrm{Bun}^{\overline{\boldsymbol{\lambda}},d/k}_{G,\overline{\mu}}(S_{\vec{k}})$ *is connected* (*possibly empty*).

Following the numerology of Section 2.9, we introduce the weights $\lambda^{(i)} := (k_i, \overline{\lambda}^{(i)}, 0) \in \Lambda^{+}_{\mathrm{aff},k_i}$, and $\mu := (k, \overline{\mu}, -\frac{1}{2k}(2d + (\overline{\mu}, \overline{\mu}) - (\overline{\lambda}, \overline{\lambda}))) \in \Lambda^{+}_{\mathrm{aff},k}$ where $\overline{\lambda} = \sum_{i=1}^{n} \overline{\lambda}^{(i)}$. We also set $\lambda := (k, \overline{\lambda}, 0)$. Now we redenote $\mathrm{Bun}^{\overline{\boldsymbol{\lambda}},d/k}_{G,\overline{\mu}}(S^{\vec{a}}_{\vec{k}})$ by $\mathrm{Bun}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}})$.

Given a representation $\varrho : G \to \mathrm{SL}(W_\varrho)$ we have a morphism $\varrho_* : \mathrm{Bun}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathrm{Bun}^{\varrho\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho),\varrho\circ\mu}(S^{\vec{a}}_{\vec{k}}) \subset \mathcal{U}^{\varrho\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho),\varrho\circ\mu}(S^{\vec{a}}_{\vec{k}})$. Here $\lambda^{(p)} = (k_p, \overline{\lambda}^{(p)}, 0)$, $\varrho \circ \lambda^{(p)} := (k_p, \varrho \circ \overline{\lambda}^{(p)}, 0)$; $\varrho \circ \boldsymbol{\lambda} = (\varrho \circ \lambda^{(1)}, \ldots, \varrho \circ \lambda^{(n)})$; $\mu = (k, \overline{\mu}, m)$, $\varrho \circ \mu := (k, \varrho \circ \overline{\mu}, \varrho_{\mathbb{Z}} m)$, and $\varrho_{\mathbb{Z}}$ is the Dynkin index of the representation ϱ (we stick to the notation of [3, 6.1]).

We define $\mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}})$ as the closure of the image of $\prod_\varrho \varrho_*(\mathrm{Bun}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}))$ inside $\prod_\varrho \mathcal{U}^{\varrho\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho),\varrho\circ\mu}(S^{\vec{a}}_{\vec{k}})$. For any ϱ we have an evident projection morphism $\mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\varrho\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho),\varrho\circ\mu}(S^{\vec{a}}_{\vec{k}})$. By an abuse of notation we will denote this morphism by ϱ_*.

Proposition 3.3. *Assume that any representation of G is a direct summand of a tensor power of ϱ* (*this is equivalent to requesting that ϱ is faithful*). *Then $\varrho_* : \mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\varrho\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho),\varrho\circ\mu}(S^{\vec{a}}_{\vec{k}})$ is a closed embedding. In particular, $\mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}})$ is of finite type.*

Proof. Let $x \in \mathcal{U}^{\varrho\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho),\varrho\circ\mu}(S^{\vec{a}}_{\vec{k}})$ be a point in the closure of the locally closed subvariety $\varrho_*(\mathrm{Bun}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}))$. There is an affine pointed curve $(C,c) \subset \mathcal{U}^{\varrho\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho),\varrho\circ\mu}(S^{\vec{a}}_{\vec{k}})$ such that $(C-c) \subset \varrho_*(\mathrm{Bun}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}))$, and $c = x$.

Let $\varsigma : G \to \mathrm{SL}(W_\varsigma)$ be another representation of G. We choose a projection $\varkappa : \varrho^{\otimes m} \twoheadrightarrow \varsigma$. According to Proposition 2.14, we consider $\bar{\tau}(C) \subset \mathcal{U}^{\varrho^{\otimes m}\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varrho^{\otimes m}),\varrho^{\otimes m}\circ\mu}(S^{\vec{a}}_{\vec{k}})$, and then $\varkappa_*\bar{\tau}(C) \subset \mathcal{U}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}})$. Since $\varkappa_*\bar{\tau}(C-c) \subset \mathrm{Bun}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}})$, we have lifted $x = c$ to a point of $\mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}})$.

It remains to prove that such a lift is unique. Let $\varkappa' : \varrho^{\otimes m'} \twoheadrightarrow \varsigma$ be another projection. Then $\varkappa_*\bar{\tau} = \varkappa'_*\bar{\tau} : (C-c) \hookrightarrow \mathrm{Bun}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}})$. Since $\mathcal{U}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}})$ is separated, it follows that $\varkappa_*\bar{\tau}(c) = \varkappa'_*\bar{\tau}(c)$.

This completes the proof of the proposition. □

3.4. Convolution morphism

The collection of convolution morphisms $\mathcal{U}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\varsigma\circ\lambda^{(1)}+\dots+\varsigma\circ\lambda^{(n)}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S'^{\vec{a}}_{\vec{k}})$ (see Section 2.9) gives rise to the convolution morphism $\varpi : \mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(S'^{\vec{a}}_{\vec{k}})$.

Lemma 3.5. *The morphism* $\varpi : \mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(S'^{\vec{a}}_{\vec{k}})$ *is birational.*

Proof. It suffices to check that ϖ is an isomorphism when restricted to the open subset $\mathrm{Bun}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(S'^{\vec{a}}_{\vec{k}}) \subset \mathcal{U}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(S'^{\vec{a}}_{\vec{k}})$. For any representation $\varsigma : G \to \mathrm{SL}(W_\varsigma)$ and the corresponding closed embedding $\varsigma_* : \mathcal{U}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(S'^{\vec{a}}_{\vec{k}}) \hookrightarrow \mathcal{U}^{\varsigma\circ\lambda^{(1)}+\dots+\varsigma\circ\lambda^{(n)}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S'^{\vec{a}}_{\vec{k}})$ we have $\varsigma_*(\mathrm{Bun}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(S'^{\vec{a}}_{\vec{k}})) \subset \mathrm{Bun}^{\varsigma\circ\lambda^{(1)}+\dots+\varsigma\circ\lambda^{(n)}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S'^{\vec{a}}_{\vec{k}})$. Any vector bundle $\mathcal{F} \in \mathrm{Bun}^{\varsigma\circ\lambda^{(1)}+\dots+\varsigma\circ\lambda^{(n)}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S'^{\vec{a}}_{\vec{k}})$ has a unique preimage $\mathcal{F}'$ under the convolution morphism $\mathcal{U}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}}) \to \mathcal{U}^{\varsigma\circ\lambda^{(1)}+\dots+\varsigma\circ\lambda^{(n)}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S'^{\vec{a}}_{\vec{k}})$, moreover, $\mathcal{F}' \in \mathrm{Bun}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}})$ since the 0-stability implies the $\zeta^\bullet_{\mathbb{R}}$-stability. Clearly, if $\mathcal{F}$ lies in the image $\varsigma_*(\mathrm{Bun}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(S'^{\vec{a}}_{\vec{k}})) \subset \mathrm{Bun}^{\varsigma\circ\lambda^{(1)}+\dots+\varsigma\circ\lambda^{(n)}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S'^{\vec{a}}_{\vec{k}})$, then $\mathcal{F}'$ lies in the image $\varsigma_*(\mathrm{Bun}^{\boldsymbol{\lambda}}_{G,\mu}(S^{\vec{a}}_{\vec{k}})) \subset \mathrm{Bun}^{\varsigma\circ\boldsymbol{\lambda}}_{\mathrm{SL}(W_\varsigma),\varsigma\circ\mu}(S^{\vec{a}}_{\vec{k}})$. So this $\mathcal{F}'$ is the unique preimage of $\mathcal{F}$ under ϖ. □

3.6. Main conjecture

We have a proper surjective morphism $\varpi : \mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S_{\vec{k}}) \to \mathcal{U}^{\lambda^{(1)}+\dots+\lambda^{(n)}}_{G,\mu}(\mathbb{A}^2/\Gamma_k)$, and we are interested in the multiplicities in $\varpi_*\mathrm{IC}(\mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S_{\vec{k}}))$. For $\mu \leqslant \nu \leqslant \lambda^{(1)} + \dots + \lambda^{(n)}$ we will denote the multiplicity of $\mathrm{IC}(\mathcal{U}^{\nu}_{G,\mu}(\mathbb{A}^2/\Gamma_k))$ in $\varpi_*\mathrm{IC}(\mathcal{U}^{\boldsymbol{\lambda}}_{G,\mu}(S_{\vec{k}}))$ by $M^{\boldsymbol{\lambda}}_{\nu,\mu}$.

Conjecture 3.7.

a) *ϖ is semismall, and hence $M^{\boldsymbol{\lambda}}_{\nu,\mu}$ is just a vector space in degree zero.*
b) *$M^{\boldsymbol{\lambda}}_{\nu,\mu}$ is independent of μ and equals the multiplicity of the $G^\vee_{\mathrm{aff}}$-module $L(\nu)$ in the tensor product $L(\lambda^{(1)}) \otimes \dots \otimes L(\lambda^{(n)})$.*

Remark 3.8. The direct image $\varpi_*\mathrm{IC}(\mathfrak{U}^{\boldsymbol{\lambda}}_{G,\mu}(S_{\vec{k}}))$ *is not* isomorphic to the direct sum $\bigoplus_\nu M^{\boldsymbol{\lambda}}_{\nu,\mu} \otimes \mathrm{IC}(\mathfrak{U}^\nu_{G,\mu}(\mathbb{A}^2/\Gamma_k))$: it contains the IC sheaves of other strata of $\mathfrak{U}^\nu_{G,\mu}(\mathbb{A}^2/\Gamma_k)$ with nonzero multiplicities. This observation is due to H. Nakajima [12].

The following proposition is essentially proved in [12].

Proposition 3.9. *Conjecture* 3.7 *for* $G = \mathrm{SL}(N)$ *holds true.*

Proof. By definition, the desired multiplicity $M^{\boldsymbol{\lambda}}_{\nu,\mu}$ can be computed on the quiver varieties, in the particular case $\zeta^\circ_{\mathbb{C}} = (0, \ldots, 0)$. It is computed in Theorem 5.15 (Eq. (5.16)) of [12] under the name $V^{\mathbf{v}^0,\emptyset}_{\mathbf{v}',\emptyset}$. Note that we are interested in the particular case $\mu = \emptyset = \lambda$, $\mathbf{v}^0 = \mathbf{v}$, thereof (we apologize for the conflicting roles of λ, μ in [12] and in the present paper). We set $d = v_1 + \cdots + v_k$, $d' = v'_1 + \cdots + v'_k$. Finally, $\bar{v}$ is associated to the pair $(\mathbf{v}', \mathbf{w})$ as in [2, 7.3], and $\nu = (k, \bar{v}, \frac{1}{2k}[2d' - 2d - (\bar{v}, \bar{v}) + (\bar{\lambda}^{(1)} + \cdots + \bar{\lambda}^{(n)}, \bar{\lambda}^{(1)} + \cdots + \bar{\lambda}^{(n)})])$.

Furthermore, in Remark 5.17.(3) of [12] the multiplicity $V^{\mathbf{v}^0,\emptyset}_{\mathbf{v}',\emptyset}$ is identified via I. Frenkel's level-rank duality with the multiplicity of the $G^\vee_{\mathrm{aff}}$-module $L(\nu)$ in the tensor product $L(\lambda^{(1)}) \otimes \cdots \otimes L(\lambda^{(n)})$. □

3.10. Digression on the Beilinson–Drinfeld Grassmannian

Let C be a smooth algebraic curve and let c be a point of C. It is well known that a choice of formal parameter at c gives rise to an identification of Gr_G with the moduli space of G-bundles on C endowed with a trivialization away from c. Similarly, for any $n \geqslant 1$ one can introduce the *Beilinson–Drinfeld Grassmannian* $\mathrm{Gr}_{C,G,n}$ as the moduli space of the following data:

1) An ordered collection of points $(c_1, \ldots, c_n) \in C^n$;
2) A G-bundle $\mathcal{F}$ on C trivialized away from $(c_1, \ldots, c_n)$.

We have an obvious map $p_n : \mathrm{Gr}_{C,G,n} \to C^n$ sending the above data to $(c_1, \ldots, c_n)$. When all the points c_i are distinct, the fiber $p_n^{-1}(c_1, \ldots, c_n)$ is non-canonically isomorphic to $(\mathrm{Gr}_G)^n$. When all the points coincide, the corresponding fiber is isomorphic to just one copy of Gr^n_G. For any $\lambda_1, \ldots, \lambda_n \in \Lambda^+$ one can define the closed subvariety $\overline{\mathrm{Gr}}^{\lambda_1,\ldots,\lambda_n}_{C,G}$ in $\mathrm{Gr}_{C,G,n}$ such that for any collection $(p_1, \ldots, p_n)$ of distinct points of C the intersection $p_n^{-1}(c_1, \ldots, c_n) \cap \overline{\mathrm{Gr}}^{\lambda_1,\ldots,\lambda_n}_{C,G}$ is isomorphic to $\overline{\mathrm{Gr}}^{\lambda_1} \times \cdots \times \overline{\mathrm{Gr}}^{\lambda_n}$ and the intersection $p_n^{-1}(c, \ldots, c) \cap \overline{\mathrm{Gr}}^{\lambda_1,\ldots,\lambda_n}_{C,G}$ is isomorphic to $\overline{\mathrm{Gr}}^{\lambda_1+\cdots+\lambda_n}$.

Similarly, given C and n as above one defines the scheme $\widetilde{\mathrm{Gr}}_{G,C,n}$ classifying the following data:

1) An element $(c_1, \ldots, c_n) \in C^n$.
2) An n-tuple $(\mathcal{F}_1, \ldots, \mathcal{F}_n)$ of G-bundles on C; we also let $\mathcal{F}_0$ denote the trivial G-bundle on C.
3) An isomorphism κ_i between $\mathcal{F}_{i-1}|_{C \setminus \{c_i\}}$ and $\mathcal{F}_i|_{C \setminus \{c_i\}}$ for each $i = 1, \ldots, n$.

We denote by $\tilde{p}_n$ the natural map from $\widetilde{\mathrm{Gr}}_{C,G,n}$ to C^n. Note that from 3) one gets a trivialization of $\mathcal{F}_n$ away from $(c_1, \ldots, c_n)$. Thus we have the natural map $\widetilde{\mathrm{Gr}}_{C,G,n} \to \mathrm{Gr}_{C,G,n}$. This map is proper and it is an isomorphism on the open subset where all the points c_i are distinct. On the other hand, the morphism $\widetilde{p}_n^{-1}(c,\dots,c)\to \mathrm{Gr}_{C,G,n}$ is isomorphic to the morphism $\underbrace{\mathrm{Gr}_G\star\cdots\star\mathrm{Gr}_G}_{n}\to\mathrm{Gr}_G$. For $\lambda_1,\dots,\lambda_n$ as above we denote by $\widetilde{\mathrm{Gr}}_{C,G}^{\lambda_1,\dots,\lambda_n}$ the closed subset of $\widetilde{\mathrm{Gr}}_{C,G,n}$ given by the condition that each κ_i lies in $\overline{\mathrm{Gr}}_G^{\lambda_i}$. Then the intersection $\widetilde{p}_n^{-1}(c,\dots,c)\cap\widetilde{\mathrm{Gr}}_{C,G}^{\lambda_1,\dots,\lambda_n}$ is isomorphic to $\overline{\mathrm{Gr}}_G^{\lambda_1}\star\cdots\star\overline{\mathrm{Gr}}_G^{\lambda_n}$.

3.11. Beilinson–Drinfeld Grassmannian for G_{aff}

Our next task will be to define an analog of (some pieces) of the Beilinson–Drinfeld Grassmannian for G_{aff} in the case when $C=\mathbb{A}^1$. The idea is that as $(a_1,\dots,a_n)\in\mathbb{A}^{n-1}$ varies, we will organize $\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(S_{\vec{k}}^{\vec{a}})$ (resp. $\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(S_{\vec{k}}'^{\vec{a}})$) into a family $\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(\mathcal{S}_{\vec{k}})$ (resp. $\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(\mathcal{S}'_{\vec{k}})$) over $\mathcal{X}=\mathbb{A}^{n-1}$ (though there is *no* family of smooth 2-dimensional stacks over $\mathcal{X}$). We will also construct a proper birational morphism $\varpi:\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(\mathcal{S}_{\vec{k}})\to\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(\mathcal{S}'_{\vec{k}})$ specializing to the morphisms ϖ of Section 3.4 for the particular values of $(a_1,\dots,a_n)$.

In case $G=\mathrm{SL}(N)$, we *define* $\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(\mathcal{S}_{\vec{k}})$ (resp. $\mathcal{U}_{G,\mu}^{\boldsymbol{\lambda}}(\mathcal{S}'_{\vec{k}})$) as the families of quiver varieties $\mathfrak{M}_{\zeta_{\mathbb{R}}^\bullet}(V,W)$ (resp. $\mathfrak{M}_0(V,W)$) over the variety $\mathcal{X}$ of moment levels $\zeta_{\mathbb{C}}^\circ$ (recall that $\zeta_{\mathbb{C}}^\circ$ is reconstructed from $(a_1,\dots,a_n)$ by the beginning of Section 2.2), see [12] between Lemma 5.12 and Remark 5.13.

For general G we repeat the procedure of Section 3.1. We only have to define the morphism $\bar\tau:\mathcal{U}_{\mathrm{SL}(N),\mu}^{\boldsymbol{\lambda}}(\mathcal{S}_{\vec{k}}^{\vec{a}})\times\mathcal{U}_{\mathrm{SL}(N'),\mu'}^{\prime\boldsymbol{\lambda}}(\mathcal{S}_{\vec{k}}^{\vec{a}})\to\mathcal{U}_{\mathrm{SL}(NN'),\mu\otimes\mu'}^{\boldsymbol{\lambda}\otimes'\boldsymbol{\lambda}}(\mathcal{S}_{\vec{k}}^{\vec{a}})$, that is to prove a relative analogue of Proposition 2.14.

To this end we consider the resolution $\mathfrak{M}_{\zeta_{\mathbb{R}}}(V,W)\to\mathfrak{M}_{\zeta_{\mathbb{R}}^\circ}(V,W)\to\mathfrak{M}_{\zeta_{\mathbb{R}}^\bullet}(V,W)$, see [12] between Lemma 5.12 and Remark 5.13. Here $\zeta_{\mathbb{R}}^\circ$ is (chosen and fixed) generic in the hyperplane $\zeta_{\mathbb{R}}^\circ\cdot\delta=0$ (see [11, 1(iii)]), and $\zeta_{\mathbb{R}}$ is in the chamber containing $\zeta_{\mathbb{R}}^\circ$ in its closure with $\zeta_{\mathbb{R}}\cdot\delta<0$. According to the Main Theorem of [11], $\mathfrak{M}_{\zeta_{\mathbb{R}}}(V,W)$ is isomorphic to the Giesecker moduli space of torsion-free sheaves on the simultaneous resolution $\widetilde{\overline{\mathcal{S}}}_{\vec{k}}$ trivialized at ℓ_∞. Now repeating the argument of Proposition 2.14 we obtain a morphism $\tilde\tau:\mathfrak{M}_{\zeta_{\mathbb{R}}^\circ}(V,W)\times\mathfrak{M}_{\zeta_{\mathbb{R}}^\circ}(V',W')\to\mathfrak{M}_{\zeta_{\mathbb{R}}^\circ}(V'',W'')$ where the Γ_k-modules V'',W'' are defined as follows: $W''=W\otimes W'$, $V''=V\otimes W'\oplus V'\otimes W\oplus V\otimes V'\otimes(Q\ominus\mathbb{C}^2)$, and Q is the tautological 2-dimensional representation of $\Gamma_k\subset\mathrm{SL}(2)$, while $\mathbb{C}^2$ is the trivial 2-dimensional representation of Γ_k. Composing it with the projection $\mathfrak{M}_{\zeta_{\mathbb{R}}^\circ}(V'',W'')\to\mathfrak{M}_{\zeta_{\mathbb{R}}^\bullet}(V'',W'')$ we obtain a morphism $\tau':\mathfrak{M}_{\zeta_{\mathbb{R}}^\circ}(V,W)\times\mathfrak{M}_{\zeta_{\mathbb{R}}^\circ}(V',W')\to\mathfrak{M}_{\zeta_{\mathbb{R}}^\bullet}(V'',W'')$. Now the argument of Proposition 2.14 proves that τ' factors through the desired morphism $\bar\tau:\mathfrak{M}_{\zeta_{\mathbb{R}}^\bullet}(V,W)\times\mathfrak{M}_{\zeta_{\mathbb{R}}^\bullet}(V',W')\to\mathfrak{M}_{\zeta_{\mathbb{R}}^\bullet}(V'',W'')$.

3.12. Semismallness of ϖ

In this subsection we speculate on a possible approach to Conjecture 3.7a) using the double affine version of the Beilinson–Drinfeld Grassmannian.

Assume k is even. We set $\mathcal{X}=\mathbb{A}^{n-1}$ with coordinates $a_1,\dots,a_n$, $k_1a_1+\cdots+k_na_n=0$. We consider the weighted projective space $\underline{\mathbb{P}}(2,k,k,2)=(\mathbb{A}^4-0)//\mathbb{G}_m$ where $\mathbb{G}_m$ acts as follows: $t(x,y,z,w)=(t^2x,t^ky,t^kz,t^2w)$. We define a relative surface $q:\underline{\overline{\mathcal{S}}'}\to\mathcal{X}$ as the hypersurface in $\underline{\mathbb{P}}(2,k,k,2)\times\mathcal{X}$ given by the equation $yz=(x-a_1w)^{k_1}\cdots(x-a_nw)^{k_n}$. The divisor at infinity is given by $w=0$. Note that q is a compactification of a subfamily of the semiuniversal deformation of the A_{k-1}-singularity constructed in [13]. Clearly, the fiber $q^{-1}(a_1,\dots,a_n)$ with the divisor at infinity removed is isomorphic to $\underline{S'}_{\vec{k}}^{\vec{a}}$. There is a family $\underline{\overline{\mathcal{S}}}\xrightarrow{p}\underline{\overline{\mathcal{S}}'}\xrightarrow{q}\mathcal{X}$ such that p is an isomorphism in a neighbourhood of the divisor at infinity, and the restriction of p to the fiber $q^{-1}(a_1, \ldots, a_n)$ with the divisor at infinity removed is nothing else than the partial resolution $\underline{S}^{\vec{a}}_{\vec{k}} \to \underline{S'}^{\vec{a}}_{\vec{k}}$ of Section 2.2.

By the axioms of Section 1.11, we should have a proper birational morphism $\varpi : \mathcal{U}_G(\mathcal{S}) \to \mathcal{U}_G(\mathcal{S}')$ whose fiber over $x = (0, \ldots, 0) \in \mathcal{X}$ coincides with ϖ of Section 3.4. This is nothing else than ϖ of Section 3.11.

Since the family $\overline{\underline{\mathcal{S}}} \to \mathcal{X}$ is equisingular, we expect the morphism $\mathfrak{p} : \mathcal{U}_G(\mathcal{S}) \to \mathcal{X}$ to be locally acyclic. Hence the specialization of the intersection cohomology sheaf $\mathrm{IC}(\mathcal{U}_G(\mathcal{S}))$ to the fiber $\mathfrak{p}^{-1}(0, \ldots, 0)$ coincides with $\mathrm{IC}(\mathcal{U}_G(S_{\vec{k}}))$. Since the specialization commutes with the direct image under proper morphisms, we obtain $\varpi_* \mathrm{IC}(\mathcal{U}_G(S_{\vec{k}})) = \mathbf{Sp}_{(0,\ldots,0)} \varpi_* \mathrm{IC}(\mathcal{U}_G(\mathcal{S})) = \mathbf{Sp}_{(0,\ldots,0)} \mathrm{IC}(\mathcal{U}_G(\mathcal{S}'))$. Here the second equality holds since ϖ is an isomorphism off the diagonals in $\mathcal{X}$. It follows that $\varpi_* \mathrm{IC}(\mathcal{U}_G(S_{\vec{k}}))$ is perverse (and semisimple, by the decomposition theorem).

Acknowledgments

The idea of using mixed Kleinian stacks in order to describe convolution in the double affine Grassmannian was suggested to us by E. Witten (a differential-geometric approach to this problem is discussed in Section 5 of [14]); we are very grateful to him for sharing his ideas with us and for numerous very interesting conversations on the subject. Both authors would like to thank H. Nakajima for a lot of very helpful discussions and in particular for his patient explanations of the contents of [11,12]. We are obliged to R. Bezrukavnikov, D. Kaledin, D. Kazhdan, and A. Kuznetsov for the useful discussions. This paper was completed when the first author was visiting the J.-V. Poncelet CNRS laboratory at the Independent University of Moscow.

A.B. was partially supported by the NSF grant DMS-0600851. M.F. was partially supported by the RFBR grant 09-01-00242, the HSE Science Foundation award No. 11-09-0033, the Ministry of Education and Science of Russian Federation, grant No. 2010-1.3.1-111-017-029, and the AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023.

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