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MARINE PHYSICS

Nonlinear Internal Waves in the Ocean Stratified in Density and Current

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Abstract—A nonlinear theory of long internal waves in the ocean stratified in density and current was developed. A nonlinear evolutionary equation of the type of the generalized Korteweg–de Vries equation with an accuracy of the third order of the perturbation theory was obtained in the framework of asymptotic methods. The procedures of decomposition and calculation of all coefficients of the evolutionary equation were automated with the help of the Maple package of symbolical calculations. Internal waves in a two-layer ocean (the upper layer moves at a constant speed) and in a three-layer ocean with a background current in the intermediate layer were considered as examples. The properties of large-amplitude solitons in a stratified ocean were investigated. The contribution of high-order terms with respect to nonlinearity and dispersion to the soliton structure was estimated.

INTRODUCTION

Internal waves of large amplitude, frequently observed in the coastal waters of the World Ocean, are usually interpreted in the framework of the weak-nonlinear theory [1, 20]. Thus, the evolution and transformation of long (in comparison with the depth of the ocean) internal waves are described with the help of the well-known Korteweg–de Vries equation. Its coefficients are determined by the vertical distribution of fluid density and shear horizontal current. As is known, the stratification of the ocean is not constant and varies in space and time, which causes the variability of the coefficients of the evolutionary equation and effects the parameters of internal waves. The calculations of the coefficients of the Korteweg–de Vries equation for various regions of the World Ocean [2, 5, 11, 21] indicate that the coefficient of quadratic nonlinearity varies especially strongly and can change its sign in the coastal zone. Basically, the sign change of the quadratic nonlinearity has long been known in the idealized model of a two-layer fluid, when the pycnocline becomes closer to the sea bottom than to the surface due to decreasing depth [13]. It is obvious that, in the zones with a small value of quadratic nonlinearity, the role of the high-order nonlinear terms in the asymptotic decomposition of the wave field becomes greater. Appropriate generalizations of the Korteweg–de Vries equation have already been reported in publications [6, 8, 13–15, 17–19]; however, the expressions for the coefficients of high-order terms become too bulky "to feel" their signs and values. It turned out, for example, that the coefficient of cubic nonlinearity can be positive [7, 9], and this has a fundamental influence on the dynamics of waves of large amplitude. In particular, the quasi-steady solitons can be transformed to quickly oscillating wave packages (breathers) and visa versa [10].

In the most consistent form, the generalized Korteweg–de Vries equations for internal waves in the ocean with a shear flow were obtained in the paper by Lamb [16]. They were reported at the symposium on internal waves (Sydney, Canada, 1998). However, in actuality, the main emphasis in this paper was made on the poor applicability of these generalizations for describing internal solitons of a large amplitude, close to the limiting one. Here, we shall consider moderate-amplitude internal waves, widespread in the coastal zone, for which we can expect that the generalizations of the Korteweg–de Vries equation of a not very high order will appear reasonably suitable. The coefficients of these equations will be given for arbitrary stratification of the ocean in density and current. The Maple package for symbolical calculations is used to get reliable results of bulky calculations, which allows one to operate with the formulas of almost any complexity and frees the researcher from having to manually write cumbersome expressions. Particular cases of two- and three-layer oceans with a background horizontal current in one of the layers are used for the calculating all the coefficients in an explicit form. The parameters of internal waves for these conditions, including the limiting amplitude of solitons and the evaluations of the contribution of various high-order terms to the structure of nonlinear internal waves, were calculated for the first time.

INITIAL EQUATIONS

The well-known nonlinear equations for the hydrodynamics of an inviscid and incompressible stratified fluid were taken as the initial ones, which, in terms of stream function ψ and buoyancy b, take the form

$$\frac{\partial \Delta\psi}{\partial t} - \frac{\partial b}{\partial x} = J(\psi, \Delta\psi), \quad (1)$$

$$\frac{\partial b}{\partial t} + N^2(z)\frac{\partial \psi}{\partial x} = J(\psi, b), \quad (2)$$

where ψ is the stream function ($u = \partial\psi/\partial z$ and $w = -\partial\psi/\partial x$), the z-axis is directed upwards, $N(z)$ is the Brunt–Vaisala frequency, and $J(A, B) = A_x B_z - A_z B_x$ is the Jacobian. The sea bottom $z = 0$ and the surface $z = H$ are assumed rigid; therefore, we use the boundary conditions of impenetrability:

$$\frac{\partial \psi}{\partial x}(z = 0) = \frac{\partial \psi}{\partial x}(z = H) = 0. \quad (3)$$

It is convenient to use the Lagrangian variable instead of the vertical coordinate z,

$$y = z - \eta(x, z, t), \quad (4)$$

where $\eta(x, z, t)$ is the vertical displacement of the isopycnal surface relative to its undisturbed position at spatial point (x, z) at time moment t. The Lagrangian coordinate allows us to trace the position of the isopycnal, which is most frequently obtained from experimental data. The vertical distribution of density does not change in this coordinate system: $\rho(x, z, t) = \rho(y)$; thus, the kinematic equation of the isopycnal displacement is supplementary to equation (1):

$$\frac{\partial z}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x} = -\frac{\partial \psi}{\partial x}. \quad (5)$$

The transition to the Lagrangian coordinate leads to a greater complexity of equation (1). If we use dimensionless values for coordinates (z/H, x/L, tU/L), Brunt–Vaisala frequency (HN/U) and stream function ($\Psi(x, y, t) = \psi/UH$), where U and L are the characteristic values of the velocity and wavelength, equation (1) takes the form

$$\frac{\partial \Theta}{\partial t} + \frac{1}{z_y}\frac{\partial \Psi}{\partial y}\frac{\partial \Theta}{\partial x} - N^2(y)\frac{z_x}{z_y} = 0, \quad (6)$$

where the dimensionless vorticity is

$$\Theta = \frac{1}{z_y^2}\frac{\partial^2 \Psi}{\partial y^2} - \frac{z_{yy}}{z_y^3}\frac{\partial \Psi}{\partial y} + \mu\left(\frac{\partial^2 \Psi}{\partial x^2} - \frac{2z_x}{z_y}\frac{\partial^2 \Psi}{\partial x \partial y} + \left(\frac{z_x}{z_y}\right)^2\frac{\partial^2 \Psi}{\partial y^2} + \left(\frac{z_x}{z_y}\left(\frac{z_x}{z_y}\right)_y - \left(\frac{z_x}{z_y}\right)_x\right)\frac{\partial \Psi}{\partial y}\right), \quad (7)$$

and equation (5) transforms to

$$\frac{\partial z}{\partial t} = -\frac{\partial \Psi}{\partial x}. \quad (8)$$

Here, $\mu = H^2/L^2$ characterizes the ratio of the ocean depth to wavelength. This parameter is small for the long waves considered in this work.

DERIVATION OF THE NONLINEAR EVOLUTIONARY EQUATION

Equations (6)–(8) were obtained by Lamb [16] and are referred to as equations in the Euler–Lagrange approach. It is noteworthy that they are exact. Considering the waves to be sufficiently weak, it is natural to introduce a small nonlinearity parameter ε and to write the wave disturbances in the form

$$z = y + \varepsilon\varsigma(x, y, t), \quad (9)$$

$$\Psi = \int_y U(y')dy' + \varepsilon\psi(x, y, t), \quad (10)$$

where $U(y)$ is the velocity of the background shear current.

In the approximation of the long waves of small amplitude ($\varepsilon \ll 1$, $\mu \ll 1$), the solution of equations (6) and (8) can be sought in the form of a series with respect to ε and μ:

$$\varsigma(x, y, t) = \varsigma_{00}(x, y, t) + \varepsilon\varsigma_{10}(x, y, t) + \mu\varsigma_{01}(x, y, t) + \varepsilon\mu\varsigma_{11}(x, y, t) + \ldots, \quad (11)$$

$$\psi(x, y, t) = \psi_{00}(x, y, t) + \varepsilon\psi_{10}(x, y, t) + \mu\psi_{01}(x, y, t) + \varepsilon\mu\psi_{11}(x, y, t) + \ldots, \quad (12)$$

where the first index corresponds to the power of the ε parameter and the second corresponds to the power of μ. According to the multiscale method, it is necessary to introduce a set of times $\tau_{ij} = \varepsilon^i\mu^j t$ and perform a transition to the system of coordinates moving at a velocity c (which is not yet determined). Then, the temporal derivatives can be transformed in the following way:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x} + \sum_{i,j}\varepsilon^i\mu^j\frac{\partial}{\partial \tau_{ij}}. \quad (13)$$

As a result, in each order of the perturbation theory (for either small parameter), we can exclude the unknown function ψ_{ij} and obtain one equation for the $\zeta_{ij}(x, y, t)$ function. It can be written in the general form as

$$\frac{\partial}{\partial y}\left[(c - U(y))^2\frac{\partial^2 \varsigma_{ij}}{\partial x \partial y}\right] + N^2(y)\frac{\partial \varsigma_{ij}}{\partial x} = R_{ij}, \quad (14)$$

where the right side is found from the solutions of previous approximations and derivatives with respect to slow times.

In the zero order of the theory of perturbations ($i = j = 0$), describing a linear internal wave without dispersion, the right-hand side in (14) is absent and the variables are separated:

$$\varsigma_{00}(x, y, \tau, \ldots) = \eta(x, \tau, \ldots)\Phi(y), \quad (15)$$

where $\Phi(y)$ is found as the solution of the boundary problem

$$L\Phi \equiv \frac{d}{dy}\left[(c - U(y))^2 \frac{d\Phi}{dy}\right] + N^2(y)\Phi = 0 \quad (16)$$

with zero boundary conditions at the bottom and sea surface. The boundary problem allows us to determine both the mode structure of the long internal wave (function Φ) and the velocity of its propagation c. In this approximation, nonlinearity and dispersion do not influence the wave form (the η function). From here on, we shall consider only one mode with the maximum propagation velocity. We also assume that the current in the ocean is stable and no critical layer exists.

The right side of (14) may be represented in the higher order approximations as

$$R_{ij} = 2\frac{\partial\eta}{\partial\tau_{ij}}\frac{d}{dy}\left[(c - U(y))\frac{d\Phi}{dy}\right] + \sum_k M_{ij}^k(\eta)S_{ij}^2(y), \quad (17)$$

where M is, generally speaking, the nonlinear differential operator with respect to x (it includes the terms of the form $\partial^n\eta^m/\partial x^n$) and S includes the corrections to the mode found from the previous approximations (we shall discuss this below). We shall not give the corresponding expressions for these functions because they are very cumbersome [3, 16]. The $\partial\eta/\partial\tau$ function is not determined, and we may use this freedom to represent the R function as the product of functions depending on x and y only. To do this, we require that

$$\frac{\partial\eta}{\partial\tau_{ij}} = \sum_k s_{ij}^k M_{ij}^k(\eta), \quad (18)$$

where the constants s for different indices will be found below. Then, the variables in the left-hand side of equation (14) are also separated:

$$\frac{\partial\varsigma_{ij}}{\partial x} = \sum_k M_{ij}^k(\eta)\Phi_{ij}^k(y), \quad (19)$$

where the functions Φ_{ij} are the solutions of the nonhomogeneous boundary problem

$$L\Phi_{ij}^k = 2s_{ij}^k\frac{d}{dy}\left[(c - U)\frac{d\Phi}{dy}\right] + S_{ij}^k(y) \quad (20)$$

with zero boundary conditions at the bottom and sea surface. It is known that the boundary problem (20) has a solution only if its right-hand side is orthogonal to the eigenfunction of the self-adjoint operator L, which leads to the determination of the previously unknown coefficients s:

$$s_{ij}^k = \frac{\int_0^H \Phi S_{ij}^k dy}{2\int_0^H (C - U)(d\Phi/dy)^2 dy}. \quad (21)$$

After this, the ordinary differential equation (20) can be solved analytically or numerically.

Thus, according to (15) and (19), the series for isopycnal displacement at different depths (11) is completely determined by a single unknown function $\eta(x, t)$ and a series for the stream function or current velocity. Some of the terms of this series are given below:

$$\begin{aligned}\zeta = {} & \eta(x, t)\Phi(y) + \varepsilon\eta^2\Phi_{10}(y) + \mu\eta_{xx}\Phi_{01}(y) \\ & + \varepsilon^2\eta^3\Phi_{20}(y) + \mu^2\eta_{xxxx}\Phi_{02}(y) \\ & + \varepsilon\mu\left(\frac{\eta_x^2}{2}\Phi_{11}^1(y) + \eta\eta_{xx}\Phi_{11}^2(y)\right) + \ldots.\end{aligned} \quad (22)$$

It is noteworthy that the solution of equation (20) is found with an accuracy of the solution of the corresponding homogeneous equation, that is, with an accuracy of the Φ function, and therefore, each term of series (22) is not constant. This was first noted in [15], where the authors suggested using this fact to minimize the difference between numerical simulations with the full and approximate nonlinear models. This method is not appropriate, however, from the point of view of applying asymptotic series, which do not require numerical solutions of initial equations. It is our opinion that the problem lies in correctly defining the physical variable used in the approximate model. From the practical point of view, it is convenient to choose a single isopycnal located in the maximum of the linear mode (let us designate this depth as y_{max}). If we impose additional conditions

$$\Phi(y_{max}) = 1, \quad \Phi_{ij}^k(y_{max}) = 0, \quad (23)$$

then series (22) at the point y_{max} breaks at the first term

$$\varsigma(x, y_{max}, t) = \eta(x, t) \quad (24)$$

and the $\eta(x, t)$ function is the isopycnal displacement at this level for any order of the perturbation method. It is evident that other conditions are also possible, but they lead to other values of the coefficients of the evolutionary equation [6, 12, 15].

The $\eta(x, t)$ function describing the wave evolution over the horizontal plane is specified by the slow-time derivatives (18). Taking (13) into account, we may rewrite it in the usual form:

$$\frac{\partial\eta}{\partial t}+c\frac{\partial\eta}{\partial x}+\varepsilon\alpha\eta\frac{\partial\eta}{\partial x}+\mu\beta\frac{\partial^3\eta}{\partial x^3}+\varepsilon^2\alpha_1\eta^2\frac{\partial\eta}{\partial x}$$
$$=-\mu^2\beta_1\frac{\partial^5\eta}{\partial x^5}-\varepsilon\mu\left(\gamma_1\eta\frac{\partial^3\eta}{\partial x^3}+\gamma_2\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}\right)+\ldots\,. \quad (25)$$

Equation (25) in the first-order approximation with respect to ε and μ is the well-known Korteweg–de Vries equation widely used for the analysis of nonlinear internal waves in the ocean. With the accuracy of the second order of nonlinearity, equation (25) is the so-called Gardner equation. In a more complete form, this equation can be called a generalized or extended Korteweg–de Vries equation. Due to its significance, we give the expressions for all coefficients of equation (25) in the dimensional form (parameters ε and μ are omitted):

$$\alpha=\frac{3}{2I}\int_0^H(c-U)^2(d\Phi/dy)^3dy, \quad (26)$$

$$\beta=\frac{1}{2I}\int_0^H(c-U)^2\Phi^2dy, \quad (27)$$

$$\alpha_1=\frac{1}{2I}\int_0^H dy\{3(c-U)^2[3(dT_n/dy)$$
$$-2(d\Phi/dy)^2](d\Phi/dy)^2+\alpha(c-U) \quad (28)$$
$$\times[5(d\Phi/dy)^2-4(dT_n/dy)](d\Phi/dy)-\alpha^2(d\Phi/dy)^2\},$$

$$\beta_1=\frac{1}{2I}\int_0^H dy\{2\beta(c-U)[\Phi^2-(d\Phi/dy)(dT_d/dy)]$$
$$-\beta^2(d\Phi/dy)^2+(c-U)^2\Phi T_d\}, \quad (29)$$

$$\gamma_1=-\frac{1}{2I}\int_0^H dy\{2(c-U)[\alpha(dT_d/dy)+2\beta(dT_n/dy)]$$
$$\times(d\Phi/dy)+2\alpha\beta(d\Phi/dy)^2-2\alpha(c-U)\Phi^2$$
$$+(c-U)^2\Phi^2(d\Phi/dy)-4\beta(c-U)(d\Phi/dy)^3 \quad (30)$$
$$-(c-U)^2[3(dT_d/dy)(d\Phi/dy)^2+2T_n\Phi]\},$$

$$\gamma_2=-\frac{1}{2I}\int_0^H dy\{(c-U)[2\beta(d\Phi/dy)^3+6\alpha\Phi^2]$$
$$-3\alpha\beta(d\Phi/dy)^2-2(c-U)^2[\Phi^2(d\Phi/dy)-3T_n\Phi] \quad (31)$$
$$-6\alpha(c-U)(dT_d/dy)(d\Phi/dy)$$
$$+3(c-U)^2dT_d/dy)(d\Phi/dy)^2\},$$

where

$$I=\int_0^H(c-U)(d\Phi/dy)^2dy \quad (32)$$

and the mode corrections of the first approximation are the solutions of the following equations:

$$LT_n=-\alpha\frac{d}{dy}\left[(c-U)\frac{d\Phi}{dy}\right]$$
$$+\frac{3}{2}\frac{d}{dy}\left[(c-U)^2\left(\frac{d\Phi}{dy}\right)^2\right] \quad (33)$$

for the nonlinear correction and

$$LT_d=-2\beta\frac{d}{dy}\left[(c-U)\frac{d\Phi}{dy}\right]-(c-U)^2\Phi \quad (34)$$

for the dispersion correction with zero boundary conditions at the sea bottom and surface, and also at the point of the maximum of the linear mode.

The procedure of deriving the generalized Korteweg–de Vries equation given above is described in [16] and reproduced here for clarity. It is rather cumbersome. We managed to automate this procedure using the Maple package, which improves the accuracy of obtaining the coefficients. It is important to emphasize that, due to the relations imposed by equations (16), (33), and (34), the expressions for the coefficients may take different forms [3, 16]. Here, we give only one form, which is more convenient for calculations.

INTERNAL WAVES IN A TWO-LAYER FLOW

The general expressions for the coefficients of the generalized Korteweg–de Vries equation written above can be very easily calculated for a two-layer ocean with a moving upper layer (Fig. 1). It is noteworthy that this problem with a zero flow was once a test to include the cubic effects into the Korteweg–de Vries equation [13]. The current has no influence on the modal structure of the internal wave (taking into account the nonlinear and dispersion corrections), but it changes the wave speed:

$$c=U_0\frac{h_2}{H}\pm\sqrt{c_0^2-U_0^2\frac{h_1h_2}{H^2}}, \quad (35)$$

where h_1 and h_2 are the thicknesses of the upper and lower layers ($H = h_1 + h_2$), U_0 is the velocity of the current in the upper layer, and c_0 is the wave speed in the ocean at rest:

$$c_0=\sqrt{g\frac{\Delta\rho}{\rho}\frac{h_1h_2}{H}}. \quad (36)$$

As expected, the internal wave is accelerated if it propagates together with the flow and is decelerated if the flow is opposite, but this is not related to the Doppler effect in the mean current U_0h_1/H. We shall assume that U_0 is always smaller than c_0 $U_0 < c_0$, and, therefore, no critical layer is present there. The dependence of the wave speed on the current velocity is shown in Fig. 2. For convenience, all coefficients are expressed using two dimensionless parameters: the thickness of the lower layer $l = h_2/H$ and the velocity of the current U_0/c (these dependencies of the nonlinearity coefficients and nonlinear dispersion are shown in Figs. 3–6):

$$\frac{\alpha}{c}H = -\frac{3}{2}\frac{1+(u-2)l}{l(l-1)}, \tag{37}$$

$$\frac{\beta}{cH^2} = -\frac{1}{6}l(l-1)\frac{ul(u-2)-(1-u)^2}{ul-1}, \tag{38}$$

$$\frac{\alpha_1}{c}H^2 = -\frac{3}{8}\frac{(4-3u)l+2(u-2)l^2+1}{l^2(l-1)^2}, \tag{39}$$

$$\begin{aligned}\frac{\beta_1}{cH^4} = -\frac{1}{360}\frac{l}{(ul-1)^3}&(4-8u-l+44l^2u^4-96l^3u\\&+204l^3u^3-36l^2u^3-101l^5u^4+164l^4u^4-36l^5u^2\\&+54l^4u^2-316l^4u^3-126l^3u^4+34l^3u^2-5lu^4\\&-48l^6u^3+204l^5u^3+24l^6u^4+36l^4u-16lu-86l^2u^2\\&+84l^2u-6l^2+30lu^2+4u^2+3l^3+8lu^3),\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{\gamma_1}{cH} = -\frac{1}{12(ul-1)^2}&(8l^3u^3-16l^3u^2-4l^2u-11l^2u^3\\&+28l^2u^2-5lu^2+14l+3lu^3-17ul-u^2-7+8u),\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{\gamma_2}{cH} = -\frac{1}{24(ul-1)^2}&(32l^3u^3-64l^3u^2-40l^2u\\&+136l^2u^2-41l^2u^3+9lu^3-53ul\\&+62l-41lu^2+5u^2+26u-31).\end{aligned} \tag{42}$$

Similar to the case of the zero current, the coefficient of quadratic nonlinearity α changes its sign depending on the depth of the pycnocline. If the current direction coincides with the wave, the zero point of α is shifted to the greater values of the thickness of the lower layer; an opposite current shifts it in the opposite direction. For undercritical currents, the latter shift is not greater than $l = H/3$. The coefficient of the cubic nonlinearity α_1 is always negative for the zero current, but if the direction of the current coincides with the wave, its value decreases. The coefficients of linear dispersion β and β_1 are positive for any values of the layer thickness, and if the direction of the current coincides with the wave, their values generally increase. The behavior of the coefficients of nonlinear dispersion γ_1

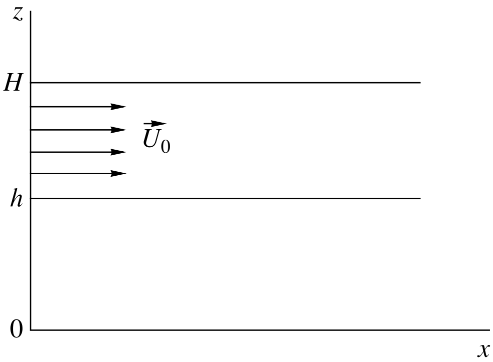

Fig. 1. Two-layer stratification.

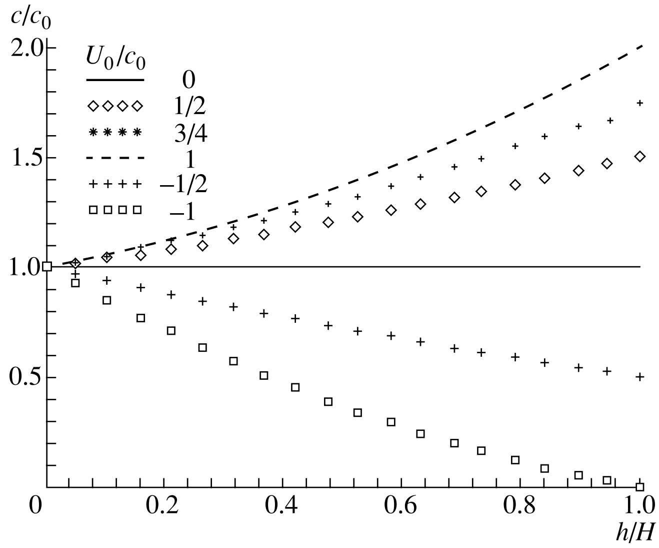

Fig. 2. Dependence of the wave propagation velocity c/c_0 on the depth of the pycnocline for different current velocities in a two-layer ocean.

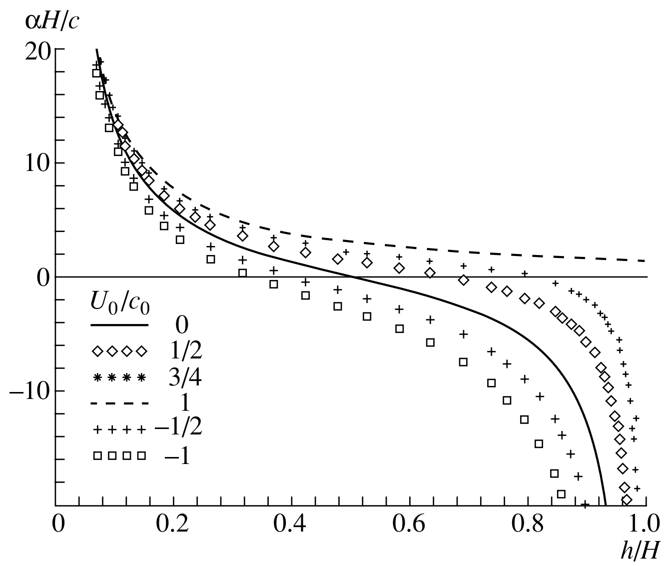

Fig. 3. Dependence of the coefficient of quadratic nonlinearity ($\alpha H/c$) on the depth of the pycnocline for different current velocitiesin a two-layer ocean.

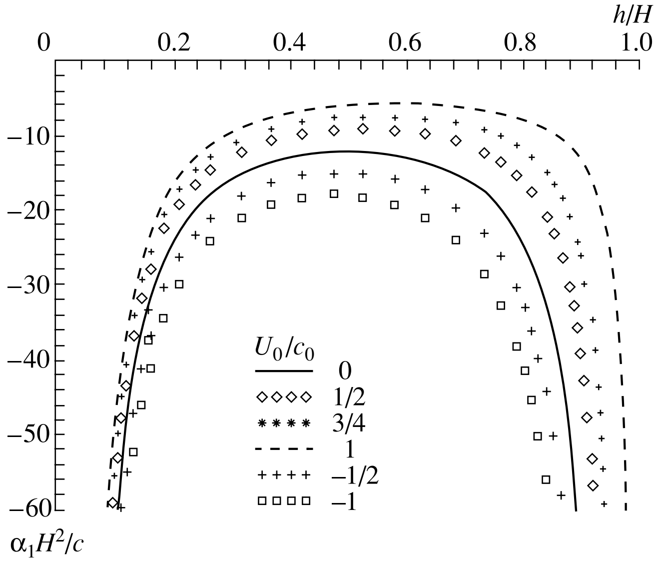

Fig. 4. Dependence of the coefficient of cubic nonlinearity ($\alpha_1 H^2/c$) on the depth of the pycnocline for different current velocities in a two-layer ocean.

and γ_2 in the presence of the current is more complex. They turn to zero at different values of the parameters (none of them coincide with the position of the zero of the quadratic nonlinearity α). When the current is absent, they are proportional to each other (γ_1/γ_2 = 31/14), and they turn to zero when the thickness of the layers is the same. When the direction of the current coincides with the wave, the increase in the dispersion coefficients allows us to conclude that, fundamentally, nonlinear effects (all other factors being the same) should be expressed to a lesser degree, although strong variations of the coefficient of quadratic nonlinearity may lead to an opposite effect.

NONLINEAR INTERNAL WAVES IN A THREE-LAYER SHEAR FLOW

Let us now consider a three-layer model of the density stratification with two symmetrical density jumps $\Delta\rho/\rho$ at each density interface. The thickness of the lower and upper layer is h, the total depth is H, and U_0 is the velocity of the horizontal current in the middle layer (Fig. 7). This kind of stratification with a zero shear flow was studied in [9]. The vertical structure of the first linear mode for this stratification is easily found:

$$Z_{00}(y) = \begin{cases} y/h, & 0 \le y \le h, \\ 1, & h \le y \le H-h, \\ (H-y)/h, & H-h < y \le H, \end{cases} \qquad (43)$$

as well as the nonlinear

$$Z_{10}(y)$$

$$= \begin{cases} -\dfrac{3y}{2h^2}\dfrac{c^2}{(c-U_0)^2}\left(1-\dfrac{H}{2h}\right), & 0 \le y < h, \\ -\dfrac{3}{2h^2}\dfrac{c^2}{(c-U_0)^2}\left(y-\dfrac{H}{2}\right), & h \le y \le H-h, \\ -\dfrac{3(y-H)}{2h^2}\dfrac{c^2}{(c-U_0)^2}\left(1-\dfrac{H}{2h}\right), & H-h < y \le H \end{cases} \qquad (44)$$

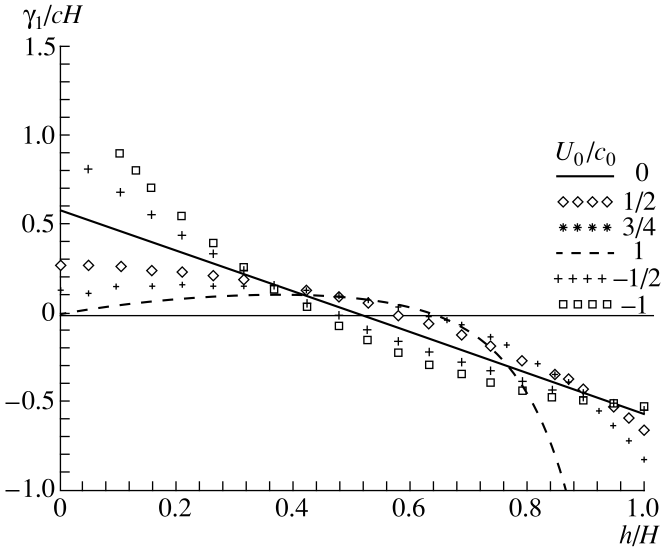

Fig. 5. Dependence of the coefficient of nonlinear dispersion (γ_1/cH) on the depth of the pycnocline for different current velocities in a two-layer ocean.

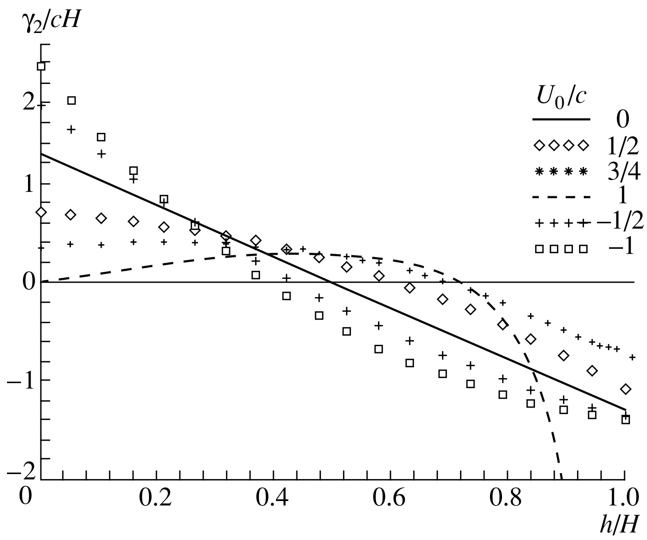

Fig. 6. Dependence of the coefficient of nonlinear dispersion (γ_2/cH) on the depth of the pycnocline for different current velocities in a two-layer ocean.

and dispersion

$$Z_{01}(y) = \begin{cases} -\frac{y}{24h}(4y^2 + 3H^2 + 8h^2 - 12Hh), & 0 \le y < h, \\ -\frac{1}{2}\left(y - \frac{H}{2}\right)^2, & h \le y \le H - h, \\ \frac{y-H}{24h}(4(y-H)^2 + 3H^2 + 8h^2 - 12Hh), & H - h < y \le H \end{cases} \quad (45)$$

modal corrections. The coefficients of the generalized Korteweg–de Vries equation are also calculated analytically

$$c = \sqrt{g\frac{\Delta\rho}{\rho}h}, \quad (46)$$

$$\alpha = 0, \quad \gamma_1 = \gamma_2 = 0, \quad (47)$$

$$\beta = -\frac{hc}{12}\left(4h - 3H + 3\frac{U_0}{c}(H - 2h)\left(2 - \frac{H_0}{c}\right)\right), \quad (48)$$

$$\alpha_1 = -\frac{3}{8}\frac{c^2}{(c - U_0)^2}\frac{c}{h^3}\left[26h - 9H + 8h\frac{U_0}{c}\left(\frac{U_0}{c} - 2\right)\right], \quad (49)$$

$$\beta_1 = \frac{hc}{1440}\left(30H^3 + 16h^3 - 45H^2h + 60\frac{U_0}{c} \times (8Hh^2 - H^3 - 8h^3) + 30\left(\frac{U_0}{c}\right)^2 \times (H^3 + 3H^2h - 22Hh^2 + 24h^3) - 45\left(\frac{U_0}{c}\right)^4 h(H - 2h)^2\right). \quad (50)$$

The dependence of the coefficient of cubic nonlinearity on the $l = h/H$ parameter for different values of U_0/c is shown in Fig. 8. It is interesting to note that the wave speed does not depend on the background flow and, due to the symmetry of the problem, the coefficients of the quadratic nonlinearity and nonlinear dispersion are equal to zero. As a result, the evolutionary equation (25) transforms to the modified Korteweg–de Vries equation with an additional linear dispersion term. The coefficient of cubic nonlinearity α_1 may be either positive or negative. This fact was noted for the first time in [7, 9] for a three-layer fluid with a zero current. The presence of a background current influences the position of the zero of function $\alpha_1(h)$: the opposite current moves it to the left, and a fair current moves it to the right. As it was expected, α_1 becomes negative for $h = H/2$ (the three-layer model with a current in the middle layer transforms to a two-layer model with the layers of equal thickness and zero current). We note that, when $U_0 \longrightarrow c$, the coefficient of the cubic nonlinearity term tends to infinity. Therefore, nonlinear effects are increased due to the fair current. The coefficients of linear dispersion β and β_1 are always positive, and the fair current increases their values, while the opposite current decreases them. Here, however, we may say about the competition of nonlinear and dispersion effects that, when $U_0 \longrightarrow c$, a fair current would generally tend to increase the role of nonlinear effects as compared to the dispersion.

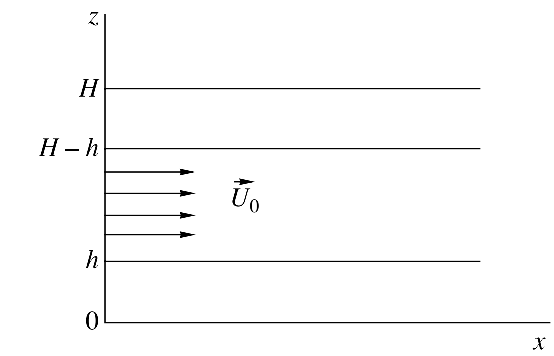

Fig. 7. Three-layer stratification with a moving middle layer.

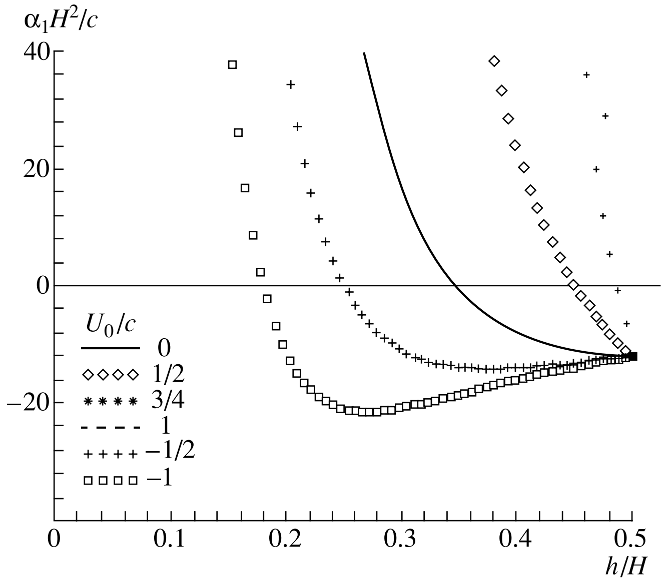

Fig. 8. Dependence of the coefficient of cubic nonlinearity (α_1H^2/c) on the depth of the pycnocline for different current velocities in a three-layer ocean.

HIERARCHY OF NONLINEAR MODELS AND ESTIMATES OF THEIR APPLICABILITY

Knowledge of the coefficients of the generalized Korteweg–de Vries equation allows us calculate the parameters of internal waves in different basins. Due to the fact that the linear dispersion described by the β coefficient is not zero for any stratification (this is seen from the quadratic property of the integrals in (27)), the contribution of the high-order dispersion (with coefficient β_1) should be small. The coefficient of quadratic nonlinearity α may change its sign or may even be equal to zero; thus, the role of cubic nonlinearity should be significant, and the term with α_1 should remain in the evolutionary equation (in those cases, when cubic nonlinearity is also absent, as was the case with one of the examples above, we have to take into account the terms with a higher order of nonlinearity). The contribution of nonlinear dispersion (terms with γ_1 and γ_2) seems to be low because it is the product of two small factors: quadratic nonlinearity and dispersion. Thus, the basic model for the calculations of internal waves should at least be based on the equation

$$\frac{\partial\eta}{\partial t} + (c + \alpha\eta + \alpha_1\eta^2)\frac{\partial\eta}{\partial x} + \beta\frac{\partial^3\eta}{\partial x^3} = 0, \quad (51)$$

which is named after Gardner. The popularity of this equation in the general theory of nonlinear waves is associated with its full integrability and the possibilities of a wide use of analytical methods (see, for example, [4]). At present, it is beginning to be applied as the base model for numerically simulating internal waves of large amplitude over the shelf [6, 12].

Let us discuss the solutions of (51) in the form of solitary stationary waves (solitons). Such a solution can very easily be found in an explicit form for $\alpha_1 < 0$:

$$\eta(x, t) = \frac{A}{1 + B\cosh(\Gamma(x - Vt))},$$
$$B^2 = 1 + \frac{\alpha_1}{\alpha}A, \quad \Gamma = \sqrt{\frac{\alpha A}{6\beta}}, \quad a = \frac{A}{1 + B}, \quad (52)$$
$$V = c + \frac{\alpha A}{6},$$

where $1/\Gamma$ is the effective width of the soliton and a is its amplitude. Any of these parameters can be considered free. The soliton's polarity is determined by the sign of the coefficient of quadratic nonlinearity α, in particular, in the two-layer model, the soliton is a depression wave if the pycnocline is located close to the sea surface and an elevation wave if the pycnocline is close to the sea bottom. It is important to emphasize that the soliton's amplitude cannot exceed (by its absolute value) the limit value

$$\alpha_{\lim} = -\frac{\alpha}{\alpha_1}, \quad (53)$$

and its form tends to rectangular (Fig. 9). The dependence of the limit amplitude on the location of the pycnocline in the two-layer model is shown in Fig. 10. A fair current increases the limit amplitude if the pycnocline is displaced closer to the bottom and decreases it if the pycnocline is at the sea surface. The limit amplitude, according to Fig. 10, can reach half of the full depth of the basin. In this case, a problem arises concerning the applicability of the described theory based on small parameters. One can easily see from (51) that the nonlinear correction to the wave speed is characterized by parameters $\alpha\eta/c$ and $\alpha_1\eta^2/c$. In the case of the limiting soliton, they become equal to each other and their value $\alpha^2/\alpha_1 c$ should be small within the asymptotic theory. If we take this value as equal to 0.3, the weak-nonlinear theory will be valid for the pycnocline location within (0.35–0.65)H (and zero current), while the maximum amplitude of the soliton will not exceed 0.15H. If the depth of the coastal zone is within 100–200 m, such limiting solitons with an amplitude of 15–30 m may be described by the theory developed here.

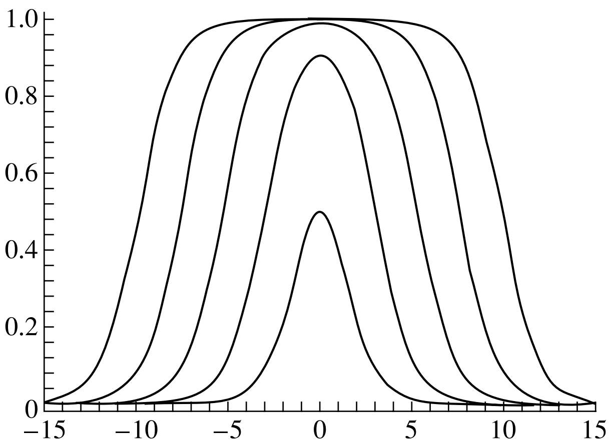

Fig. 9. Form of the soliton for negative cubic nonlinearity.

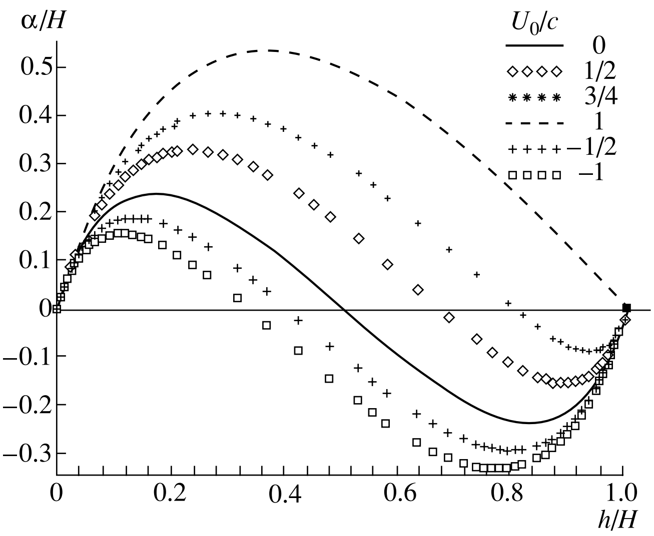

Fig. 10. Dependence of the limiting amplitude of the soliton (α/H) on the depth of the pycnocline for different current velocities in a two-layer ocean.

To demonstrate these estimates we calculated the contribution of different terms in the generalized Korteweg–de Vries equation to the structure of a solitary wave. In the first runs of the model, the thickness of the upper layer was 220 m, and the thickness of the lower layer was 280 m (the total depth of the basin was 500 m). In this case, the polarity of the soliton is negative and its maximal amplitude is (by the absolute value) 29.8 m. It follows from the estimates given above that the Gardner equation (51) should be a good approximation of the generalized Korteweg–de Vries equation. The calculations naturally confirmed this. For example, when the amplitude of the soliton is close to the critical value, both quadratic and cubic non-linearities are of the order of 10^{-4}, the Korteweg–de Vries dispersion is 10^{-5}, and all other terms do not exceed 8×10^{-7}. In the second example, the thickness of the upper layer is 100 m, and the thickness of the lower layer is 400 m. The limiting amplitude of the soliton is 117 m and exceeds the thickness of the upper layer (the soliton has negative polarity). The calculations show that, if the soliton's amplitude is moderate (up to 10–20 m), all corrections to the classic Korteweg–de Vries equations are small, and it is not necessary to include them into the numerical model for internal waves. If, however, the soliton's amplitude is 50 m, all terms in the generalized Korteweg–de Vries equation are of the same order (10^{-3}) and, therefore, the asymptotic series does not converge. This was also shown by Lamb [16] in his calculations using a full nonlinear model for the waves of almost limiting amplitude. Therefore, in numerical simulations of the internal wave field using the models of the Korteweg–de Vries type, it is necessary to analyze their applicability in advance. Our experience in the simulation of actual internal tides over the shelves shows that all situations are possible including those within the framework of the simplified model. In particular, the Korteweg–de Vries situation is realized for internal waves up to their critical height over the shelves of Australia [12] and Ireland [22].

If the cubic nonlinearity (α_1) is positive, then the soliton-wise solution of equation (51) is still described by (52), and formally has no limitation on the wave amplitude. Physically, this limitation should be related to the small nonlinear and dispersion corrections to the wave speed, which lead to the convergence of the asymptotic series. It is interesting to note that solution (52) with negative values of the B parameter is also possible. It corresponds to a soliton of opposite polarity. Such a soliton is, however, only possible if its amplitude (by the absolute value) is greater than $2|\alpha/\alpha_1|$. Therefore, when the amplitudes are small, the soliton has only one polarity (determined by its sign α), while, when the amplitudes are large, its polarity can be either positive or negative. The structure of the soliton, when the cubic nonlinearity is positive, is illustrated in Fig. 11. The influence of the current leads to a variation in the soliton length. In particular, in the three-layer model considered above, the soliton length is reduced when the current direction coincides with the wave and grows when the current is opposite to the wave.

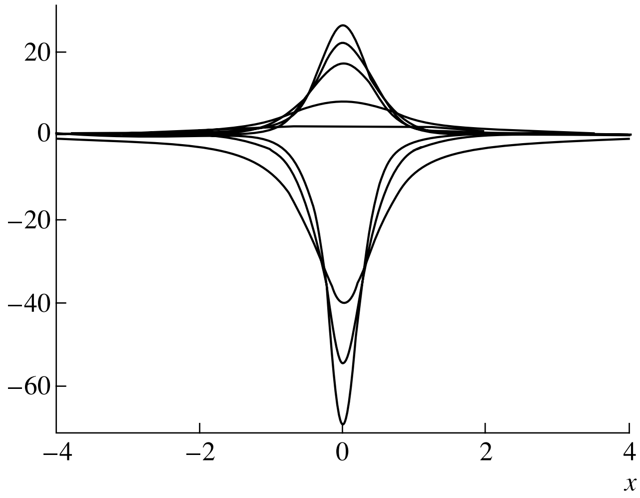

Fig. 11. Form of the soliton for positive cubic nonlinearity.

CONCLUSION

The generalized Korteweg–de Vries equation for nonlinear internal waves in an ocean stratified in density and velocity was obtained using an asymptotic procedure. The application of the Maple package of symbolical calculations allowed us to automate the process of obtaining higher approximations and increased the reliability of the cumbersome calculations. All coefficients of the generalized Korteweg–de Vries equation were calculated analytically for a two-layer ocean with a moving upper layer and for a three-layer ocean with a background current in the middle layer. The properties of solitary waves (solitons) in the stratified ocean were studied, and the influence of the current on the soliton parameters was investigated. The estimates of the corrections for higher orders of nonlinearity and dispersion on the structure of solitary waves were given. It was shown that taking cubic nonlinearity into account in the Korteweg–de Vries equation is, in many cases, enough to calculate the evolution of an internal wave over the shelf.

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