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Superposition Operators in the Algebra of Functions of Two Variables with Finite Total Variation

By

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Abstract. We show that the Hardy space of functions of two variables with finite total variation is a Banach algebra under the pointwise operations and a suitably chosen norm. Then we characterize Nemytskii superposition operators, which map the Hardy space into itself and satisfy the global Lipschitz condition.

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1. Introduction

The purpose of this paper is to characterize Lipschitzian superposition (Nemytskii) operators in the Hardy space of functions of two variables with finite total variation. More details on the definitions and results of this paper are given in Section 2. Here we review some facts known for functions of one variable.

Let $I = [a, b] \subset \mathbb{R}$ be a closed interval, $\mathbb{R}^I$ be the algebra of all functions $f : I \to \mathbb{R}$ under the usual pointwise operations and $h : I \times \mathbb{R} \to \mathbb{R}$ be a given function. The *superposition* (*Nemytskii*) *operator* $H = H_h : \mathbb{R}^I \to \mathbb{R}^I$ is defined by

$$(Hf)(x) \equiv H(f)(x) = h(x, f(x)), \qquad f \in \mathbb{R}^I, \quad x \in I. \tag{1}$$

The function h is called the *generator* of H. Let $B(I) \subset \mathbb{R}^I$ be a Banach function space with the norm $\|\cdot\|$. We are interested in finding conditions on the generator h in order for the operator $H : B(I) \to B(I)$ to be *Lipschitzian*: there exists a constant $\mu > 0$ such that

$$\|H(f_1) - H(f_2)\| \leqslant \mu \|f_1 - f_2\| \quad \text{for all} \quad f_1, f_2 \in B(I). \tag{2}$$

The operator H with the property (2) (and $\mu < 1$) is closely connected with the solution of the functional equation $f(x) = h(x, f(x))$, $x \in I$, written also as $f = Hf$, with respect to $f \in B(I)$ via the classical Banach fixed point theorem. For instance, if $x \in I$ and $h(x, u) = \sin u$, $u \in \mathbb{R}$, the corresponding operator H is Lipschitzian in the space of continuous functions $C(I)$ (with the supremum norm) and in the space $L^p(I)$ of Lebesgue p-summable functions on I (with the standard norm), $p \geqslant 1$. In contrast with this, Matkowski [12] proved that if $B(I) = \mathrm{Lip}(I)$ is the space of Lipschitz functions on I with respect to the usual Lipschitzian norm, then condition (2) implies that the generator h of H is of the form $h(x, u) = h_0(x) + h_1(x)u$,

$x \in I$, $u \in \mathbb{R}$, for some functions $h_0, h_1 \in \mathrm{Lip}(I)$. For the space $B(I) = BV(I)$ of functions of bounded (Jordan) variation on I with the standard norm Matkowski and Miś [16] (cf. also Appell and Zabrejko [3, Theorem 6.14]) showed that the generator h of a Lipschitzian superposition operator H satisfies the condition:

$$h^*(x,u) = h_0(x) + h_1(x)u, \qquad x \in (a,b], \quad u \in \mathbb{R}, \tag{3}$$

where $h^*(x,u) = \lim_{y \to x-0} h(y,u)$ is the left regularization of h in the first variable and functions h_0, $h_1 \in BV(I)$ are continuous from the left.

The last two results assert that the sets of Lipschitzian superposition operators on spaces $\mathrm{Lip}(I)$ and $BV(I)$ are much poorer than those on spaces $C(I)$ and $L^p(I)$ in that the corresponding generators are necessarily linear in the second variable; in particular, the above functional equation cannot be solved by directly applying Banach's contraction principle if h depends on the second variable $u \in \mathbb{R}$ nonlinearly (in this case one should invoke Schauder's fixed point theorem or a similar more powerful tool). The above two results have been extended later to different spaces of functions of one variable: [3]–[7], [14], [15], [17] (see also [13] for the case of Lipschitz maps).

In this paper we first show that the Hardy space of functions of two variables with finite total variation is a Banach algebra under the usual pointwise operations and a suitably chosen norm (Theorem 1). Then we present a complete description of Lipschitzian superposition operators H on the Hardy space – the representation for the corresponding generators will be of the form similar to (3) (Theorem 2). Also, we show that the representation of type (3) is exact in that in general one cannot omit the asterisk in h^* (Theorem 3). Finally, in Section 4 we present some generalizations of the above results when functions of two variables under consideration have their values in normed linear or Banach spaces (Theorem 4). Methods of proof used in this paper are consistent with those from [5]–[7] applied for functions (and maps) of one variable.

We have chosen the basic case of functions of two variables since the principal difference with the case of functions of one variable is more clearly seen. Corresponding results for functions of $N > 2$ real variables with finite total variation from the Hardy space will be published elsewhere.

2. Banach Algebra $BV(I_a^b;\ \mathbb{R})$. Main Results

Let $I_a^b = [a_1, b_1] \times [a_2, b_2]$ be the basic rectangle (the domain of functions) with $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{R}^2$ such that $a_1 < b_1$ and $a_2 < b_2$. For the sake of brevity we will write $x = (x_1, x_2)$, $y = (y_1, y_2)$ for $x, y \in \mathbb{R}^2$, and $x \leqslant y$ if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$. Let $\xi = \{t_i\}_{i=0}^m$ and $\eta = \{s_j\}_{j=0}^n$ be partitions of $[a_1, b_1]$ and $[a_2, b_2]$, respectively (i.e., $m, n \in \mathbb{N}$, $a_1 = t_0 < t_1 < \cdots < t_{m-1} < t_m = b_1$ and $a_2 = s_0 < s_1 < \cdots < s_{n-1} < s_n = b_2$). The (two-dimensional) *Hardy-Vitali variation* ([8], [19]) of a function $f\colon I_a^b \to \mathbb{R}$ is defined by

$$V_2(f, I_a^b) = \sup_{(\xi,\eta)} \sum_{i=1}^m \sum_{j=1}^n |f(t_{i-1}, s_{j-1}) + f(t_i, s_j) - f(t_{i-1}, s_j) - f(t_i, s_{j-1})|,$$

where the supremum is over all pairs $(m,n) \in \mathbb{N}^2$ and (ξ, η) with ξ a partition of $[a_1, b_1]$ and η a partition of $[a_2, b_2]$ of the above form. If $x_2 \in [a_2, b_2]$ is fixed and $[x_1, y_1]$ is a subinterval of $[a_1, b_1]$, the *Jordan variation* of the function $f(\cdot, x_2)$ of one variable defined by $f(\cdot, x_2)(t) = f(t, x_2)$, $t \in [a_1, b_1]$, on the interval $[x_1, y_1]$ is the quantity:

$$V_{x_1}^{y_1}(f(\cdot, x_2)) = \sup_{\xi} \sum_{i=1}^{m} |f(t_i, x_2) - f(t_{i-1}, x_2)|,$$

where the supremum is over all partitions $\xi = \{t_i\}_{i=0}^m$ $(m \in \mathbb{N})$ of $[x_1, y_1]$. A similar definition applies to the Jordan variation $V_{x_2}^{y_2}(f(x_1, \cdot))$ if $x_1 \in [a_1, b_1]$ is fixed and $[x_2, y_2]$ is a subinterval of $[a_2, b_2]$.

We define the *total variation* of $f : I_a^b \to \mathbb{R}$ by

$$TV(f, I_a^b) = V_{a_1}^{b_1}(f(\cdot, a_2)) + V_{a_2}^{b_2}(f(a_1, \cdot)) + V_2(f, I_a^b), \tag{4}$$

and the Hardy space of functions with finite total variation – by

$$BV(I_a^b; \mathbb{R}) = \{f : I_a^b \to \mathbb{R} \mid TV(f, I_a^b) < \infty\}.$$

The value $V_2(f, I_a^b)$ was usually defined (cf. Adams and Clarkson [1], [2] and Hardy [8]) with the supremum over all partitions of I_a^b (by a finite number of arbitrary nonoverlapping subrectangles of I_a^b with the edges parallel to the coordinate axes, and not necessarily of the form $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$ as above) and the Hardy space of functions was defined so that the three variations in (4) are finite. It was shown by Hildebrandt [9, III.4.2] and Leonov [11, Corollary 4] that all these definitions of the Hardy space coincide. Recall that the space $BV(I_a^b; \mathbb{R})$ is essential for the representation of continuous linear functionals on the space of continuous functions on I_a^b via the Riemann- or Lebesgue-Stieltjes integrals (cf. Shilov and Gurevich [18, Ch. II, Sect. 5]). The total variation (4) was effectively used ([9, III.6.5], [10, Theorem 3.2], [11, Theorem 4]) to obtain a Helly selection principle for functions from the Hardy space $BV(I_a^b; \mathbb{R})$.

It is known ([9, III.6.3], [11, Corollary 2], see also Section 4.1 below) that $BV(I_a^b; \mathbb{R})$ is a Banach space with respect to the norm:

$$\|f\| = |f(a)| + TV(f, I_a^b), \qquad f \in BV(I_a^b; \mathbb{R}). \tag{5}$$

The main ingredient in the proof of this assertion is the following inequality (due to Leonov [11, Corollaries 1 and 5] for functions of N variables) which will also be needed below: if $f \in BV(I_a^b; \mathbb{R})$ and $x \leqslant y$ in I_a^b, then

$$|f(y) - f(x)| \leqslant TV(f, I_x^y); \tag{6}$$

in fact, this is straightforward for functions of two variables:

$$\begin{aligned} |f(y_1, y_2) - f(x_1, x_2)| &\leqslant |f(y_1, x_2) - f(x_1, x_2)| + |f(x_1, y_2) - f(x_1, x_2)| \\ &\quad + |f(x_1, x_2) + f(y_1, y_2) - f(x_1, y_2) - f(y_1, x_2)| \\ &\leqslant V_{x_1}^{y_1}(f(\cdot, x_2)) + V_{x_2}^{y_2}(f(x_1, \cdot)) + V_2(f, I_x^y) = TV(f, I_x^y). \end{aligned}$$

The first main result of this paper is the following theorem which will be proved later in this section:

Theorem 1. *The space $BV(I_a^b; \mathbb{R})$ is a Banach algebra with respect to the usual pointwise operations and norm* (5), *and the following inequality holds*:

$$\|f \cdot g\| \leqslant 4\, \|f\| \cdot \|g\|, \qquad f, g \in BV(I_a^b; \mathbb{R}). \tag{7}$$

Given $f\in BV(I_a^b;\mathbb{R})$, we define its *left–left regularization* $f^*:\ I_a^b\to\mathbb{R}$ by

$$f^*(x_1,x_2)=\begin{cases}\lim\limits_{(y_1,y_2)\to(x_1-0,x_2-0)} f(y_1,y_2) & \text{if}\quad a_1<x_1\leqslant b_1 \text{ and } a_2<x_2\leqslant b_2,\\ \lim\limits_{(y_1,y_2)\to(x_1-0,a_2+0)} f(y_1,y_2) & \text{if}\quad a_1<x_1\leqslant b_1 \text{ and } x_2=a_2,\\ \lim\limits_{(y_1,y_2)\to(a_1+0,x_2-0)} f(y_1,y_2) & \text{if}\quad x_1=a_1 \text{ and } a_2<x_2\leqslant b_2,\\ \lim\limits_{(y_1,y_2)\to(a_1+0,a_2+0)} f(y_1,y_2) & \text{if}\quad x_1=a_1 \text{ and } x_2=a_2.\end{cases}$$

It is to be noted that $(y_1,y_2)\to(x_1-0,x_2-0)$ means that $(y_1,y_2)\in I_a^b, y_1<x_1$, $y_2<x_2$ and $(y_1,y_2)\to(x_1,x_2)$ in $\mathbb{R}^2$, and similarly for the other three limits. The existence of all these limits was proved, e.g., in [9, III.5.3].

A function $f:I_a^b\to\mathbb{R}$ is said to be *left-left continuous* if

$$\lim_{(y_1,y_2)\to(x_1-0,x_2-0)} f(y_1,y_2)=f(x_1,x_2)\quad \text{for all } x_1\in(a_1,b_1] \text{ and } x_2\in(a_2,b_2].$$

We denote by $BV^*(I_a^b;\mathbb{R})$ the subspace of $BV(I_a^b;\mathbb{R})$ of those functions which are left–left continuous on $(a_1,b_1]\times(a_2,b_2]$.

The second main result reads as follows:

Theorem 2. *Let* $H:\mathbb{R}^{I_a^b}\to\mathbb{R}^{I_a^b}$ *be a superposition operator with the generator* $h:I_a^b\times\mathbb{R}\to\mathbb{R}$ (*cf.* (1) *with* $I=I_a^b$).

If H *maps* $BV(I_a^b;\mathbb{R})$ *into itself and is Lipschitzian in the sense of* (2), *then*

$$|h(x,u_1)-h(x,u_2)|\leqslant 2\mu\,|u_1-u_2|,\qquad x\in I_a^b,\quad u_1,u_2\in\mathbb{R},\tag{8}$$

and there exist two functions $h_0,\ h_1\in BV^*(I_a^b;\mathbb{R})$ *such that*

$$h^*(x,u)=h_0(x)+h_1(x)u,\qquad x\in I_a^b,\quad u\in\mathbb{R},\tag{9}$$

where $h^*(x,u)$ *is the left–left regularization of the function* $x\mapsto h(x,u)$ *for each fixed* $u\in\mathbb{R}$.

Conversely, if $h_0,\ h_1\in BV(I_a^b;\mathbb{R})$ *and* $h(x,u)=h_0(x)+h_1(x)u$, $x\in I_a^b$, $u\in\mathbb{R}$, *then* H *maps* $BV(I_a^b;\mathbb{R})$ *into itself and is Lipschitzian.*

Theorem 2 will be proved in Section 3. Now we present a corollary:

Corollary 1. *Suppose that the function* $h:I_a^b\times\mathbb{R}\to\mathbb{R}$ *is such that* $h^*=h$ *on* $I_a^b\times\mathbb{R}$, *and the superposition operator* H *is generated by* h. *Then the following two conditions are equivalent*:

(a) *H maps the space* $BV(I_a^b;\mathbb{R})$ *into itself and is Lipschitzian*;

(b) *there exist functions* $h_0,\ h_1\in BV^*(I_a^b;\mathbb{R})$ *such that* $h(x,u)=h_0(x)+h_1(x)u$ *for all* $x\in I_a^b$ *and* $u\in\mathbb{R}$.

Another corollary of Theorem 2 is given after the proof of Theorem 2.

In general the function h^* in the representation (9) cannot be replaced by h. This is shown in the next theorem (its proof is given in Section 3) which is a modification of the one-dimensional ideas in [16, Example on p. 157].

Theorem 3. *Let* $\{p_k\}_{k=1}^{\infty}$ *and* $\{q_\ell\}_{\ell=1}^{\infty}$ *be sequences of distinct rational numbers of the intervals* $[a_1, b_1]$ *and* $[a_2, b_2]$, *respectively. Let* $h : I_a^b \times \mathbb{R} \to \mathbb{R}$ *be defined for* $(x_1, x_2) \in I_a^b$ *and* $u \in \mathbb{R}$ *by*:

$$h(x_1, x_2, u) = \begin{cases} 2^{-k-\ell} \sin u, & \text{if } x_1 = p_k \text{ and } x_2 = q_\ell,\ k,\ \ell \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the superposition operator H generated by h maps $BV(I_a^b; \mathbb{R})$ *into itself and is Lipschitzian.* (*Note that the left–left regularization of h is given by* $h^* \equiv 0$ *which is of the form* (9).)

In Section 4 we generalize Theorems 1 and 2 to the case when functions of two variables under consideration have values in normed linear spaces (Section 4.1 and Theorem 4).

Now we turn to the proof of Theorem 1.

Proof of Theorem 1. By (4) and (5), we have:

$$\|fg\| = |(fg)(a)| + V_{a_1}^{b_1}((fg)(\cdot, a_2)) + V_{a_2}^{b_2}((fg)(a_1, \cdot)) + V_2(fg, I_a^b). \tag{10}$$

For the second term we apply the following well known estimate:

$$V_{a_1}^{b_1}((fg)(\cdot, a_2)) \leqslant \Big(\sup_{[a_1, b_1]} |f(\cdot, a_2)|\Big) V_{a_1}^{b_1}(g(\cdot, a_2)) + V_{a_1}^{b_1}(f(\cdot, a_2)) \Big(\sup_{[a_1, b_1]} |g(\cdot, a_2)|\Big).$$

Since

$$\sup_{[a_1, b_1]} |f(\cdot, a_2)| \leqslant |f(a)| + V_{a_1}^{b_1}(f(\cdot, a_2)), \qquad \sup_{[a_1, b_1]} |g(\cdot, a_2)| \leqslant |g(a)| + V_{a_1}^{b_1}(g(\cdot, a_2)),$$

we get:

$$\begin{aligned} V_{a_1}^{b_1}((fg)(\cdot, a_2)) \leqslant{} & |f(a)|\, V_{a_1}^{b_1}(g(\cdot, a_2)) + V_{a_1}^{b_1}(f(\cdot, a_2))|g(a)| \\ & + 2\, V_{a_1}^{b_1}(f(\cdot, a_2)) V_{a_1}^{b_1}(g(\cdot, a_2)), \end{aligned} \tag{11}$$

and similarly for the third term:

$$\begin{aligned} V_{a_2}^{b_2}((fg)(a_1, \cdot)) \leqslant{} & |f(a)| V_{a_2}^{b_2}(g(a_1, \cdot)) + V_{a_2}^{b_2}(f(a_1, \cdot))|g(a)| \\ & + 2\, V_{a_2}^{b_2}(f(a_1, \cdot)) V_{a_2}^{b_2}(g(a_1, \cdot)). \end{aligned} \tag{12}$$

In order to estimate the fourth term $V_2(fg, I_a^b)$, let $\{t_i\}_{i=0}^m$ and $\{s_j\}_{j=0}^n$ be partitions of $[a_1, b_1]$ and $[a_2, b_2]$, respectively. We note that for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ the following equality holds:

$$\begin{aligned} &(fg)(t_{i-1}, s_{j-1}) + (fg)(t_i, s_j) - (fg)(t_{i-1}, s_j) - (fg)(t_i, s_{j-1}) \\ &\quad = [f(t_{i-1}, s_{j-1}) + f(t_i, s_j) - f(t_{i-1}, s_j) - f(t_i, s_{j-1})]\, g(t_{i-1}, s_{j-1}) \\ &\qquad + f(t_i, s_j)\, [g(t_{i-1}, s_{j-1}) + g(t_i, s_j) - g(t_{i-1}, s_j) - g(t_i, s_{j-1})] \\ &\qquad + [f(a_1, s_j) - f(a_1, s_{j-1})]\, [g(t_i, a_2) - g(t_{i-1}, a_2)] \\ &\qquad + [f(a_1, s_j) - f(a_1, s_{j-1})]\, [g(t_{i-1}, a_2) + g(t_i, s_{j-1}) - g(t_{i-1}, s_{j-1}) - g(t_i, a_2)] \\ &\qquad + [f(a_1, s_{j-1}) + f(t_i, s_j) - f(a_1, s_j) - f(t_i, s_{j-1})]\, [g(t_i, a_2) - g(t_{i-1}, a_2)] \end{aligned}$$

$$
\begin{aligned}
&+ [f(a_1,s_{j-1}) + f(t_i,s_j) - f(a_1,s_j) - f(t_i,s_{j-1})] \\
&\qquad \times [g(t_{i-1},a_2) + g(t_i,s_{j-1}) - g(t_{i-1},s_{j-1}) - g(t_i,a_2)] \\
&+ [f(t_i,a_2) - f(t_{i-1},a_2)]\,[g(a_1,s_j) - g(a_1,s_{j-1})] \\
&+ [f(t_i,a_2) - f(t_{i-1},a_2)]\,[g(a_1,s_{j-1}) + g(t_{i-1},s_j) - g(a_1,s_j) - g(t_{i-1},s_{j-1})] \\
&+ [f(t_{i-1},a_2) + f(t_i,s_j) - f(t_{i-1},s_j) - f(t_i,a_2)]\,[g(a_1,s_j) - g(a_1,s_{j-1})] \\
&+ [f(t_{i-1},a_2) + f(t_i,s_j) - f(t_{i-1},s_j) - f(t_i,a_2)] \\
&\qquad \times [g(a_1,s_{j-1}) + g(t_{i-1},s_j) - g(a_1,s_j) - g(t_{i-1},s_{j-1})] \\
= {} & S_1^{ij} + S_2^{ij} + S_3^{ij} + S_4^{ij} + S_5^{ij} + S_6^{ij} + S_7^{ij} + S_8^{ij} + S_9^{ij} + S_{10}^{ij}.
\end{aligned}
$$

Let us estimate the sums $\sum_{i=1}^m \sum_{j=1}^n |S_k^{ij}|$, $k = 1, \dots, 10$, separately. By virtue of (5) and (6), we have: $|f(t,s)| \leqslant \|f\|$, $|g(t,s)| \leqslant \|g\|$, $(t,s) \in I_a^b$, and so,

$$
\sum_{i=1}^m \sum_{j=1}^n |S_1^{ij}| \leqslant V_2(f, I_a^b)\|g\|, \qquad \sum_{i=1}^m \sum_{j=1}^n |S_2^{ij}| \leqslant \|f\| V_2(g, I_a^b).
$$

Clearly,

$$
\sum_{i=1}^m \sum_{j=1}^n |S_3^{ij}| \leqslant V_{a_2}^{b_2}(f(a_1,\cdot)) V_{a_1}^{b_1}(g(\cdot,a_2)),
$$

$$
\sum_{i=1}^m \sum_{j=1}^n |S_7^{ij}| \leqslant V_{a_1}^{b_1}(f(\cdot,a_2)) V_{a_2}^{b_2}(g(a_1,\cdot)).
$$

Since the variation $V_2(f,\cdot)$ is *additive* [9, III.4.2] (i.e. *for all* partitions $\{t_i\}_{i=0}^m$ and $\{s_j\}_{j=0}^n$ of $[a_1,b_1]$ and $[a_2,b_2]$ as above and $I_{i,j} = [t_{i-1},t_i] \times [s_{j-1},s_j]$ the following equality holds: $V_2(f, I_a^b) = \sum_{i=1}^m \sum_{j=1}^n V_2(f, I_{i,j})$), we have:

$$
\begin{aligned}
&\sum_{i=1}^m \sum_{j=1}^n |S_4^{ij}| \leqslant V_{a_2}^{b_2}(f(a_1,\cdot)) V_2(g, I_a^b), &&\sum_{i=1}^m \sum_{j=1}^n |S_5^{ij}| \leqslant V_2(f, I_a^b) V_{a_1}^{b_1}(g(\cdot,a_2)), \\
&\sum_{i=1}^m \sum_{j=1}^n |S_8^{ij}| \leqslant V_{a_1}^{b_1}(f(\cdot,a_2)) V_2(g, I_a^b), &&\sum_{i=1}^m \sum_{j=1}^n |S_9^{ij}| \leqslant V_2(f, I_a^b) V_{a_2}^{b_2}(g(a_1,\cdot)), \\
&\sum_{i=1}^m \sum_{j=1}^n |S_6^{ij}| \leqslant V_2(f, I_a^b) V_2(g, I_a^b), &&\sum_{i=1}^m \sum_{j=1}^n |S_{10}^{ij}| \leqslant V_2(f, I_a^b) V_2(g, I_a^b).
\end{aligned}
$$

Thus, $V_2(fg, I_a^b)$ is estimated from above as follows:

$$
\begin{aligned}
V_2(fg, I_a^b) \leqslant {} & |f(a)| V_2(g, I_a^b) + 2V_{a_1}^{b_1}(f(\cdot,a_2)) V_2(g, I_a^b) + 2V_{a_2}^{b_2}(f(a_1,\cdot)) V_2(g, I_a^b) \\
& + V_2(f, I_a^b)|g(a)| + 2V_2(f, I_a^b) V_{a_1}^{b_1}(g(\cdot,a_2)) + 2V_2(f, I_a^b) V_{a_2}^{b_2}(g(a_1,\cdot)) \\
& + V_{a_1}^{b_1}(f(\cdot,a_2)) V_{a_2}^{b_2}(g(a_1,\cdot)) + V_{a_2}^{b_2}(f(a_1,\cdot)) V_{a_1}^{b_1}(g(\cdot,a_2)) \\
& + 4V_2(f, I_a^b) V_2(g, I_a^b).
\end{aligned}
$$

Taking into account (10), (11), (12) and the last estimate we arrive at the desired inequality (7). □

Remark 1. As it is known from the theory of Banach algebras, the norm (5) can always be replaced by an equivalent norm $|||\cdot|||$ on $BV(I_a^b;\mathbb{R})$ such that

$$|||f\cdot g|||\leqslant|||f|||\cdot|||g|||,\qquad f,g\in BV(I_a^b;\mathbb{R}).\tag{13}$$

In fact, given $f\in BV(I_a^b;\mathbb{R})$, consider the linear continuous operator M_f from $BV(I_a^b;\mathbb{R})$ into itself defined by $M_f(g)=f\cdot g$ whenever $g\in BV(I_a^b;\mathbb{R})$. Then the operator norm $|||f|||=\|M_f\|\overset{\text{def}}{=}\sup\{\|f\cdot g\|;\|g\|=1\}$ is the desired norm on $BV(I_a^b;\mathbb{R})$ satisfying (13) and $\|f\|\leqslant|||f|||\leqslant 4\|f\|$ for all $f\in BV(I_a^b;\mathbb{R})$.

3. Lipschitzian Superposition Operators

In order to prove Theorem 2 we need a lemma.

Lemma 1. *If $f\in BV(I_a^b;\mathbb{R})$, then $f^*\in BV^*(I_a^b;\mathbb{R})$.*

Proof. From the first line of the definition of f^* it is clear that f^* is left–left continuous on $(a_1,b_1]\times(a_2,b_2]$, so we prove that $f^*\in BV(I_a^b;\mathbb{R})$. Let $a_1=t_0<t_1<\cdots<t_{m-1}<t_m=b_1$, $a_2=s_0<s_1<\cdots<s_{n-1}<s_n=b_2$, and let us fix $\varepsilon>0$. By the definition of f^*, there exist $t_i'\in(t_{i-1},t_i)$, $i=1,\dots,m$, $s_j'\in(s_{j-1},s_j)$, $j=1,\dots,n$, $t_0'\in(a_1,t_1')$ and $s_0'\in(a_2,s_1')$ such that

$$\begin{aligned}&|f^*(t_{i-1},s_{j-1})+f^*(t_i,s_j)-f^*(t_{i-1},s_j)-f^*(t_i,s_{j-1})|\\&\quad\leqslant|f(t_{i-1}',s_{j-1}')+f(t_i',s_j')-f(t_{i-1}',s_j')-f(t_i',s_{j-1}')|+(\varepsilon/mn)\end{aligned}$$

for all $i=1,\dots,m$ and $j=1,\dots,n$. Summing over these i and j, we get:

$$\sum_{i=1}^m\sum_{j=1}^n|f^*(t_{i-1},s_{j-1})+f^*(t_i,s_j)-f^*(t_{i-1},s_j)-f^*(t_i,s_{j-1})|\leqslant V_2(f,I_a^b)+\varepsilon,$$

which means that $V_2(f^*,I_a^b)$ is finite.

To prove that $V_{a_1}^{b_1}(f^*(\cdot,a_2))<\infty$, let $a_1=t_0<t_1<\cdots<t_{m-1}<t_m=b_1$. By the definition of f^*, there exist $t_i'\in(t_{i-1},t_i)$, $i=1,\dots,m$, $t_0'\in(a_1,t_1')$ and $s_0\in(a_2,b_2)$ such that

$$|f^*(t_i,a_2)-f^*(t_{i-1},a_2)|\leqslant|f(t_i',s_0)-f(t_{i-1}',s_0)|+(\varepsilon/m),$$

so that summing over $i=1,\dots,m$ we have:

$$\begin{aligned}\sum_{i=1}^m|f^*(t_i,a_2)-f^*(t_{i-1},a_2)|&\leqslant\sum_{i=1}^m|f(t_i',s_0)-f(t_{i-1}',s_0)|+\varepsilon\\&\leqslant V_{a_1}^{b_1}(f(\cdot,s_0))+\varepsilon\\&\leqslant V_{a_1}^{b_1}(f(\cdot,a_2))+V_2(f,I_a^b)+\varepsilon,\end{aligned}\tag{14}$$

which means that $V_{a_1}^{b_1}(f^*(\cdot,a_2))$ is finite. The last inequality in (14) can be proved as follows. Since, for any $a_1\leqslant\tau\leqslant t\leqslant b_1$,

$$f(t,s_0)-f(\tau,s_0)=[f(t,a_2)-f(\tau,a_2)]+[f(\tau,a_2)+f(t,s_0)-f(\tau,s_0)-f(t,a_2)],$$

setting $\tau = t_{i-1}$ and $t = t_i$, summing over $i = 1, \dots, m$ and applying the triangle inequality, we find that

$$\sum_{i=1}^{m} |f(t_i, s_0) - f(t_{i-1}, s_0)| \leqslant \sum_{i=1}^{m} |f(t_i, a_2) - f(t_{i-1}, a_2)|$$

$$+ \sum_{i=1}^{m} |f(t_{i-1}, a_2) + f(t_i, s_0) - f(t_{i-1}, s_0) - f(t_i, a_2)|$$

$$\leqslant V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2(f, I_{a_1,a_2}^{b_1,s_0}) \leqslant V_{a_1}^{b_1}(f(\cdot, a_2)) + V_2(f, I_a^b), \tag{15}$$

which proves the desired inequality.

Similar arguments apply to prove that $V_{a_2}^{b_2}(f^*(a_1, \cdot)) < \infty$. □

Proof of Theorem 2. Using definitions (5) and (4) we rewrite inequality (2) more explicitly as (functions f_1 and f_2 are arbitrary in $BV(I_a^b; \mathbb{R})$):

$$\begin{aligned} &|(Hf_1 - Hf_2)(a)| + V_{a_1}^{b_1}((Hf_1 - Hf_2)(\cdot, a_2)) + V_{a_2}^{b_2}((Hf_1 - Hf_2)(a_1, \cdot)) \\ &\quad + V_2(Hf_1 - Hf_2, I_a^b) \\ &\leqslant \mu[|(f_1 - f_2)(a)| + V_{a_1}^{b_1}((f_1 - f_2)(\cdot, a_2)) + V_{a_2}^{b_2}((f_1 - f_2)(a_1, \cdot)) \\ &\quad + V_2(f_1 - f_2, I_a^b)]. \end{aligned} \tag{16}$$

For any $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define auxiliary Lipschitz functions $\eta_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ by

$$\eta_{\alpha,\beta}(t) = \begin{cases} 0 & \text{if} \quad t \leqslant \alpha, \\ (t - \alpha)/(\beta - \alpha) & \text{if} \quad \alpha \leqslant t \leqslant \beta, \\ 1 & \text{if} \quad t \geqslant \beta. \end{cases}$$

1. First we prove inequality (8). We consider the following four cases: (*i*) $a_1 < x_1 \leqslant b_1$ and $a_2 < x_2 \leqslant b_2$; (*ii*) $a_1 < x_1 \leqslant b_1$ and $x_2 = a_2$; (*iii*) $x_1 = a_1$ and $a_2 < x_2 \leqslant b_2$; and (*iv*) $x_1 = a_1$ and $x_2 = a_2$.

Let u_1, $u_2 \in \mathbb{R}$ be arbitrary numbers and let $\mathscr{H} = Hf_1 - Hf_2$.

Case (*i*). We define two functions $f_1, f_2 \in BV(I_a^b; \mathbb{R})$ by

$$f_j(y_1, y_2) = [\eta_{a_1,x_1}(y_1) + \eta_{a_2,x_2}(y_2)]u_j/2, \qquad a_j \leqslant y_j \leqslant b_j, \quad j = 1, 2.$$

Since $f_j(a) = 0, j = 1, 2$, $V_{a_1}^{b_1}((f_1 - f_2)(\cdot, a_2)) = V_{a_2}^{b_2}((f_1 - f_2)(a_1, \cdot)) = |u_1 - u_2|/2$ and $V_2(f_1 - f_2, I_a^b) = 0$, the norm $\|f_1 - f_2\|$ on the right hand side of (16) is equal to $\|f_1 - f_2\| = |u_1 - u_2|$. Noting that $\mathscr{H}(a_1, a_2) = (Hf_1 - Hf_2)(a) = 0$ and taking into account (16) we have:

$$\begin{aligned} &|h(x_1, x_2, u_1) - h(x_1, x_2, u_2)| \\ &\quad = |(Hf_1 - Hf_2)(x_1, x_2)| = |\mathscr{H}(x_1, x_2)| \\ &\quad \leqslant |\mathscr{H}(x_1, a_2) - \mathscr{H}(a_1, a_2)| + |\mathscr{H}(a_1, x_2) - \mathscr{H}(a_1, a_2)| \\ &\qquad + |\mathscr{H}(a_1, a_2) + \mathscr{H}(x_1, x_2) - \mathscr{H}(a_1, x_2) - \mathscr{H}(x_1, a_2)| \\ &\quad \leqslant V_{a_1}^{b_1}(\mathscr{H}(\cdot, a_2)) + V_{a_2}^{b_2}(\mathscr{H}(a_1, \cdot)) + V_2(\mathscr{H}, I_a^b) \\ &\quad \leqslant \|Hf_1 - Hf_2\| \leqslant \mu |u_1 - u_2|, \end{aligned}$$

and inequality (8) follows.

Cases (*ii*) *and* (*iii*). In case (*ii*) we set $f_j(y_1,y_2) = \eta_{a_1,x_1}(y_1)u_j$ for $a_j \leqslant y_j \leqslant b_j$, $j = 1,2$, and note that $f_j(a) = 0$, $j = 1,2$, $V_{a_1}^{b_1}((f_1 - f_2)(\cdot,a_2)) = |u_1 - u_2|$, $V_{a_2}^{b_2}((f_1 - f_2)(a_1,\cdot)) = 0$ and $V_2(f_1 - f_2, I_a^b) = 0$, so that $\|f_1 - f_2\| = |u_1 - u_2|$. Since $f_j(x_1,a_2) = u_j$, $j = 1,2$, inequality (16) yields:

$$
\begin{aligned}
|h(x_1,a_2,u_1) - h(x_1,a_2,u_2)| &= |(Hf_1 - Hf_2)(x_1,a_2)| = |\mathscr{H}(x_1,a_2)| \\
&= |\mathscr{H}(x_1,a_2) - \mathscr{H}(a_1,a_2)| \leqslant V_{a_1}^{b_1}(\mathscr{H}(\cdot,a_2)) \leqslant \|Hf_1 - Hf_2\| \leqslant \mu|u_1 - u_2|.
\end{aligned}
$$

Similarly, in case (*iii*) we set $f_j(y_1,y_2) = \eta_{a_2,x_2}(y_2)u_j$ for $a_j \leqslant y_j \leqslant b_j$, $j = 1,2$.

Case (*iv*). Setting

$$
f_j(y_1,y_2) = [2 - \eta_{a_1,b_1}(y_1) - \eta_{a_2,b_2}(y_2)]u_j/2, \qquad a_j \leqslant y_j \leqslant b_j, \quad j = 1,2,
$$

we have: $f_j(a) = u_j$, $j = 1,2$, $V_{a_1}^{b_1}((f_1 - f_2)(\cdot,a_2)) = V_{a_2}^{b_2}((f_1 - f_2)(a_1,\cdot)) = |u_1 - u_2|/2$ and $V_2(f_1 - f_2, I_a^b) = 0$, and so, $\|f_1 - f_2\| = 2|u_1 - u_2|$. Since $\mathscr{H}(b_1,b_2) = (Hf_1 - Hf_2)(b) = 0$, it follows from (16) that

$$
\begin{aligned}
|h(a_1,a_2,u_1) - h(a_1,a_2,u_2)| &= |(Hf_1 - Hf_2)(a_1,a_2)| \\
&= |\mathscr{H}(a_1,a_2)| \leqslant |\mathscr{H}(b_1,a_2) - \mathscr{H}(a_1,a_2)| \\
&\quad + |\mathscr{H}(a_1,b_2) - \mathscr{H}(a_1,a_2)| + |\mathscr{H}(a_1,a_2) \\
&\quad + \mathscr{H}(b_1,b_2) - \mathscr{H}(a_1,b_2) - \mathscr{H}(b_1,a_2)| \\
&\leqslant V_{a_1}^{b_1}(\mathscr{H}(\cdot,a_2)) + V_{a_2}^{b_2}(\mathscr{H}(a_1,\cdot)) + V_2(\mathscr{H}, I_a^b) \\
&\leqslant 2\mu|u_1 - u_2|,
\end{aligned}
$$

which completes the proof of inequality (8).

2. Now we prove the validity of the representation (9). For this, we first fix $x_1 \in (a_1,b_1]$ and $x_2 \in (a_2,b_2]$ and set $x = (x_1,x_2)$. Also, let $m \in \mathbb{N}$, $a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < x_1$ and $a_2 < \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \cdots < \overline{\alpha}_m < \overline{\beta}_m < x_2$. Inequality (16) implies, in particular, that

$$
\sum_{i=1}^m |\mathscr{H}(\beta_i,a_2) - \mathscr{H}(\alpha_i,a_2)| + \sum_{i=1}^m |\mathscr{H}(a_1,\overline{\beta}_i) - \mathscr{H}(a_1,\overline{\alpha}_i)| + V_2(\mathscr{H}, I_a^b) \leqslant \mu\|f_1 - f_2\|. \tag{17}
$$

Let us define two auxiliary functions $\eta_m : [a_1,b_1] \to [0,1]$ and $\overline{\eta}_m : [a_2,b_2] \to [0,1]$ as follows:

$$
\eta_m(t) = \begin{cases} 0 & \text{if} \quad a_1 \leqslant t \leqslant \alpha_1, \\ \eta_{\alpha_i,\beta_i}(t) & \text{if} \quad \alpha_i \leqslant t \leqslant \beta_i, \ \ i = 1,\dots,m, \\ 1 - \eta_{\beta_i,\alpha_{i+1}}(t) & \text{if} \quad \beta_i \leqslant t \leqslant \alpha_{i+1}, \ \ i = 1,\dots,m-1, \\ 1 & \text{if} \quad \beta_m \leqslant t \leqslant b_1, \end{cases} \tag{18}
$$

$$
\overline{\eta}_m(s) = \begin{cases} 0 & \text{if} \quad a_2 \leqslant s \leqslant \overline{\alpha}_1, \\ \eta_{\overline{\alpha}_i,\overline{\beta}_i}(s) & \text{if} \quad \overline{\alpha}_i \leqslant s \leqslant \overline{\beta}_i, \ \ i = 1,\dots,m, \\ 1 - \eta_{\overline{\beta}_i,\overline{\alpha}_{i+1}}(s) & \text{if} \quad \overline{\beta}_i \leqslant s \leqslant \overline{\alpha}_{i+1}, \ \ i = 1,\dots,m-1, \\ 1 & \text{if} \quad \overline{\beta}_m \leqslant s \leqslant b_2. \end{cases}
$$

For arbitrary numbers $u_1,\ u_2\in\mathbb{R}$ we set

$$f_j(y_1,y_2)=\frac{1}{2}[\eta_m(y_1)+\overline{\eta}_m(y_2)]u_1+(2-j)u_2,\qquad a_j\leqslant y_j\leqslant b_j,\quad j=1,2,$$

and note that $f_1-f_2\equiv u_2$, and so, $\|f_1-f_2\|=|u_2|$. Since for $i=1,\dots,m$ we have (recall that $\mathscr{H}=Hf_1-Hf_2$):

$$\begin{aligned}|\mathscr{H}(\beta_i,\overline{\beta}_i)-\mathscr{H}(\alpha_i,\overline{\alpha}_i)|\leqslant&|\mathscr{H}(\beta_i,a_2)-\mathscr{H}(\alpha_i,a_2)|+|\mathscr{H}(a_1,\overline{\beta}_i)-\mathscr{H}(a_1,\overline{\alpha}_i)|\\&+|\mathscr{H}(\alpha_i,a_2)+\mathscr{H}(\beta_i,\overline{\beta}_i)-\mathscr{H}(\alpha_i,\overline{\beta}_i)-\mathscr{H}(\beta_i,a_2)|\\&+|\mathscr{H}(a_1,\overline{\alpha}_i)+\mathscr{H}(\alpha_i,\overline{\beta}_i)-\mathscr{H}(a_1,\overline{\beta}_i)-\mathscr{H}(\alpha_i,\overline{\alpha}_i)|,\end{aligned}$$

summing over $i=1,\dots,m$ and taking into account (17) we get:

$$\sum_{i=1}^{m}|(Hf_1-Hf_2)(\beta_i,\overline{\beta}_i)-(Hf_1-Hf_2)(\alpha_i,\overline{\alpha}_i)|\leqslant\mu|u_2|.$$

As $f_1(\beta_i,\overline{\beta}_i)=u_1+u_2$, $f_2(\beta_i,\overline{\beta}_i)=u_1$, $f_1(\alpha_i,\overline{\alpha}_i)=u_2$ and $f_2(\alpha_i,\overline{\alpha}_i)=0$, the last inequality can be rewritten in the form:

$$\sum_{i=1}^{m}|h(\beta_i,\overline{\beta}_i,u_1+u_2)-h(\beta_i,\overline{\beta}_i,u_1)-h(\alpha_i,\overline{\alpha}_i,u_2)+h(\alpha_i,\overline{\alpha}_i,0)|\leqslant\mu|u_2|.\tag{19}$$

Since constant functions of two variables on I_a^b belong to $BV(I_a^b;\mathbb{R})$ and H maps $BV(I_a^b;\mathbb{R})$ into itself, the function $h(\cdot,u)=[x\mapsto h(x,u)]$ is in $BV(I_a^b;\mathbb{R})$ for all $u\in\mathbb{R}$. Hence, by Lemma 1, its left–left regularization in the first two variables $h^*(\cdot,u)$ is in $BV^*(I_a^b;\mathbb{R})$ for all $u\in\mathbb{R}$. Passing to the limit as $(\alpha_1,\overline{\alpha}_1)\to(x_1-0,\ x_2-0)$ in the inequality (19), we find that

$$\sum_{i=1}^{m}|h^*(x_1,x_2,u_1+u_2)-h^*(x_1,x_2,u_1)-h^*(x_1,x_2,u_2)+h^*(x_1,x_2,0)|\leqslant\mu|u_2|,$$

whence

$$|h^*(x,u_1+u_2)-h^*(x,u_1)-h^*(x,u_2)+h^*(x,0)|\leqslant\mu|u_2|/m.$$

Due to the arbitrariness of $m\in\mathbb{N}$ this implies the equality:

$$h^*(x,u_1+u_2)-h^*(x,u_1)-h^*(x,u_2)+h^*(x,0)=0,\tag{20}$$

which holds for all $a_1<x_1\leqslant b_1$, $a_2<x_2\leqslant b_2$ and $u_1,\ u_2\in\mathbb{R}$.

Now let $a_1<x_1\leqslant b_1$ and $x_2=a_2$. If $m\in\mathbb{N}$, $a_1<\alpha_1<\beta_1<\cdots<\alpha_m<\beta_m<x_1$ and $a_2<\overline{\alpha}_1<\overline{\beta}_1<\cdots<\overline{\alpha}_m<\overline{\beta}_m<b_2$, the above arguments provide the estimate (19). Taking the limit as $(\alpha_1,\overline{\beta}_m)\to(x_1-0,a_2+0)$ in (19) we arrive at the equality (20). The cases when $x_1=a_1$ and $a_2<x_2\leqslant b_2$ or $x_1=a_1$ and $x_2=a_2$ are treated similarly.

Thus, equality (20) holds for all $x\in I_a^b$ and all $u_1,\ u_2\in\mathbb{R}$.

The rest of the proof of (9) is standard (cf. [12]). For each fixed $x\in I_a^b$ we define the function $T_x:\mathbb{R}\to\mathbb{R}$ by

$$T_x(u)=h^*(x,u)-h^*(x,0),\qquad u\in\mathbb{R},$$

so that equality (20) can be rewritten as

$$T_x(u_1+u_2)=T_x(u_1)+T_x(u_2), \qquad u_1,u_2\in\mathbb{R},$$

which shows that T_x is an additive function. By inequality (8) and the definition of $h^*(\cdot,u)$, we have:

$$|T_x(u_1)-T_x(u_2)|\leqslant 2\mu|u_1-u_2|, \qquad u_1,u_2\in\mathbb{R},$$

and so, T_x is (Lipschitz) continuous on $\mathbb{R}$. Therefore there exists a function $h_1: I_a^b\to\mathbb{R}$ such that $T_x(u)=h_1(x)u$ for all $x\in I_a^b$ and $u\in\mathbb{R}$. Setting $h_0(x)=h^*(x,0)$, $x\in I_a^b$, we get the representation (9). Since $h_0(\cdot)=h^*(\cdot,0)$ and $h_1(\cdot)=h^*(\cdot,1)-h^*(\cdot,0)$, we conclude (from Lemma 1) that $h_0, h_1\in BV^*(I_a^b;\mathbb{R})$, and this completes the proof of the first part of Theorem 2.

3. To prove the second part of Theorem 2, we note that the superposition operator H is given by:

$$(Hf)(x)=h_0(x)+h_1(x)f(x), \qquad x\in I_a^b, \quad f\in BV(I_a^b;\mathbb{R}),$$

and since $BV(I_a^b;\mathbb{R})$ is an algebra according to Theorem 1, it follows that H maps $BV(I_a^b;\mathbb{R})$ into itself. Applying inequality (7) we have:

$$\|H(f_1)-H(f_2)\|=\|h_1(f_1-f_2)\|\leqslant 4\,\|h_1\|\cdot\|f_1-f_2\|, \quad f_1,f_2\in BV(I_a^b;\mathbb{R}), \quad (21)$$

and so H is a Lipschitzian operator. This completes the proof. □

Remark 2. A theorem similar to Theorem 2 holds for the right–right, right–left and left–right regularizations of $h(\cdot,u)$, $u\in\mathbb{R}$.

Remark 3. If h_0, $h_1\in BV(I_a^b;\mathbb{R})$ and $\|h_1\|<1/4$, then, by Banach's contraction principle and (21), there exists a unique function $f\in BV(I_a^b;\mathbb{R})$ such that $f(x)=h_0(x)+h_1(x)f(x)$ for all $x\in I_a^b$.

Given $N\in\mathbb{N}$, let $(\mathbb{R}^N)^{I_a^b}=(\mathbb{R}^{I_a^b})^N$ be the algebra of all functions $f: I_a^b\to\mathbb{R}^N$, $h: I_a^b\times\mathbb{R}^N\to\mathbb{R}$ be a function of $N+2$ variables, $h=h(x_1,x_2,u_1,\ldots,u_N)$, and let $H_h:(\mathbb{R}^N)^{I_a^b}\to\mathbb{R}^{I_a^b}$ be the *superposition operator* defined by

$$(H_hf)(x)=h(x,f_1(x),\ldots,f_N(x)), \quad x\in I_a^b, \quad f=(f_1,\ldots,f_N)\in(\mathbb{R}^N)^{I_a^b}.$$

We endow the Cartesian product

$$BV(I_a^b;\mathbb{R})^N=\underbrace{BV(I_a^b;\mathbb{R})\times\cdots\times BV(I_a^b;\mathbb{R})}_{N\text{ times}}$$

with the product norm $\|f\|_N=\sum_{i=1}^N\|f_i\|$ for $f=(f_1,\ldots,f_N)\in BV(I_a^b;\mathbb{R})^N$. Clearly, the space $BV(I_a^b;\mathbb{R})^N$ is a Banach algebra with respect to the componentwise pointwise operations, and for all f, $g\in BV(I_a^b;\mathbb{R})^N$ the following inequality holds: $\|f\cdot g\|_N\leqslant 4\|f\|_N\|g\|_N$.

Corollary 2. *Let $h: I_a^b\times\mathbb{R}^N\to\mathbb{R}$ be a given function. If H_h maps the space $BV(I_a^b;\mathbb{R})^N$ into $BV(I_a^b;\mathbb{R})$ and is Lipschitzian (in the obvious sense), then for the left–left regularization of $h(\cdot,u_1,\ldots,u_N)$ we have: $h^*(x,u_1,\ldots,u_N)=h_0(x)+\sum_{i=1}^N h_i(x)u_i$, $x\in I_a^b$, $(u_1,\ldots,u_N)\in\mathbb{R}^N$, for some functions $h_0,h_1,\ldots,h_N\in BV^*(I_a^b;\mathbb{R})$. Conversely, if $h_0,h_1,\ldots,h_N\in BV(I_a^b;\mathbb{R})$ and*

$h(x,u_1,\dots,u_N)=h_0(x)+\sum_{i=1}^N h_i(x)u_i$, $x\in I_a^b$, $(u_1,\dots,u_N)\in\mathbb{R}^N$, then the superposition operator H_h maps $BV(I_a^b;\mathbb{R})^N$ into $BV\ (I_a^b;\mathbb{R})$ and is Lipschitzian.

Proof of Theorem 3. First we show that H maps $BV(I_a^b;\mathbb{R})$ *into itself. Let* $a_1=t_0<t_1<\cdots<t_{m-1}<t_m=b_1$ and $a_2=s_0<s_1<\cdots<s_{n-1}<s_n=b_2$. For any function $f\in BV(I_a^b;\mathbb{R})$ we have:

$$\begin{aligned}\sum_{i=1}^m\sum_{j=1}^n &|h(t_{i-1},s_{j-1},\ f(t_{i-1},s_{j-1}))+h(t_i,s_j,f(t_i,s_j))\\ &\qquad -h(t_{i-1},s_j,f(t_{i-1},s_j))-h(t_i,s_{j-1},f(t_i,s_{j-1}))|\\ &\leqslant 4\sum_{i=0}^m\sum_{j=0}^n|h(t_i,s_j,\ f(t_i,s_j))|\leqslant 4\sum_{k=1}^\infty\sum_{\ell=1}^\infty|h(p_k,q_\ell,\ f(p_k,q_\ell))|\\ &=4\sum_{k=1}^\infty\sum_{\ell=1}^\infty 2^{-k-\ell}|\sin\ f(p_k,q_\ell)|\leqslant 4\sum_{k=1}^\infty\sum_{\ell=1}^\infty 2^{-k-\ell}=4,\end{aligned}$$

so that $V_2(Hf,I_a^b)\leqslant 4$. Also, we have:

$$\begin{aligned}\sum_{i=1}^m &|h(t_i,a_2,\ f(t_i,a_2))-h(t_{i-1},a_2,\ f(t_{i-1},a_2))|\\ &\leqslant 2\sum_{i=0}^m|h(t_i,a_2,\ f(t_i,a_2))|\\ &\leqslant 2\sum_{k=1}^\infty|h(p_k,a_2,\ f(p_k,a_2))|\\ &\leqslant 2\sum_{k=1}^\infty 2^{-k}|\sin\ f(p_k,a_2)|\\ &\leqslant 2\sum_{k=1}^\infty 2^{-k}=2,\end{aligned}$$

i.e. $V_{a_1}^{b_1}((Hf)(\cdot,a_2))\leqslant 2$, and similarly, $V_{a_2}^{b_2}((Hf)(a_1,\cdot))\leqslant 2$. This proves that $TV(Hf,I_a^b)\leqslant 8$, and so H maps $BV(I_a^b;\mathbb{R})$ into $BV(I_a^b;\mathbb{R})$.

Now we prove that H is a Lipschitzian operator. For any two functions f_1, $f_2\in BV(I_a^b;\mathbb{R})$ we have, by virtue of (5) and (6),

$$\begin{aligned}\sum_{i=1}^m\sum_{j=1}^n &|(Hf_1-Hf_2)(t_{i-1},s_{j-1})+(Hf_1-Hf_2)(t_i,s_j)\\ &\qquad -(Hf_1-Hf_2)(t_{i-1},s_j)-(Hf_1-Hf_2)(t_i,s_{j-1})|\\ &\leqslant 4\sum_{i=0}^m\sum_{j=0}^n|(Hf_1-Hf_2)(t_i,s_j)|\\ &=4\sum_{i=0}^m\sum_{j=0}^n|h(t_i,s_j,f_1(t_i,s_j))-h(t_i,s_j,f_2(t_i,s_j))|\\ &\leqslant 4\sum_{k=1}^\infty\sum_{\ell=1}^\infty|h(p_k,q_\ell,f_1(p_k,q_\ell))-h(p_k,q_\ell,f_2(p_k,q_\ell))|\end{aligned}$$

$$= 4\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} 2^{-k-\ell}|\sin f_1(p_k,q_\ell) - \sin f_2(p_k,q_\ell)|$$

$$\leqslant 4\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} 2^{-k-\ell}|(f_1-f_2)(p_k,q_\ell)| \leqslant 4\|f_1-f_2\|,$$

so that $V_2(Hf_1 - Hf_2, I_a^b) \leqslant 4\|f_1 - f_2\|$. In a similar manner we have:

$$\sum_{i=1}^{m}|(Hf_1-Hf_2)(t_i,a_2) - (Hf_1-Hf_2)(t_{i-1},a_2)|$$

$$\leqslant 2\sum_{i=0}^{m}|h(t_i,a_2,f_1(t_i,a_2)) - h(t_i,a_2,f_2(t_i,a_2))|$$

$$\leqslant 2\sum_{k=1}^{\infty}2^{-k}|\sin f_1(p_k,a_2) - \sin f_2(p_k,a_2)| \leqslant 2\|f_1-f_2\|,$$

i.e. $V_{a_1}^{b_1}((Hf_1 - Hf_2)(\cdot,a_2)) \leqslant 2\|f_1 - f_2\|$, and a similar estimate holds for the Jordan variation of $(Hf_1 - Hf_2)(a_1,\cdot)$ over $[a_2,b_2]$. Finally, noting that

$$|(Hf_1-Hf_2)(a)| = |h(a,f_1(a)) - h(a,f_2(a))| \leqslant |\sin f_1(a) - \sin f_2(a)|$$
$$\leqslant |f_1(a) - f_2(a)| \leqslant \|f_1 - f_2\|,$$

we find that $\|H(f_1) - H(f_2)\| \leqslant 9\|f_1 - f_2\|$. □

4. Some Generalizations

1. Theorem 1 is valid (with the same proof) if we replace the target space $\mathbb{R}$ in it by any Banach algebra $(\mathbb{U}, |\cdot|)$; the definition of the space $BV(I_a^b;\mathbb{U})$ is straightforward. More generally, let $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{W}$ be normed linear spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and the norms denoted by the same symbol $|\cdot|$ (which won't lead to ambiguities). Suppose that there exists a bilinear map $M : \mathbb{U}\times\mathbb{V} \to \mathbb{W}$, called a *multiplication*, such that $|M(u,v)| \leqslant |u|\cdot|v|$ for all $u\in\mathbb{U}$ and $v\in\mathbb{V}$. The following generalization of Theorem 1 holds: *if* $f\in BV(I_a^b;\mathbb{U})$ *and* $g\in BV(I_a^b;\mathbb{V})$, *then the product function* $f\cdot g : I_a^b \to \mathbb{W}$ *defined by* $(f\cdot g)(x) = M(f(x),g(x))$, $x\in I_a^b$, *belongs to* $BV(I_a^b;\mathbb{W})$, *and inequality* (7) *holds.*

Let us prove that if $\mathbb{U}$ is a Banach space, then $BV(I_a^b;\mathbb{U})$ is also a Banach space with respect to the norm (5). The linearity of $BV(I_a^b;\mathbb{U})$ and the axioms of a norm for $\|\cdot\|$ are clear (due to (6)). In order to prove the completeness of $BV(I_a^b;\mathbb{U})$, let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $BV(I_a^b;\mathbb{U})$. By virtue of (6), $|f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|$, $x\in I_a^b$, n, $m\in\mathbb{N}$, and so, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{U}$, so that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists in $\mathbb{U}$. Since $f_n - f_m$ tends to $f_n - f$ in $\mathbb{U}$ pointwise on I_a^b as $m\to\infty$,

$$\|f_n - f\| \leqslant \liminf_{m\to\infty}\|f_n - f_m\| = \lim_{m\to\infty}\|f_n - f_m\|,$$

whence

$$\limsup_{n\to\infty}\|f_n - f\| \leqslant \limsup_{n\to\infty}\lim_{m\to\infty}\|f_n - f_m\| = \lim_{n\to\infty}\lim_{m\to\infty}\|f_n - f_m\| = 0.$$

It follows that $\lim_{n\to\infty}\|f_n - f\| = 0$. Since $\{f_n\}_{n=1}^{\infty}$ is Cauchy in $BV(I_a^b;\mathbb{U})$, it is bounded, and so there exists a constant $C\geqslant 0$ such that $\|f_n\|\leqslant C$ for all $n\in\mathbb{N}$. The pointwise convergence of f_n to f implies

$$\|f\|\leqslant \liminf_{n\to\infty}\|f_n\|\leqslant C,$$

i.e. $f\in BV(I_a^b;\mathbb{U})$. This proves that $BV(I_a^b;\mathbb{U})$ is complete.

2. In order to generalize Theorem 2 for $BV(I_a^b;\mathbb{U})$ with $(\mathbb{U},|\cdot|)$ a Banach space, we have to show that the limits in the definition of the left–left regularization f^* of $f\in BV(I_a^b;\mathbb{U})$ exist and prove an analogue of Lemma 1. To do this, we first prove, in addition to (6), that if $f\in BV(I_a^b;\mathbb{U})$ and $x=(x_1,x_2)\leqslant(y_1,y_2)=y$ in I_a^b, then

$$TV(f,I_x^y)\leqslant TV(f,I_a^y) - TV(f,I_a^x). \tag{22}$$

In fact, similar to (14) (more exactly cf. the first inequality in (15)) we have:

$$V_{x_1}^{y_1}(f(\cdot,x_2))\leqslant V_{x_1}^{y_1}(f(\cdot,a_2)) + V_2(f,I_{x_1,a_2}^{y_1,x_2}),$$
$$V_{x_2}^{y_2}(f(x_1,\cdot))\leqslant V_{x_2}^{y_2}(f(a_1,\cdot)) + V_2(f,I_{a_1,x_2}^{x_1,y_2}).$$

By the additivity of $V_2(f,\cdot)$,

$$V_2(f,I_a^y) = V_2(f,I_a^x) + V_2(f,I_x^y) + V_2(f,I_{a_1,x_2}^{x_1,y_2}) + V_2(f,I_{x_1,a_2}^{y_1,x_2}). \tag{23}$$

Applying these (in)equalities, the additivity of the Jordan variation and definition (4) we get (omitting the straightforward calculation):

$$\begin{aligned} TV(f,I_x^y) &= V_{x_1}^{y_1}(f(\cdot,x_2)) + V_{x_2}^{y_2}(f(x_1,\cdot)) + V_2(f,I_x^y)\\ &\leqslant TV(f,I_a^y) - TV(f,I_a^x). \end{aligned}$$

As the second step, we assert that the total variation function $\nu_f : I_a^b\to\mathbb{R}$ defined by $\nu_f(x) = TV(f,I_a^x)$, $x\in I_a^b$, belongs to $BV(I_a^b;\mathbb{R})$ for any function $f\in BV(I_a^b;\mathbb{U})$ and, moreover, $TV(\nu_f,I_a^b) = TV(f,I_a^b)$. Indeed, the functions $\nu_f(t,a_2) = V_{a_1}^t(f(\cdot,a_2))$ and $\nu_f(a_1,s) = V_{a_2}^s(f(a_1,\cdot))$ are nondecreasing for $t\in[a_1,b_1]$ and $s\in[a_2,b_2]$, respectively, and for any x, $y\in I_a^b$ with $x\leqslant y$, making use of (4), (23) and the (additivity) equalities:

$$V_2(f,I_{a_1,a_2}^{x_1,y_2}) = V_2(f,I_a^x) + V_2(f,I_{a_1,x_2}^{x_1,y_2}),$$
$$V_2(f,I_{a_1,a_2}^{y_1,x_2}) = V_2(f,I_a^x) + V_2(f,I_{x_1,a_2}^{y_1,x_2}),$$

we have:

$$\nu_f(x_1,x_2) + \nu_f(y_1,y_2) - \nu_f(x_1,y_2) - \nu_f(y_1,x_2) = V_2(f,I_x^y),$$

which implies $V_2(\nu_f,I_a^b) = V_2(f,I_a^b)$.

Now, (6) and (22) yield the estimate:

$$|f(y) - f(x)|\leqslant \nu_f(y) - \nu_f(x), \qquad x\leqslant y \text{ in } I_a^b. \tag{24}$$

Since $\nu_f\in BV(I_a^b;\mathbb{R})$, it has one-sided limits shown in the definition of f^* above ([9, III.5.3]). By the Cauchy criterion for the existence of a limit, (24) and the completeness of $\mathbb{U}$, any function $f\in BV(I_a^b;\mathbb{U})$ has the limits defining f^*. Thus, in this more general situation Lemma 1 holds as well.

3. If $(\mathbb{U}, |\cdot|_{\mathbb{U}})$ and $(\mathbb{V}, |\cdot|_{\mathbb{V}})$ are normed linear spaces, we denote by $L(\mathbb{U};\mathbb{V})$ the normed linear space of all linear bounded operators from $\mathbb{U}$ into $\mathbb{V}$. Denote by $\mathbb{U}^{I_a^b}$ the space of all functions $f: I_a^b \to \mathbb{U}$ mapping I_a^b into $\mathbb{U}$. Given $h: I_a^b \times \mathbb{U} \to \mathbb{V}$, the *superposition operator* $H: \mathbb{U}^{I_a^b} \to \mathbb{V}^{I_a^b}$ is defined as in (1) with $f \in \mathbb{R}^I$ and $x \in I$ replaced by $f \in \mathbb{U}^{I_a^b}$ and $x \in I_a^b$. Let $P_j([a_j, b_j]; \mathbb{U}) \subset \mathbb{U}^{[a_j, b_j]}$ be a family of functions having the following property: for all u_1, $u_2 \in \mathbb{U}$, $m \in \mathbb{N}$ and $a_j < \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m < b_j$ the polygonal function defined by $[a_j, b_j] \ni t \mapsto \eta_m(t)\ u_1 + u_2 \in \mathbb{U}$ (for the definition of η_m see (18)) belongs to $P_j([a_j, b_j]; \mathbb{U})$, $j = 1, 2$. Clearly, $P(I_a^b; \mathbb{U}) = P_1([a_1, b_1]; \mathbb{U}) + P_2([a_2, b_2]; \mathbb{U})$ is a subspace of $BV(I_a^b; \mathbb{U})$.

The analysis of the proof of Theorem 2 shows that the following counterpart and generalization of Theorem 2 holds:

Theorem 4. *Suppose that the superposition operator* $H: \mathbb{U}^{I_a^b} \to \mathbb{V}^{I_a^b}$ *is generated by a function* $h: I_a^b \times \mathbb{U} \to \mathbb{V}$.

If $\mathbb{U}$ *is a real normed linear space*, $\mathbb{V}$ *is a Banach space and H maps the space* $P(I_a^b; \mathbb{U})$ *into* $BV(I_a^b; \mathbb{V})$ *and is Lipschitzian* (*in the sense of the norms in these spaces*), *then there exists a constant* $\mu_0 > 0$ *such that*

$$|h(x, u_1) - h(x, u_2)|_{\mathbb{V}} \leqslant \mu_0 |u_1 - u_2|_{\mathbb{U}}, \qquad x \in I_a^b, \quad u_1, u_2 \in \mathbb{U},$$

and there exist two functions $h_0 \in BV^*(I_a^b; \mathbb{V})$ *and* $h_1: I_a^b \to L(\mathbb{U}; \mathbb{V})$ *with the property that* $[x \mapsto h_1(x)u] \in BV^*(I_a^b; \mathbb{V})$ *for all* $u \in \mathbb{U}$ *such that*

$$h^*(x, u) = h_0(x) + h_1(x)u \quad \text{in} \quad \mathbb{V}, \qquad x \in I_a^b, \quad u \in \mathbb{U}.$$

Conversely, if $\mathbb{U}$ *and* $\mathbb{V}$ *are normed linear spaces*, $h_0 \in BV(I_a^b; \mathbb{V})$, $h_1 \in BV(I_a^b; L(\mathbb{U}; \mathbb{V}))$ *and* $h(x, u) = h_0(x) + h_1(x)u$, $x \in I_a^b$, $u \in \mathbb{U}$, *then H maps* $BV(I_a^b; \mathbb{U})$ *into* $BV(I_a^b; \mathbb{V})$ *and is Lipschitzian.*

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