An Approximation Scheme for the $1|r_j|\sum T_j$ Scheduling Problem with Guaranteed Absolute Error

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Keywords: scheduling algorithm, total tardiness, metric, approximation scheme.

1 Introduction

In the paper, we consider the following problem. Suppose we have a set $N = \{1, 2, \ldots, n\}$ of n jobs to be processed on a single machine. Preemptions are not allowed. The machine is available since time $t_0 = 0$ and can handle only one job at a time. Job $j \in N$ is available for processing since its *release date* $r_j \geq 0$, its processing requires *processing time* $p_j \geq 0$ time units and should ideally be completed before its *due date* d_j. We will call an instance the set of given parameters: release dates, processing times, and due dates. We will use superscripts to distinguish parameters belonged to different instances. Note that an instance $A = \{r_1^A, \ldots, r_n^A, p_1^A, \ldots, p_n^A, d_1^A, \ldots, d_n^A\}$ can be considered as a vector in $3n$-dimensional space.

Let $S_j(\pi)$ and $C_j(\pi)$ be a *starting and a completion time* of job $j \in N$ in schedule π, respectively. We will consider only *early schedules* (sequences), i.e., if $\pi = (j_1, \ldots, j_n)$, then $S_{j_1} = \max\{0, r_{j_1}\}$, $S_{j_k} = \max\{r_{j_k}, C_{j_{k-1}}\}, k = 2, 3, \ldots, n$, and $C_j(\pi) = S_j(\pi) + p_j, j \in N$. Thus an early schedule is uniquely determined by a permutation of the jobs of set N. Then let $T_j(\pi) = \max\{0, C_j(\pi) - d_j\}$ be a *tardiness* of job j in schedule π.

The objective is to find an *optimal schedule* $\bar{\pi}$ which minimizes the *total tardiness*, i.e., objective function is $F(\pi) = \sum\limits_{j=1}^{n} T_j(\pi)$. In the notation introduced by Graham *et. al.* (1979) the problem is denoted by $1|r_j|\sum T_j$. The problem is NP-hard in ordinary sense (Du and Leung 1990).

In the paper we propose an approximation scheme for the minimizing total tardiness problem. In the scheme we construct a polynomially solvable instance B and apply its solution to the given instance A. To evaluate the error of the solution we construct a metric for the considered problem. For the $1||\sum T_j$ problem the metric was constructed by Lazarev and Kvaratskheliya (2010). For the problem $1|r_j|\sum T_j$ we propose a metric $\rho(A, B)$

$$\rho(A,B) = n \cdot \max_{j \in N} |r_j^A - r_j^B| + n \cdot \sum_{j=1}^{n} |p_j^A - p_j^B| + \sum_{j=1}^{n} |d_j^A - d_j^B|.$$

2 Approximation scheme

Lemma 1 *The function*

$$\rho(A,B) = n \cdot \max_{j \in N} |r_j^A - r_j^B| + n \cdot \sum_{j=1}^{n} |p_j^A - p_j^B| + \sum_{j=1}^{n} |d_j^A - d_j^B|.$$

satisfies the metric axioms.

Theorem 1 *Let $\bar{\pi}^A$ and $\bar{\pi}^B$ be an optimal schedules for instances A and B, respectively. Moreover, let $\tilde{\pi}^B$ be an approximate schedule, subject to*

$$\sum_{j=1}^{n} T_j^B(\tilde{\pi}^B) - \sum_{j=1}^{n} T_j^B(\bar{\pi}^B) \leq \delta.$$

Then

$$\sum_{j=1}^{n} T_j^A(\tilde{\pi}^B) - \sum_{j=1}^{n} T_j^A(\bar{\pi}^A) \leq \delta + 2\rho(A, B).$$

The idea of the approximated scheme is to find the least distanced in the metric from the given instance A polynomially solvable instance B. Then, by applying known polynomial algorithm to the instance B, one obtains a schedule $\bar{\pi}^B$ which can be used as an approximate solution for instance A with error no greater than $2\rho(A, B)$. One can also use approximate solution for the instance B with an absolute error δ as an approximate solution for instance A, in this case the error is not greater that $2\rho(A, B) + \delta$.

Thereby, the problem $1|r_j|\sum T_j$ is reduced to the problem of minimizing the function $\rho(A, B)$.

Let us search for the instance B in the polynomially solvable class defined by the system of linear inequalities

$$\mathcal{A} \cdot R^B + \mathcal{B} \cdot P^B + \mathcal{C} \cdot D^B \leq H,$$

where $R^B = (r_1^B, \ldots, r_n^B)^T, P^B = (p_1^B, \ldots, p_n^B)^T$, $D^B = (d_1^B, \ldots, d_n^B)^T$, $p_j^B \geq 0, r_j^B \geq 0$, $j \in N$, ^{T} is transposition symbol, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ – $m \times n$ matrices, and H – a column of m elements.

Then the problem of finding the least distanced from A instance of the given polynomially solvable class can be formulated as follows

$$\text{minimize } f = n \cdot (y^r - x^r) + n \cdot \sum_{j=1}^{n} (y_j^p - x_j^p) + \sum_{j=1}^{n} (y_j^d - x_j^d),$$

subject to

$$\begin{gathered} x^r \leq r_j^A - r_j^B \leq y^r, \\ x_j^p \leq p_j^A - p_j^B \leq y_j^p, \\ x_j^d \leq d_j^A - d_j^B \leq y^d, \\ r_j^B \geq 0, p_j^B \geq 0, j \in N, \\ \mathcal{A} \cdot R^B + \mathcal{B} \cdot P^B + \mathcal{C} \cdot D^B \leq H. \end{gathered}$$

It is the problem of the linear programming, with $7n + 2$ variables: $r_j^B, p_j^B, d_j^B, x_j^p, y_j^p$, $x_j^d, y_j^d, x^r, y^r, j = 1, \ldots, n$.

However, it is not necessary to use algorithms of the linear programming, if there are less complicated ways.

3 Examples of the approximation scheme

Let $\mathcal{PR}$ denote the class of instances with $r_j = r, p_j = p, j = 1, \ldots, n$ and $\mathcal{PD}$ denote the class with $p_j = p, d_j = d, j = 1, \ldots, n$. Both classes are polynomially solvable. In the optimal schedules jobs are processed in the increasing order of their due dates for $\mathcal{PR}$ class and in the increasing order of their release dates for $\mathcal{PR}$.

If in the approximation scheme one searches for the instance B in class $\mathcal{PR}$, one has to minimize the function

$$f(r,p) = n \cdot \max_{j \in N} |r_j^A - r| + n \cdot \sum_{j=1}^{n} |p_j^A - p|.$$

Lemma 2 *Function $f(r,p)$ has minimum at point $(r = \frac{r_{max}^A + r_{min}^A}{2}, p \in \{p_1^A, \ldots, p_n^A\})$, where $r_{max}^A = \max_{j \in N} r_j^A$, $r_{min}^A = \min_{j \in N} r_j^A$, therefore minimum can be found in $O(n)$ operations.*

And if in the approximation scheme one searches for the instance B in class $\mathcal{PD}$, one has to minimize the function

$$g(p,d) = n \cdot \sum_{j=1}^{n} |p_j^A - p| + \sum_{j=1}^{n} |d_j^A - d|.$$

Lemma 3 *Function $g(p,d)$ has minimum at point $(p \in \{p_1^A, \ldots, p_n^A\}, d \in \{d_1^A, \ldots, d_n^A\})$, therefore minimum can be found in $O(n)$ operations.*

So, we have two variants of the proposed scheme: with the use of $\mathcal{PR}$ and $\mathcal{PD}$ classes. To evaluate approximated solutions for both cases we have run computational experiments. 10000 instances were generated for each value of n. Experiments were performed for $n = 4, 5, \ldots, 10$. For each instance, processing times p_j were generated randomly in the interval $[1, 100]$, due dates d_j were generated in the interval $[-100, 100]$, and release dates r_j were generated in the interval $[0, 100]$. We used proposed scheme to find an approximate solution with value of objective function F_a for each instance, and branch & bound algorithm to find an optimal solution with value of objective function F_o. After we estimated experimental error $\Delta = F_a - F_o$ in percentage of the theoretical error, which is doubled value of function $f(r,p)$ or $g(p,d)$ for cases with $\mathcal{PR}$ and $\mathcal{PD}$ classes, respectively.

All obtained distributions are bell-shaped. The typical distribution of experimental error is shown in Fig. 1. In both cases distributions narrow with increasing of n. In the $\mathcal{PR}$-case experimental errors averages near 19% of the theoretical. In $\mathcal{PR}$-case we have obtain that error does not exceed 30% of theoretical, though its average grows from 5% to 10% with increasing of n. Obtained average errors are shown in Table 1.

Table 1. Average experimental error in percentage of the theoretical error

n	**Average error in $\mathcal{PR}$-case**	**Average error in $\mathcal{PD}$-case**
4	19%	4,5%
5	19,5%	6,2%
6	19,2%	7,3%
7	19,6%	8,5%
8	19,3%	9,2%
9	19,4%	10%
10	19%	10,5%

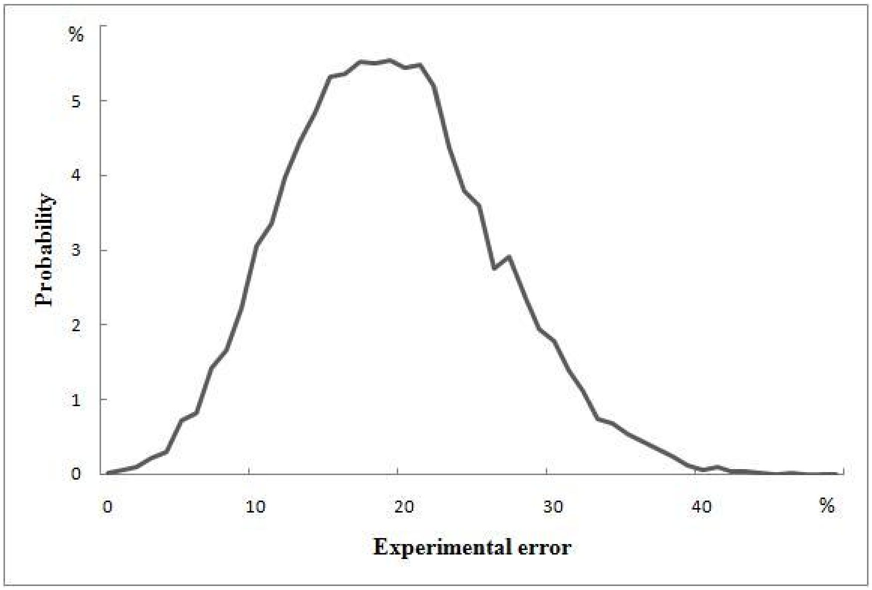

Fig. 1. Distribution of experimental error in percentage of the theoretical error

4 Conclusion

In the paper we have proposed the new approximation scheme for the minimizing total tardiness problem. The scheme is based on search for the polynomially solvable instance which has a minimal distance in the metric from the original instance.

In further research the scheme can be applied to other scheduling problems. One can also improve the scheme by constructing new metrics and finding new polynomially solvable cases of scheduling problems.

References

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