

LABELED DOUBLE PANTS DECOMPOSITIONS

ANNA FELIKSON AND SERGEY NATANZON

To the memory of Vladimir Igorevich Arnold

ABSTRACT. A double pants decomposition of a 2-dimensional surface is a collection of two pants decomposition of this surface introduced by the authors. There are two natural operations acting on double pants decompositions: flips and handle-twists. It is shown by the authors that the groupoid generated by flips and handle-twists acts transitively on admissible double pants decompositions, where the class of admissible decompositions has a natural topological and combinatorial description. In this paper, we label the curves of double pants decompositions and show that for all but one surfaces the same groupoid acts transitively on all labeled admissible double pants decompositions. The only exclusion is a sphere with two handles, where the groupoid has 15 orbits.

2000 MATH. SUBJ. CLASS. 57M50.

KEY WORDS AND PHRASES. Pants decomposition, Mapping class group.

INTRODUCTION

Consider a 2-dimensional orientable surface S of genus g with n holes. A pants decomposition of S is a decomposition into 3-holed spheres (called “pairs of pants”). In [2] we considered double pants decompositions of surfaces as a union of two pants decompositions (with an additional assumption that the homology classes of the curves contained in the double pants decomposition generate the whole homology lattice $H_1(S, \mathbb{Z})$). We introduced a simple groupoid acting on double pants decompositions (the groupoid is generated by transformations of two types called flips and handle-twists, each flip or handle-twist changing only one curve of double pants decomposition) and proved that this groupoid acts transitively on all admissible double pants decompositions. The class of admissible double pants decompositions has a simple combinatorial definition (see Definition 1.10 below) as well as a nice description in terms of Heegaard splittings of 3-manifolds.

More precisely, for each pants decomposition P of S one may construct a handlebody S_+ such that S is the boundary of S_+ and all curves of P are contractible

Received September 13, 2010.

Supported in part by RFBR grant 11-01-00289-a. The second named author also supported in part by grants RFBR 10-01-00678, NSh 709.2008.1 and the Russian government grant 11.F34.31.0005.

inside S_+ . A union of two pants decompositions of the same surface define two different handlebodies bounded by S , attaching this handlebodies along S one obtains a Heegaard splitting of some 3-manifold. This connection of two pants decompositions to a Heegaard splitting was investigated in a row of papers ([1], [6], [8] and many others, see [3] for further references). The two pants decompositions are considered usually as two vertices in a pants complex, using as the main tool the Hempel distance.

The admissible double pants decompositions defined in [2] are exactly ones resulting in Heegaard splittings of a 3-sphere. So, the transitive action of flip and handle-twists groupoid on admissible double pants decompositions may be interpreted as an action on Heegaard splitting of 3-sphere.

In this paper we consider double pants decompositions with curves labeled by distinct integer numbers. We define a trivial action of flips and handle-twists on the labels: all labels are preserved by these transformations, in particular, the label of the flipped or twisted curve coincides with its initial label. We consider the action of the flip-twist groupoid generated by flips and handle-twist on labeled admissible double pants decompositions and obtain the following theorem:

Theorem A. (Main Theorem) *The flip-twist groupoid acts transitively on labeled admissible double pants decompositions of $S_{g,n}$, $2g + n > 2$, unless $(g, n) = (2, 0)$. The action of flip-twist groupoid on labeled admissible double pants decompositions of $S_{2,0}$ has 15 orbits.*

Furthermore, we also may restrict ourselves to the case of one labeled pants decomposition. It was shown by Hatcher and Thurston [5], [4] that there are two types of transformations called flips and S -moves which are sufficient to connect all pants decompositions in the unlabeled case. We extend the statement to the labeled case:

Theorem B. *The groupoid generated by flips and S -moves acts transitively on labeled pants decompositions of $S_{g,n}$, $2g + n > 2$.*

The paper is organized as follows. In Section 1, we recall from [2] the definitions concerning double pants decompositions and flip-twist groupoid. We also introduce the notion of labeled double pants decomposition and define the action of flip-twist groupoid on the labels. For the aims of proofs we consider also a notion of strictly labeled double pants decompositions for which the action of flip-twist groupoid is unable to intermix the labels of one pants decomposition with the labels of another. In Section 2, we consider labeled pants decomposition and prove Theorem B. In Section 3, we prove transitivity of flip-twist groupoid on strictly labeled decompositions (to be exact, the groupoid acts transitively unless $(g, n) = (2, 0)$ and has 6 orbits otherwise). Finally, in Section 4, we use the result of Section 3 to prove the Main Theorem.

Acknowledgments. We are grateful to Robert Penner for suggestion to consider the orbits of labeled double pants decompositions. We also thank the anonymous referee for careful reading of the paper and helpful comments.

1. DOUBLE PANTS DECOMPOSITIONS

In this section we introduce double pants decompositions and their transformations.

1.1. Pants decompositions. Let $S = S_{g,n}$ be an oriented closed surface of genus $g \geq 0$ with n holes. A *curve* c on S is an embedded closed non-contractible non-self-intersecting curve considered up to a homotopy of S .

Given a set of curves we always assume that there are no “unnecessary intersections”, so that if two curves of this set intersect each other in k points then there are no homotopy equivalent pair of curves intersecting in less than k points.

For a pair of curves c_1 and c_2 we denote by $|c_1 \cap c_2|$ the number of (geometric) intersections of c_1 with c_2 .

Definition 1.1 (Pants decomposition). A *pants decomposition* of S is a set of (non-oriented) mutually disjoint curves $P = \{c_1, \dots, c_m\}$ decomposing S into pairs of pants (i.e., into spheres with 3 holes).

It is easy to see that any pants decomposition of a surface $S_{g,n}$ consists of $m = 3g - 3 + n$ curves. To simplify formulas we will always write m instead of $3g - 3 + n$.

Note that we do allow self-folded pants, two of whose boundary components are identified in S . A surface which consists of one self-folded pair of pants will be called a *handle*.

Definition 1.2 (Lagrangian plane of pants decomposition). Let $P = \{c_1, \dots, c_m\}$ be a pants decomposition. A Lagrangian plane $\mathcal{L}(P) \subset H_1(S, \mathbb{Z})$ is a subspace spanned by the homology classes of c_i , $i = 1, \dots, m$ (here c_i is taken with any orientation).

Definition 1.3 (Flip). Let $P = \{c_1, \dots, c_m\}$ be a pants decomposition. Define a *flip of P in the curve c_i* as a replacing of a regular curve $c_i \in P$ by any curve c'_i satisfying the following properties:

- c'_i does not coincide with any of c_1, \dots, c_m ;
- $|c'_i \cap c_i| = 2$;
- $c'_i \cap c_j = \emptyset$ for all $j \neq i$.

See Fig. 1.1 for an example of a flip. Clearly, an inverse operation to a flip is also a flip (so that the set of flips compose a groupoid acting on pants decompositions).

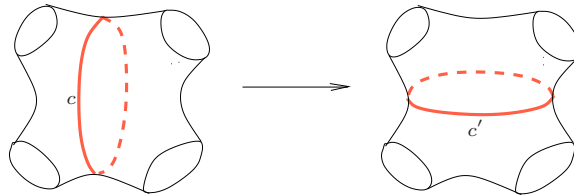


FIGURE 1.1. Flips of pants decomposition.

1.2. Double pants decompositions.

Definition 1.4 (Lagrangian planes in general position). Two *Lagrangian planes* \mathcal{L}_1 and \mathcal{L}_2 are in *general position* if $\mathcal{L}_1 \cap \mathcal{L}_2 = 0$ and $H_1(S, \mathbb{Z}) = \langle \mathcal{L}_1, \mathcal{L}_2 \rangle$, where $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ is the sublattice spanned by \mathcal{L}_1 and \mathcal{L}_2 .

See Fig. 1.2 for an example of two pants decompositions spanning a pair of Lagrangian planes in general position.

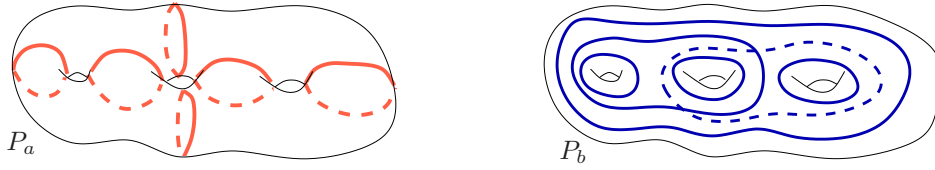


FIGURE 1.2. Pair of pants decompositions (P_a, P_b) .

Definition 1.5 (Double pants decomposition). A *double pants decomposition* $DP = (P_a, P_b)$ is a pair of pants decompositions P_a and P_b of the same surface such that the Lagrangian planes $\mathcal{L}_a = \mathcal{L}(P_a)$ and $\mathcal{L}_b = \mathcal{L}(P_b)$ spanned by these pants decompositions are in general position.

There are several natural transformations on the set of double pants decompositions:

- flips of P_a ;
- flips of P_b ;
- handle-twists (see Definition 1.6 below).

Definition 1.6 (Handle-twists). For a double pants decomposition $DP = (P_a, P_b)$ we define an additional transformation which may be performed if P_1 and P_2 contain the same curve $a_i = b_i$ separating the same handle \mathfrak{h} , see Fig. 1.3(b). Let $a \subset \mathfrak{h}$ ($b \subset \mathfrak{h}$) be the unique curve in \mathfrak{h} from P_a (P_b , resp.). Then a *handle-twist* $T_a(b)$ (respectively, $T_b(a)$) is a Dehn twist along a (respectively, b) in any of two directions (see Fig. 1.3(b)).

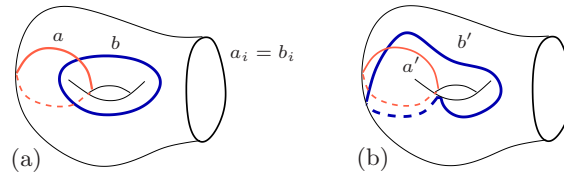


FIGURE 1.3. Handle-twists: (a) Double self-folded pair of pants; (b) The same pair of pants after a handle-twist $T_a(b)$

Notice that both flips and handle-twists are reversible transformations, hence flips and handle-twists generate a groupoid acting on the set of double pants decompositions.

Definition 1.7 (Flip-twist groupoid). A *flip-twist groupoid* FT is a groupoid generated by flips and twists.

Definition 1.8 (Double curve). A curve $c \in (P_a, P_b)$ is called *double* if $c \in P_a \cap P_b$.

Definition 1.9 (Standard decomposition). A double pants decomposition (P_a, P_b) is *standard* if there exist g double curves $c_1, \dots, c_g \in (P_a, P_b)$ such that c_i cuts out of S a handle \mathfrak{h}_i .

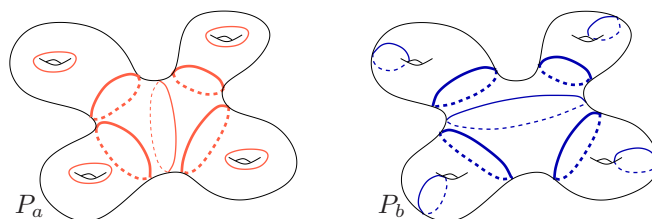


FIGURE 1.4. A standard double pants decomposition (P_a, P_b) .

Definition 1.10 (Admissible decomposition). Let us say that a double pants decomposition (P_a, P_b) is *admissible* if it is possible to transform (P_a, P_b) to a standard double pants decomposition by a sequence of flips.

The following theorem is the main result of [2].

Theorem 1.11 [2]. A *flip-twist groupoid* acts transitively on admissible double pants decompositions of $S = S_{g,n}$ (for any (g, n) such that $2g + n > 2$).

1.3. Labeled double pants decompositions. We say that a pants decomposition $P = \{c_1, \dots, c_m\}$ is *labeled* if each curve $c_i \in P$ is labeled by a number $x_i \in \{1, \dots, m\}$, $x_i \neq x_j$ for $i \neq j$, in other words we assign to the curves of P distinct numbers $\{1, \dots, m\}$. A labeled pants decomposition will be denoted $P = \{c_1, \dots, c_m\}_l$ (with index l).

Similarly, a double pants decomposition $DP = (P_a, P_b) = \{c_1, \dots, c_m; c_{m+1}, \dots, c_{2m}\}$ is *labeled* if each curve $c_i \in DP$ is labeled by a number $x_i \in \{1, \dots, 2m\}$. The notation for the labeled double pants decomposition will also contain the index l : $DP = \{c_1, \dots, c_m; c_{m+1}, \dots, c_{2m}\}_l$.

In unlabeled version of double pants decomposition we consider $\{c_1, \dots, c_m\}$ as a set without any additional structure, so we do not distinguish between two pants decompositions shown in Fig. 1.5. In labeled version $\{c_1, \dots, c_m\}_l$ these two decompositions are considered as different: they differ by their *labelings* (in figures, we show the labelings by the numbers written near the curves).

Flips and handle-twists preserve labelings, i.e., the new curve carries the same number as the deleted one had. In case of a double curve $c = c_i = c_j \in DP$ we assign two labels x_i and x_j to the same curve. Flipping the double curve we can not differ between the labels, so for any given topological flip f of a double curve we define two labeled flips f_1 and f_2 : both result in the same set of curves on the surface as f does, but the curve c is labeled by x_i after f_1 and by x_j after f_2 .

In [2] we have proved transitivity of flip-twist groupoid on admissible double pants decompositions. It is natural to ask if this groupoid act transitively on labeled admissible double pants decompositions.

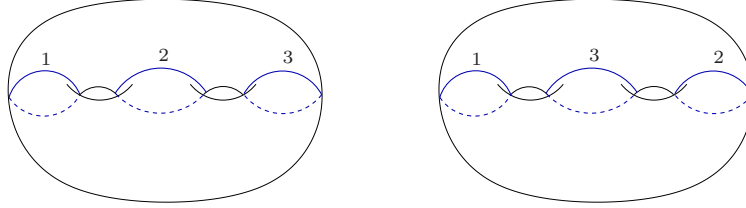


FIGURE 1.5. These two labeled decompositions are different. The numbers in the figure

1.4. Strictly labeled double pants decompositions. In a labeled double pants decomposition the two labels of a double curve are unordered: we do not know which of these labels belongs to which of the two pants decompositions. It would be convenient for the proof of our theorem to define also *strictly labeled* double pants decompositions where each label of a double curve remembers to which of two pants decompositions it belongs (so that flips and twists do not intermix labels of P_a with labels of P_b).

A double pants decomposition $DP = (P_a, P_b) = \{c_1, \dots, c_m; c_{m+1}, \dots, c_{2m}\}$ is *strictly labeled* if each curve $c_i \in P_a$ is labeled by a number $x_i \in \{1, \dots, m\}$ and each curve $c_i \in P_b$ is labeled by a number $x_i \in \{m+1, \dots, 2m\}$. In other words, instead of the whole permutation group S_{2m} the set of labels is permuted only by $S_m \times S_m$. In the strictly labeled case we write $\{c_1, \dots, c_m; c_{m+1}, \dots, c_{2m}\}_{sl}$, with index *sl*.

The flips and handle-twists preserve the labeling, i.e., the new curve carries the same number as the deleted one had. In case of a double curve $c = c_i = c_j \in DP$ we do not mix c_i with c_j since one of these curves belong to P_a and another belong to P_b . This means that if $c_i \in P_a$ and we make a flip of c_i , then the new curve $f(c_i) \in P_a$ has the same label as c_i , not as c_j . So the labels assigned to one component always stay together.

We will first work with strictly labeled double pants decompositions and then in Section 4 extend the results to the labeled double pants decompositions.

2. TRANSITIVITY FOR LABELED PANTS DECOMPOSITIONS

Let P be a pants decomposition of a surface $S = S_{g,n}$. It was shown in [5] and [4] that P may be transformed to any other pants decomposition of S via a sequence of flips and S -moves, where an S -move is defined as in Fig. 2.1. In this section we show that the groupoid generated by flips and S -moves acts transitively on labeled pants decompositions.

First, we will prove transitivity for the case of sufficiently large surfaces, namely for $S_{g,n}$ satisfying $2g + n > 4$, or in other words, for the surfaces whose pants decomposition contain at least three pairs of pants.

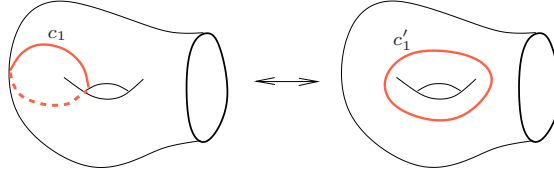


FIGURE 2.1. *S*-move: if some handle h is separated by by a curve $c \in P$ then a curve $c_1 \in P$ contained in h may be exchanged by any curve c'_1 such that $|c_1 \cap c'_1| = 1$.

Lemma 2.1. *Let P be a pants decomposition of a surface $S_{g,n}$, $2g + n > 4$. If $c_1, c_2 \in P$ are two curves in the boundary of the same pair of pants then the label of c_1 may be swapped with the label of c_2 by a sequence of flips.*

Proof. Let p be a pair of pants containing both c_1 and c_2 as boundary components. First, we will assume that p is not self-folded, i.e., that p is not a handle cut out by c_1 or c_2 (the case of the handle will be derived from this case at the end of the proof).

Let p_1 and p_2 be the adjacent pairs of pants ($c_i = p \cap p_i$). There are two possibilities: either p_1 and p_2 are two distinct pairs of pants or they coincide.

If $p_1 \neq p_2$ then the labels on c_1 and c_2 are swapped by a sequence of 5 flips shown in Fig. 2.2 (usually called “pentagon relation”). Notice that some of five boundary components of the “pentagon” may be identified, but this does not affect the procedure.

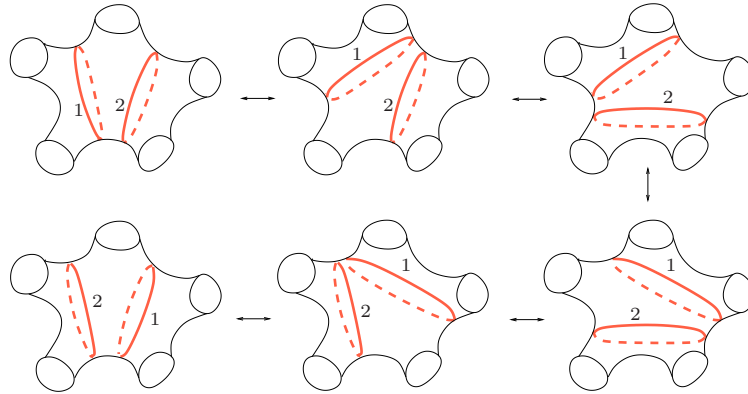
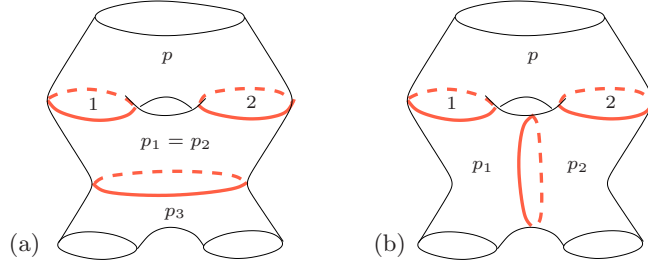


FIGURE 2.2. “Pentagon relation” exchanges the labels

Suppose that $p_1 = p_2$. Since S contains at least 3 pairs of pants, there exists a pair of pants p_3 such that $p \cup p_1 \cup p_3$ looks as shown in Fig. 2.3, left (up to possible interchange of p and p_1 and possible identification of some boundary components). Then after one flip we obtain a configuration on Fig. 2.3, right, which suits to the case $p_1 \neq p_2$ considered above.

FIGURE 2.3. Reduction of the case $p_1 = p_2$ to the case $p_1 \neq p_2$

Now, we are left to consider the case when p is a self-folded pair of pants, i.e., a handle cut out by c_1 or c_2 , say by c_1 . Then we apply a flip f to the curve c_1 obtaining a non-self-folded pair of pants p' with c_2 , $f(c_1)$ and some curve c_3 in the boundary. As it is proved above, there exists a sequence ψ of flips which swaps the labels of c_2 and $f(c_1)$. Applying the flip f^{-1} to $f(c_1)$, we will obtain a sequence $f^{-1}\psi$ of flips which swaps the labels of c_1 and c_2 . \square

Corollary 2.2. *Let P be a labeled pants decomposition of $S = S_{g,n}$, where $2g + n > 4$. Then flips act transitively on labeling of P .*

Proof. The statement follows immediately from Lemma 2.1 and the fact that S is connected. \square

It is clear that Corollary 2.2 together with transitivity of flips and S -moves on unlabeled pants decompositions imply transitivity on the labeled pants decompositions. So, we are left to consider only finitely many surfaces $S_{g,n}$ satisfying the inequality $2g + n \leq 4$, i.e., $S_{0,3}$, $S_{0,4}$, $S_{1,1}$, $S_{1,2}$ and $S_{2,0}$. Notice that pants decompositions of surfaces $S_{0,3}$, $S_{0,4}$ and $S_{1,1}$ contain at most one curve, so there is nothing to prove in these cases. A pants decomposition of $S_{1,2}$ contains two curves whose labels could be swapped as in Fig. 2.4. So, the only question left concerns $S_{2,0}$. A pants decomposition of $S_{2,0}$ contains 3 curves. After at most one flip we may assume that all three curves are homologically non-trivial. Then we cut $S_{2,0}$ along one of the curves and use the procedure described above for $S_{1,2}$ to swap the labels of the other two curves. Thus, we have all transpositions of the labels on P , and hence, all permutations.

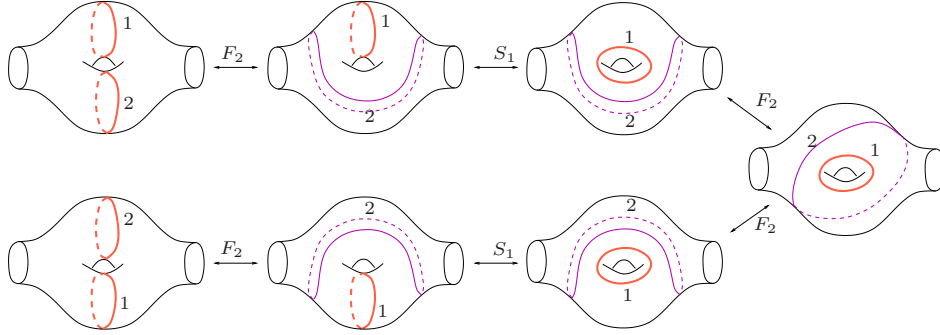
We summarize the results of this section in the following theorem:

Theorem 2.3. *For any surface $S_{g,n}$, $2g + n > 2$ flips and S -moves act transitively on labeled pants decompositions of S .*

3. TRANSITIVITY FOR STRICTLY LABELED DOUBLE PANTS DECOMPOSITIONS

Our next step is to prove transitivity of flip-twist groupoid on the strictly labeled double pants decompositions.

Theorem 3.1. *Let DP and DP' be two strictly labeled admissible double pants decompositions of $S = S_{g,n}$, where $2g + n > 4$. Then there exists a sequence of flips and handle-twists transforming DP to DP' .*


 FIGURE 2.4. Exchange of labels in $S_{1,2}$

Proof. By Theorem 1.11 there exists a sequence ψ of flips and handle-twists which takes $DP = (P_a, P_b)$ to $DP' = (P'_a, P'_b)$ as an unlabeled decomposition. There are two possibilities: either ψ takes P_a to P'_a and P_b to P'_b or ψ changes the components P_a and P_b .

Suppose that $\psi(P_a) = P'_a$, $\psi(P_b) = P'_b$. Then we apply Corollary 2.2 to take labeling of $\psi(P_a)$ and $\psi(P_b)$ to labeling of P'_a and P'_b respectively.

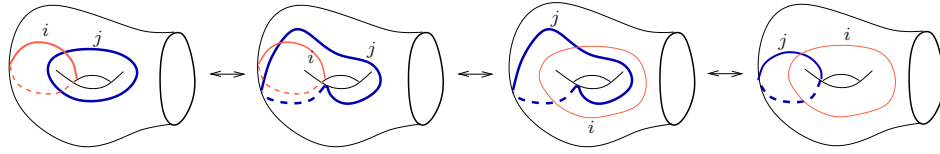


FIGURE 3.1. Exchange of labels in a handle via 3 handle-twists

Suppose that $\psi(P_a) = P'_b$, $\psi(P_b) = P'_a$. Consider a sequence of flips and handle-twists φ which takes (P'_a, P'_b) to a standard double pants decomposition (it does exist by Theorem 1.11). In a standard double pants decomposition we may swap the curve of P_a with the curve of P_b in each handle separately: see Fig. 3.1. \square

In view of Corollary 2.2, we are left to consider only finitely many surfaces $S_{g,n}$ satisfying the inequality $2g + n \leq 4$, i.e., $S_{0,3}$, $S_{0,4}$, $S_{1,1}$, $S_{1,2}$ and $S_{2,0}$. Below, we consider these five surfaces one by one.

3.1. Surfaces $S_{0,3}$, $S_{0,4}$ and $S_{1,1}$. A pants decomposition of any of the surfaces $S_{0,3}$, $S_{0,4}$ and $S_{1,1}$ consists of at most one curve, so the transitivity of flip-twist groupoid on strictly labeled pants decompositions follows trivially from the unlabeled version.

3.2. Surface $S_{1,2}$. A pants decomposition of $S_{1,2}$ consists of two curves, a double pants decomposition consists of two pairs of curves. Clearly, it is sufficient to check that there exists a sequence of flips and handle-twists which exchanges the labels of two curves of P_a and preserves the labels of P_b . This sequence is shown in Fig. 3.2.

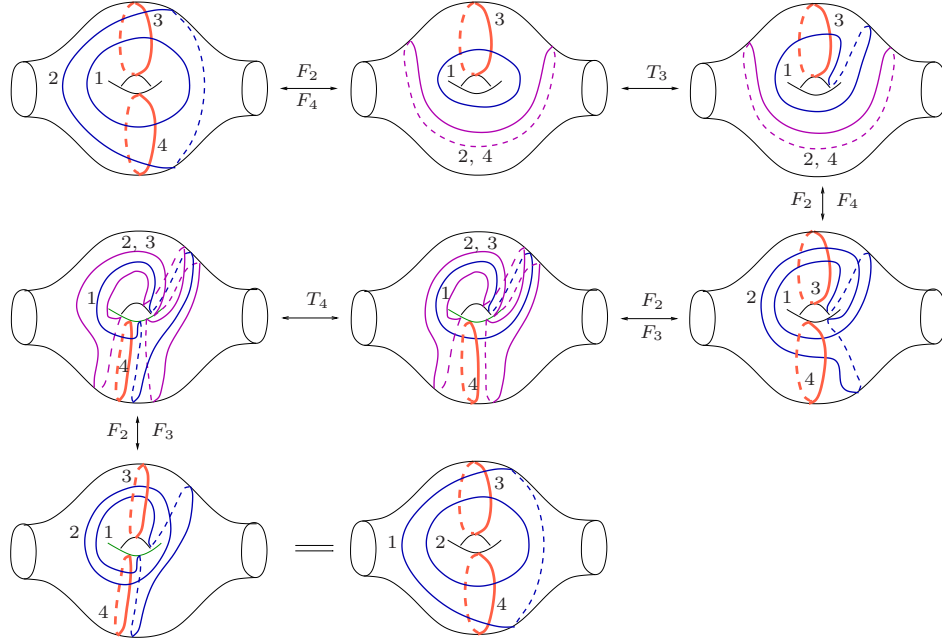


FIGURE 3.2. Changing the labels of P_a and preserving the labels of P_b (F_i stays for a flip of i -th curve, T_i stays for a handle-twist along i -th curve)

3.3. Surface $S_{2,0}$. This is the only surface where flip-twist groupoid does not act transitively on strictly labeled admissible double pants decompositions. To investigate this action we consider the double pants decompositions of the combinatorial type shown in Fig. 3.3 (we call this type of double pants decomposition *hexagonal*). This type of double pants decomposition may be characterized by the following property: we have $|a_i \cap b_{i+1}| = 1$, $|b_i \cap a_{i+1}| = 1$, $|a_i \cap b_i| = 0$, where indexes are considered modulo 3. In other words, the curves $[a_1, b_3, a_2, b_1, a_3, b_2]$ compose a “hexagon” where the neighbours do intersect and other sides do not (here we use the square brackets to show that the curves of the hexagonal decomposition are ordered). More precisely, the curves of the hexagonal decomposition decompose the surface $S_{2,0}$ into 4 hexagons.

Our aim now is to label the curves $[1, 4, 2, 5, 3, 6]$ and check which permutations of these labels may be performed via flips and handle-twists.

Consider any of the four hexagons on $S_{2,0}$ and read the labels on its sides in a clockwise direction. The obtained sequence should be considered modulo cyclic shifts of all labels and modulo reversing of the order. We obtain a *cyclic order* of the labeling. Notice that the cyclic order does not depend on the choice of one of the four hexagons (the choice of any of the two adjacent hexagons reverse the order of the labels in the sequence, the choice of the hexagon opposite to the initial one does not affects the order).

Whenever we need to compare two labeling of the same hexagonal double pants decomposition we always refer to the same hexagon on the surface, so that the notions of rotation of the labels and reversing of the order of the labels make sense. Comparing the cyclic orders of two distinct hexagonal decompositions we think of the cyclic order modulo reversing of the order and cyclic shifts of the labels.

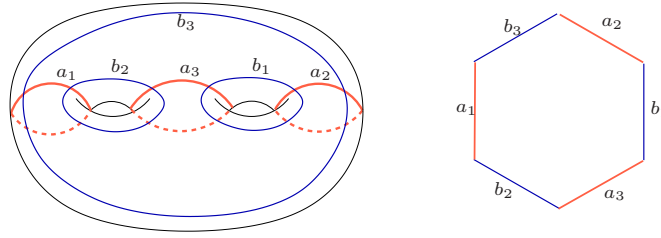


FIGURE 3.3. A hexagonal double pants decomposition

Definition 3.2 (Hexagonal twist). Let $[a_1, b_3, a_2, b_1, a_3, b_2]$ be a hexagonal double pants decomposition. A *hexagonal twist* T_{a_i} (or T_{b_i}) is a Dehn twist along a_i (respectively, along b_i), $i = 1, 2, 3$.

Lemma 3.3. (1) Any hexagonal twist is a composition of flips and handle-twists;
 (2) Any handle-twist is a composition of flips and hexagonal twists;
 (3) A hexagonal twist preserves the cyclic order in a hexagonal double pants decomposition.

Proof. Parts (1) and (2) follow from the commutative diagram shown in Fig 3.4, part (3) is evident. \square

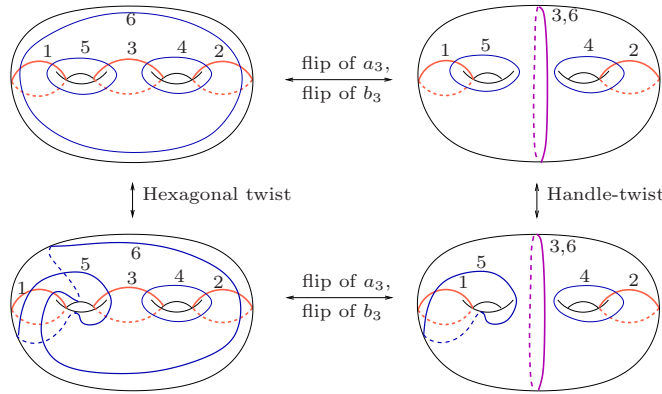


FIGURE 3.4. Hexagonal twist as a composition of flips and handle-twists

Lemma 3.4. There exists a sequence of flips and handle-twists which takes the hexagon $[a_1, b_3, a_2, b_1, a_3, b_2]$ to $[b_2, a_1, b_3, a_2, b_1, a_3]$ and the labels $[1, 4, 2, 5, 3, 6]$ to $[6, 1, 4, 2, 5, 3]$.

Proof. Consider the system $[a_1, b_3, a_2, b_1, a_3, b_2]$ and apply five hexagonal twists in a row, namely T_2, T_4, T_3, T_5, T_1 , where T_i is a twist in the curve labeled by i . Then we return to the same set of curves, but the labels are shifted, see Fig. 3.5. \square

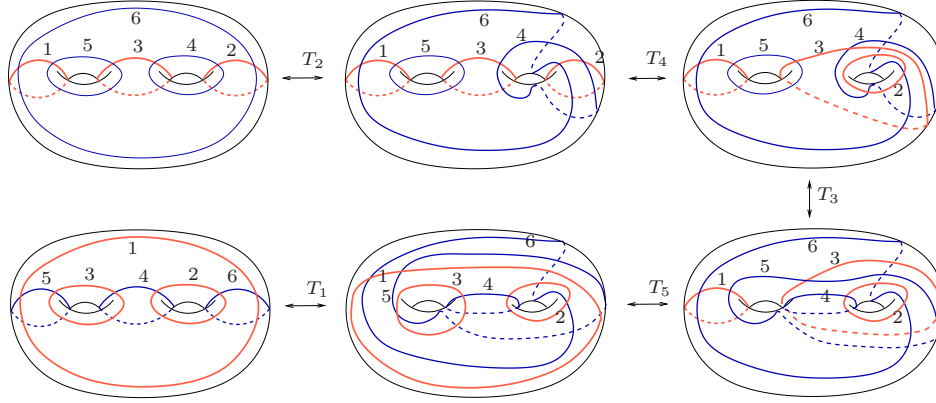


FIGURE 3.5. Rotation of the hexagon realized by 5 hexagonal twists.

Lemma 3.5. *There exists a sequence of flips and handle-twists which takes the hexagon $[a_1, b_3, a_2, b_1, a_3, b_2]$ to $[b_2, a_3, b_1, a_2, b_3, a_1]$ and the labels $[1, 4, 2, 5, 3, 6]$ to $[6, 3, 5, 2, 4, 1]$.*

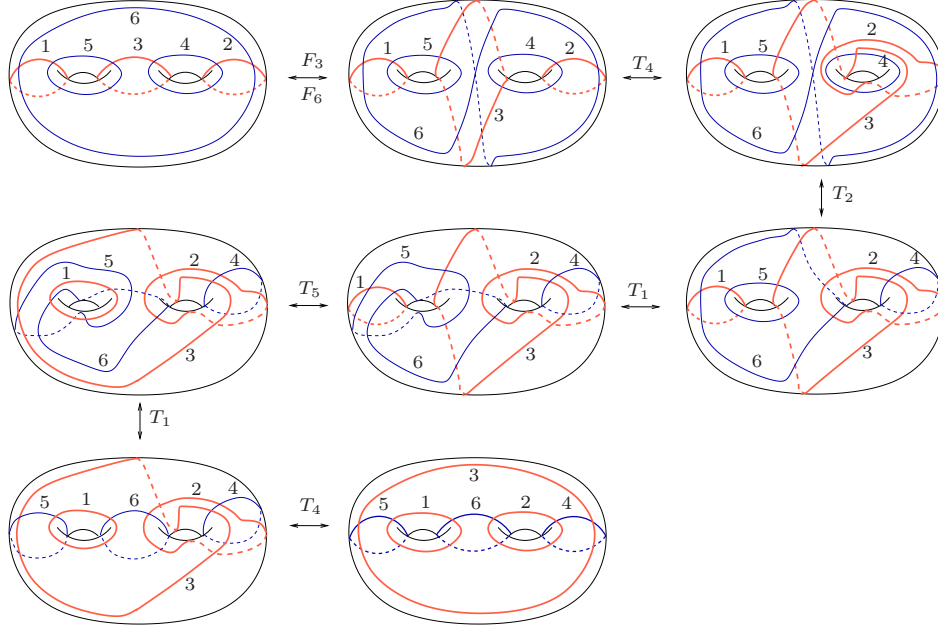
Proof. First, we make a flip in 3 and 6, and then apply hexagonal twists in 4, 2, 1, 5, 1, 4, see Fig. 3.6. \square

Lemmas 3.4 and 3.5 show that each element of the dihedral group D_6 acting on the labels of hexagon $[a_1, b_3, a_2, b_1, a_3, b_2]$ may be realized as a sequence of flips and handle-twists (Lemmas 3.4 and 3.5 represent a rotation of order 6 and a reflection of the hexagon, respectively).

In Lemma 3.7 we show that no other permutations of labels of the hexagon can be realized by flips and handle-twists. For the proof we will consider $H_1(S, \mathbb{Z}_2)$. Denote by $[c]_{\mathbb{Z}_2} \in H_1(S, \mathbb{Z}_2)$ a \mathbb{Z}_2 -homology class of the curve c . Notice that a \mathbb{Z} -homology class of c is defined only up to a change of sign (depending on the orientation of c), however, the class $[c]_{\mathbb{Z}_2}$ is well defined. In particular, if a, b, c are three boundary curves of the same pair of pants, then $[a]_{\mathbb{Z}_2} = [b]_{\mathbb{Z}_2} + [c]_{\mathbb{Z}_2}$.

Lemma 3.6. *Let $DP = [a_1, b_3, a_2, b_1, a_3, b_2]$ be a hexagonal double pants decomposition of $S_{2,0}$. Let ψ be a sequence of flips of $P_a = \{a_1, a_2, a_3\}_l$ such that $\psi(P_a)$ contains no homologically trivial curves. Then $[\psi(a_i)]_{\mathbb{Z}_2} = [a_i]_{\mathbb{Z}_2}$, $i = 1, 2, 3$.*

Proof. Label the curves a_1, a_2, a_3 of P_a by numbers 1, 2, 3 respectively and consider the sequence ψ as a composition of subsequences $\psi_1 \circ \dots \circ \psi_k$, where ψ_i is a composition of flips of the curve with the same label, while ψ_i and ψ_{i+1} are compositions of flips of curves with different labels. It is easy to see that after applying any subsequence $\psi_1 \circ \dots \circ \psi_j$, $0 \leq j \leq k$ the pants decomposition P_a turns in a


 FIGURE 3.6. Reflection of the hexagon $[a_1, b_3, a_2, b_1, a_3, b_2]$.

decomposition without homologically trivial curves (otherwise ψ_j and ψ_{j+1} can not flip the curve with distinct labels). So, it is sufficient to prove the statement of the lemma for one subsequence ψ_i .

Now, suppose that all flips in ψ_1 change the curve labeled 1. Consider a pair of pants p in $\psi_1(P_a)$ containing the curve $\psi_1(a_1)$. Then the boundary of p consists of curves $\psi_1(a_1)$, a_2 and a_3 (notice that $\partial p \neq \psi_1(a_1) \cup a_i$, $i = 2$ or 3 , since $\psi_1(a_1)$ is homologically non-trivial). This implies that

$$[\psi_1(a_1)]_{\mathbb{Z}_2} = [a_2]_{\mathbb{Z}_2} + [a_3]_{\mathbb{Z}_2} = [a_1]_{\mathbb{Z}_2},$$

and the statement for ψ_1 is proved. Applying this k times we obtain the lemma. \square

Lemma 3.7. *Let φ be a sequence of flips and handle-twists transforming the hexagonal set of curves $[a_1, b_3, a_2, b_1, a_3, b_2]$ to itself and permuting the labels $[1, 4, 2, 5, 3, 6]$ of these curves. Then the permutation coincides with some permutation obtained by an action of the dihedral group D_6 on the hexagon $[a_1, b_3, a_2, b_1, a_3, b_2]$.*

Proof. Consider a sequence φ of flips and handle-twists transforming the set of curves $\{a_1, b_3, a_2, b_1, a_3, b_2\}_{\text{sl}}$ to itself. By Lemma 3.3 any handle-twist is a composition of flips and hexagonal twists. So, φ is a composition of flips and hexagonal twists. By Lemma 3.3 hexagonal twist does not change cyclic order. We will show that if a sequence of flips takes a hexagonal double pants decomposition to a hexagonal one, then it either preserves the cyclic order or changes it to the opposite. Then the statement of the lemma follows.

So, it is left to show that if a sequence of flips φ takes a hexagonal double pants decomposition to a hexagonal one, then φ does not change the cyclic order. By Lemma 3.6, φ preserves \mathbb{Z}_2 -homology classes of all curves of P_a and P_b . This implies that $\varphi(a_i)$ intersects $\varphi(b_j)$ if and only if a_i intersects b_j . So, the cyclic order of curves $[a_1, b_3, a_2, b_1, a_3, b_2]$ is either preserved by φ or reversed. \square

Corollary 3.8. *The action of flip and twist groupoid on strictly labeled admissible double pants decompositions of $S_{2,0}$ has 6 orbits.*

Proof. Consider a hexagonal double pants decomposition. Notice that the labels from the set $\{1, 2, 3\}$ alternate with the labels from the set $\{3, 4, 5\}$, since curves from the same pants decomposition do not intersect each other.

Now, the action of the dihedral group D_6 is sufficient to put the labels 1, 2, 3 of P_a into required position. There are 6 possibilities left to put the labels 4, 5, 6. Distinct possibilities result in distinct cyclic orders, so the obtained labeling are not equivalent under the action of flip and twist groupoid. \square

The results of Sections 3.1–3.3 may be summarized in the following theorem:

Theorem 3.9. *The flip-twist groupoid acts transitively on strictly labeled admissible double pants decompositions of $S_{g,n}$, $2g + n > 2$, unless $(g, n) = (2, 0)$. The action of flip-twist groupoid on strictly labeled admissible double pants decompositions of $S_{2,0}$ has 6 orbits.*

4. TRANSITIVITY ON LABELED DOUBLE PANTS DECOMPOSITIONS

Now, we will extend Theorem 3.9 from the class of strictly labeled decompositions to the class of labeled decompositions.

Instead of thinking that two labels x_1 and x_2 of a double curve c do not know to which of P_a and P_b they do belong, we will add one more transformation, which we will call *switch*: a switch in c just changes the places of x_i and x_j .

Theorem 4.1. *The flip-twist groupoid acts transitively on labeled admissible double pants decompositions of $S_{g,n}$ unless $(g, n) = (2, 0)$. The action of flip-twist groupoid on labeled admissible double pants decompositions of $S_{2,0}$ has 15 orbits.*

Proof. First, consider all surfaces $S_{g,n}$, $(g, n) \neq (2, 0)$. By Theorem 3.9 flip-twist groupoid act transitively on strictly labeled double pants decompositions of these surfaces. So, it is sufficient to check that applying flips, handle-twists and switches in double curves we may arrive in a double pants decomposition $DP = (P_a, P_b)$ where the labels of P_a contain any given set of m numbers from $\{1, 2, \dots, 2m\}$. Clearly, it is sufficient to check that we may change the places of two labels, assigned to curves in P_a and P_b respectively. More precisely, if $c_i \in P_a$, $c_j \in P_b$, then we may change the labels i and j of the curves c_i and c_j in the following way:

- (1) find an admissible decomposition DP' with a double curve c (we can get to DP' from DP by flips and handle-twists in view of Theorem 1.11);
- (2) use transitivity of flip-twist groupoid on labeled double pants decompositions to get to the decomposition DP' with labels i and j on the double curve c ;

- (3) switch i and j ;
- (4) return to the decomposition DP with c_i and c_j labeled j and i respectively.

This implies that if $(g, n) \neq (2, 0)$ then the flip-twist groupoid acts transitively on labeled admissible double pants decompositions.

Now consider the case $(g, n) = (2, 0)$. The switch of the labels on the double curve in terms of hexagonal decomposition describes as exchange of the labels of two opposite sides of the hexagon. This implies that under the action of flip-twist groupoid the labels of opposite sides always remain the opposite. There are 15 possibilities to split 6 labels into 3 pairs (the label 1 may be paired with each of 5 other labels, for each of these 5 possibilities the smallest of the remaining labels may be paired with any of 3 other labels).

Choose one of the possible pairings, for example, $(1, 4)$, $(2, 5)$, $(3, 6)$. Using switches we make the labels 1, 2, 3 mutually non-adjacent in the hexagon. Using rotations and reflections as in Lemmas 3.4 and 3.5 we may put the labels 1, 2, 3 in any three mutually non-adjacent positions. The pairing then determines the positions of all remaining labels. \square

REFERENCES

- [1] A. Casson and C. Gordon, *Manifolds with irreducible Heegaard splittings of arbitrarily high genus*, Preprint.
- [2] A. Felikson and S. Natanzon, *Double pants decompositions of 2-surfaces*, Preprint [arXiv:1005.0073](https://arxiv.org/abs/1005.0073) [math.GT].
- [3] C. Gordon, *Workshop on Heegaard Splittings*, Geometry & Topology Monographs, vol. 12, ch. Problems, pp. 401–411, Geometry & Topology Publications, Coventry, 2007. MR [2404079](#)
- [4] A. Hatcher, *Pants decompositions of surfaces*, Preprint <http://www.math.cornell.edu/~hatcher/Papers/pantsdecomp.pdf>.
- [5] A. Hatcher and W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, Topology **19** (1980), no. 3, 221–237. MR [579573](#)
- [6] J. Hempel, *3-manifolds as viewed from the curve complex*, Topology **40** (2001), no. 3, 631–657. MR [1838999](#)
- [7] J. Johnson, *Heegaard splittings and the pants complex*, Algebr. Geom. Topol. **6** (2006), 853–874. MR [2240918](#)
- [8] F. Luo, *On Heegaard diagrams*, Math. Res. Lett. **4** (1997), no. 2–3, 365–373. MR [1453066](#)

A.F.: INDEPENDENT UNIVERSITY OF MOSCOW, B. VLASIEVSKI, 11, 119002 MOSCOW, RUSSIA
 Current address: Jacobs University Bremen, School of Engineering and Science, Campus Ring 1, 28759 Bremen, Germany
 E-mail address: a.felikson@jacobs-university.de

S.N.: HIGHER SCHOOL OF ECONOMICS, FACULTY OF MATHEMATICS, UL. VAVILOVA 7, 117312, MOSCOW, RUSSIA
 BELOZERSKY INSTITUTE OF PHYSICO-CHEMICAL BIOLOGY, MOSCOW STATE UNIVERSITY,
 INSTITUTE THEORETICAL AND EXPERIMENTAL PHYSICS, and
 LABORATORY OF GEOMETRICAL METHODS OF MATHEMATICAL PHYSICS, MSU
 E-mail address: natanzons@mail.ru