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BRIEF COMMUNICATIONS

Examples of Families of Strebel Differentials on Hyperelliptic Curves*

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Abstract. The paper contains an explicit construction of Strebel differentials on one-parameter families of hyperelliptic curves of even genus. Descriptions of the corresponding separatrices are presented.

Key words: Riemann surface, Strebel differential, Abelian integral.

0. A Strebel differential on a compact Riemann surface is a nonzero quadratic differential such that, in the metric defined by it, the surface is realized as a union of right cylinders patched along arcs of their boundaries. This concept was introduced by Strebel and Jenkins in the 1950–60s (see [1] and [2]); in 1967, Strebel established the existence of Strebel differentials on an arbitrary surface, and in 1975, Douady and Hubbard [3] proved that the Strebel differentials are dense in the space of quadratic differentials.

The cited results are existence theorems and do not provide examples of Strebel differentials. Apparently, the first explicit example was constructed by Amburg [4] in 2002; it is defined by a rather complicated formula, and the corresponding curve is defined over the quadratic field $\mathbb{Q}(\sqrt{2})$. The aim of the present paper is to construct simpler families of curves depending on one real parameter and Strebel differentials on them. By assigning rational values to the parameter, one obtains a countable set of Strebel differentials over $\mathbb{Q}$.

1. Let $\ddot{X}_p$ be the affine curve defined by the equation $y^2 = x^{2g+2} - 2px^{g+1} - 1$, and let X_p be its complete smooth model.

Theorem. *Let g be an even positive integer. For each $p \in \mathbb{R}$, the quadratic differential $x^{g-1}(\frac{dx}{y})^2$ is Strebel on X_p.*

2. Let $C_\pm$ be the "points at infinity" of the curve $\ddot{X}_p$, i.e., the points such that $\ddot{X}_p = X_p \setminus \{C_\pm\}$. We also introduce the points $A_\pm$ defined by the relations $x(A_\pm) = 0$ and $y(A_\pm) = i$.

The main character of the paper is the quadratic differential

$$\eta := x^{g-1}\left(\frac{dx}{y}\right)^2,$$

whose Strebel property we wish to establish.

The divisor relations

$$(x) = A_+ + A_- - C_+ - C_-, \qquad (\eta) = (g-1)(A_+ + A_- + C_+ + C_-)$$

are obvious.

3. Let L_k be the rays

$$\left\{x \in \mathbb{C} \mid x = te^{2i\pi k/(g+1)},\ t \in \left[((p^2+1)^{1/2} + p)^{1/(g+1)}, \infty\right)\right\}$$

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on the x-plane, and let M_k be the segments (Fig. 1)

$$\big[0, e^{i\pi(2k+1)/(g+1)}((p^2+1)^{1/2}-p)^{1/(g+1)}\big].$$

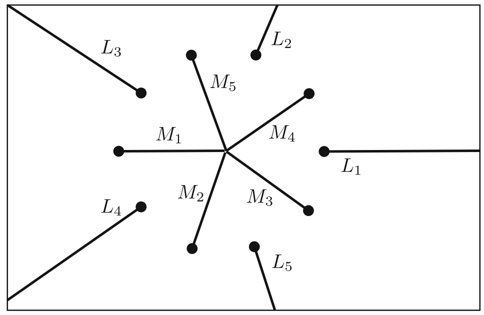

Fig. 1. The projection of the separatrix graph onto the x-plane for $g = 4$

Proposition. *For each* $k \in \{1, \dots, g+1\}$ *and any* $x_1, x_2 \in L_k$ *or* $x_1, x_2 \in M_k$, *the integrals* $\int_{x_1}^{x_2} \sqrt{\frac{x^{g-1}}{x^{2g+2}-2px^{g+1}-1}}\,dx$ *are real* (*regardless of the choice of a branch of the root*). *In other words, the integrals* $\int\sqrt{\eta}$ *along the arcs projected into the unions of the segments* M_k *and* $L_k \cup \infty$, *are real.*

The integrals $\int\sqrt{\eta}$ turn into the usual Abelian integrals after being pulled back to the *Riemann surface of the square root of a quadratic differential*, introduced in [3] as the subset defined in the cotangent bundle T^*X_p by the relation $\{\omega \mid \omega^2 = \eta\}$. In our case, this Riemann surface, which we denote by $X_p[\sqrt{\eta}]$, can be defined by the equation $v^2 = u^{4g+4} - 2pu^{2g+2} - 1$, and the projection $\pi\colon X_p[\sqrt{\eta}] \to X_p$ is defined by the relations $\pi^*x = u^2$ and $\pi^*y = v$; one can verify that $\pi^*\eta = \omega^2$, where $\omega := 2u^g\frac{du}{v}$.

Corollary. *The curves* $\{P \in X \mid x(P) \in L_k\} \cup \{C_\pm\}$ *and* $\{P \in X \mid x(P) \in M_k\}$ *are separatrices of the differential* η.

By considering the orders of zeros of the differential η, we conclude that it has no other separatrices.

It follows from the above-stated results that the separatrices of η begin and end at its zeros and that all these separatrices have finite lengths in the metric $\mathrm{d}s^2 = |\eta|$; hence their union is compact in this metric. It follows that the differential η is Strebel.

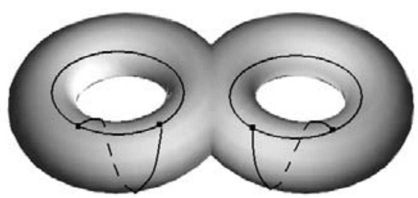

Fig. 2. The separatrix graph of the differential η for $g = 2$

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References

[1] J. A. Jenkins, Ann. Math., **66** (1957), 440–453.
[2] K. Strebel, Quadratic Differentials, Springer-Verlag, Berlin, 1984.
[3] A. Douady and J. Hubbard, Invent. Math., **30**:2 (1975), 175–179.

[4] N. Ya. Amburg, Uspekhi Mat. Nauk, **57**:5 (2002), 145–146; English transl.: Russian Math. Surveys, **57**:5 (2002), 987–988.
[5] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978.

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