ISSN 1064–5624, Doklady Mathematics, 2009, Vol. 79, No. 3, pp. 437–439. © Pleiades Publishing, Ltd., 2009.
Published in Russian in Doklady Akademii Nauk, 2009, Vol. 426, No. 3, pp. 318–320.

COMPUTER SCIENCE

On Stable Solutions to the Ordinal Social Choice Problem[¶]

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Presented by Academician S.N. Vasil'ev July 24, 2008

Received December 3, 2008

Abstract—A concept of *k*-stable alternatives is introduced. Relationship of classes of *k*-stable alternatives with dominant, uncovered and weakly stable sets is established.

DOI: 10.1134/S1064562409030375

INTRODUCTION

Ordinal Social Choice problem is defined for a finite number of alternatives, over which a finite number of agents have preferences. Individual preferences of each agent are binary relations on the set of alternatives. Solution to the problem is a rule determining an alternative (alternatives), which is (are) the "best" to the group. The so called Condorcet winner [1], an alternative that is more preferable for the majority of agents than any other alternative in pairwise comparison, is usually regarded as the best choice. However the conditions, under which a Condorcet winner exists, are extremely restrictive [2], and a Condorcet winner is absent in general case.

Numerous attempts were made to extend the set of chosen alternatives up to a certain always non-empty subset of the universal set, defined through majority relation μ, where alternative *a* is preferred to an alternative *b*, if majority of actors prefers so. Being different incarnations of an idea of optimal social choice this solution sets enable one to compare and evaluate social choice procedures. Also, when there is a connection of a solution set with a particular procedure, this solution helps to reduce the number of potential collective choices, i.e., enables one to make predictions with respect to choice results.

In the present paper for such class of majority relation μ as tournaments (complete ($\forall x, y \Rightarrow x\mu y \vee y\mu x$) and asymmetric ($\forall(x, y)$: $(x, y) \notin \mu \Rightarrow (y, x) \in \mu$) relations), three solution concepts are considered: dominant set [3–6], weakly stable set [7], uncovered set [5, 8]. It is demonstrated that a hierarchy of dominant sets might be regarded as a macrostructure of a tournament. A criterion to determine whether an alternative belongs to a minimal weakly stable set is established. As a result it is shown that for tournaments an uncovered set is always a subset of a union of minimal weakly stable sets. Then the concept of stability is employed to generalize the notions of weakly stable and uncovered sets. The concept of *k*-stable alternatives is introduced. Their properties and relations with aforementioned solution concepts are determined.

MAIN CONCEPTS

A finite set *A* of alternatives is given,[1] A, $|A| > 2$. Agents from a finite set $N = \{1, 2, \ldots, n\}$, $|N| > 1$, have preferences over alternatives from the set *A*. In general case these preferences are arbitrary binary relation. In Social Choice Theory they are usually assumed to be weak orders P_i, $i \in N$, i.e., relations, where it is always possible to say about any two alternatives that either one is more preferable than the other or they are of equal value.

Formally, strong preference relation P_i satisfies antireflexivity ($x\bar{P}_i x$), transitivity (xP_iy & $yP_iz \Rightarrow xP_iz$) and negative transitivity ($x\bar{P}_i y$ & $y\bar{P}_i z \Rightarrow x\bar{P}_i z$). Indifference relation $I_i = A \times A \backslash (P_i \cup P_i^-)$ for P_i, where $P_i^- = \{(x, y) | (y, x) \in P_i\}$, is an equivalence relation, i.e. reflexive (xI_ix), symmetric ($xI_iy \Rightarrow yI_ix$) and transitive relation.

[¶] The article was translated by the authors.

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[1] The terminology, definitions and notation given in this Section are derived mainly from [7]. Lowercase letters (except CW) denote alternatives; capital letters denote sets of alternatives.

Majority relation is a binary relation μ, $\mu \subset A \times A$, constructed such that $(x, y) \in \mu$ if x is strongly preferred to y by majority, whichever defined, of all agents. For absolute majority $x\mu y \Leftrightarrow \text{card}\{i \in N, xP_iy\} > \text{card}\{i \in N, y(P_i \cup I_i)x\}$. If $x\mu y$ then it is said that x dominates y, and y is dominated by x. By assumption μ is asymmetric: $(x, y) \in \mu \Rightarrow (y, x) \notin \mu$.

A relation μ is called a tournament if it is complete, i.e., $(y, x) \notin \mu \Rightarrow (x, y) \in \mu$. In the present paper only tournaments are considered.

An ordered pair $x \to y$ $(x \to y \Leftrightarrow x\mu y)$ is also called a step. A path $x \to y_1 \to y_2 \to \ldots \to y_{k-2} \to y_{k-1} \to y$ from x to y is an ordered sequence of steps starting at x and ending at y, such that the second alternative in each step coincides with the first alternative of the next step. In other words a path is an ordered sequence of alternatives $x, y_1, y_2, \ldots, y_{k-2}, y_{k-1}, y$, such that each alternative dominates the following one: $x\mu y_1$, $y_1\mu y_2$, $\ldots$, $y_{k-2}\mu y_{k-1}$, $y_{k-1}\mu y$. The number of steps in a path is called path's length. An alternative y is called reachable in k steps from x if there is a path of length k from x to y.

Lower contour set of an alternative x is a set $L(x)$ of all alternatives dominated by x, $L(x) = \{y \in A: x\mu y\}$. Correspondingly, upper contour set of an alternative x is a set $D(x)$ of all alternatives dominating x, $D(x) = \{y \in A: y\mu x\}$. Since μ is a tournament $L(x) \cup D(x) \cup \{x\} = A$.

A Condorcet winner CW is an alternative dominating all other alternatives, $\forall x: x \neq \text{CW} \Rightarrow \text{CW}\mu x$.

A set $D \subseteq A$, is called a dominant set (also "majority set" [3]) if each alternative in D dominates each alternative outside, i.e., D is a dominant set D $(\Leftrightarrow (\forall (x, y): (x \in D \;\&\; y \in A\backslash D) \Rightarrow (x, y) \in \mu)$ [2, 3]. A dominant set *MD* is called a minimal dominant set ("undominated (Condorcet) set" [6], "GETCHA" [9]; "weak top cycle" [10]), if none of its proper subsets is a dominant set [5, 6, 10]. *MD* always exists and is unique [6].

We say x covers y, if $x\mu y$ and $D(x) \subseteq D(y)$ [5, 8]. Thus x is uncovered $\Leftrightarrow (\forall y: y\mu x \Rightarrow \exists z: (x\mu z \;\&\; z\mu y))$. The uncovered set [8] *UC* is comprised of all alternatives that are not covered. *UC* is always non-empty and is a subset of *MD*, $UC \neq \emptyset$, $UC \subseteq MD$ [8].

WEAKLY STABLE SET AND STABLE ALTERNATIVES

A set *WS* is called a weakly stable set [7], if it has the following property: if x belongs to a weakly stable set, then for any alternative y outside the weakly stable set, which dominates x, there is an alternative z in the weakly stable set, which dominates y; i.e., *WS* is weakly stable $\Leftrightarrow (\forall x, y: (x \in WS \;\&\; y \in A\backslash WS \;\&\; y\mu x) \Rightarrow \exists z: (z \in WS \;\&\; z\mu y)$. In terms of $D(x)$ and $L(x)$ *WS* is weakly stable $\Leftrightarrow (\forall y: (y \notin WS \;\&\; WS \cap L(y) \neq \emptyset) \Rightarrow WS \cap D(y) \neq \emptyset)$. A weakly stable set *MWS* is called a minimal weakly stable set if none of its proper subsets is a weakly stable set. If such set is not unique, then the social choice is defined as a union of these sets [7], which will be denoted *UMWS*.

It also follows from the definitions that any dominant set is at the same time a weakly stable set. Thus non-emptiness of *MD* implies non-emptiness of *UMWS*.

Lemma. *Any weakly stable set is a subset of MD,* $MWS \subseteq MD$.

Corollary. Since *UMWS* and *UC* are always nonempty, then if there is a Condorcet winner CW, sets *MD, UMWS* and *UC* coincide and contain only one alternative—CW, $MD = UMWS = UC = \{\text{CW}\}$.

The definition of a minimal weakly stable set given in [7] is global. For practical calculations one needs a criterion to determine whether an alternative belongs to a minimal weakly stable set or not. Theorem 1 formulates such a criterion.

Theorem 1. *An alternative x belongs to a union of minimal weakly stable sets UMWS iff* (1) *either x is uncovered or* (2) *some alternative from x's lower contour set L(x) is uncovered, i.e.,* $x \in UMWS \Leftrightarrow (x \in UC \vee \exists y: (y \in L(x) \;\&\; y \in UC))$.

Corollary. *The uncovered set is a subset of the union of minimal weakly stable sets,* $UC \subseteq UMWS$.

Consequently, all sets considered in the present paper are related through inclusion: $UC \subseteq UMWS \subseteq MD \subseteq A$. It is possible to demonstrate that there are tournaments, where all inclusions are strict, $UC \subset UMWS \subset MD \subset A$.

It is possible to generalize the concepts of the uncovered and weakly stable sets if we consider the relative stability of alternatives and sets of alternatives. An alternative x will be called generally stable (or simply stable) if every other alternative in A is reachable from x, otherwise x is unstable. Every alternative in A is reachable from x iff x belongs to a minimal dominant set [6], thus all alternatives of a minimal dominant set and only they are generally stable.

Since A is finite, if y is reachable from x, then there is a path from x to y with a minimal length. Let $l(x, y)$ denote a minimal length function. The function $l(x, y)$ has the following property: $l(x, y) > 1 \Rightarrow l(y, x) = 1$.

For x and y, such that $x \in D$, $y \in A\backslash D$, where D is a dominant set, $l(y, x)$ is not defined, as x is not reachable from y. For such cases let $l(y, x) = \infty$. If x belongs to a minimal dominant set, $l(x, y)$ is defined and has a finite value for all $y \in A\backslash\{x\}$. Let $l(x, x) = 0$ for $l(x, y)$ to be defined on the whole set A. In terms of $l(x, y)$ x is generally stable when $\forall y: y \in A \Rightarrow l(x, y) < \infty$.

Let $l_{\max}(x)$ denote a function of x defined as $l_{\max}(x) = \max\limits_{y \in A} l(x, y)$. If $l_{\max}(x) = k < \infty$ then it is possible to reach any alternative in A from x in no more than k steps, but there is at list one alternative reachable from x in less than k steps. Let the value of $l_{\max}(x)$ be called a degree of stability of x. If the degree of stability of an alternative x is k, $k < \infty$, x will be called k-stable.[2] Let $SP_{(k)}$ denote a class of k-stable alternatives in A, $x \in SP_{(k)} \Leftrightarrow l_{\max}(x) = k$.

If follows from the definition that an alternative x has the degree of stability $x = 1$ iff x is a Condorcet winner, x = CW. Therefore $SP_{(1)} = \{\text{CW}\}$. It is also evident that if $SP_{(1)} \neq \emptyset$, then all $SP_{(k>1)} = \emptyset$, since CW is not reachable from any other alternative in A.

If follows from the definition of covering relation that an alternative x has the degree of stability $k = 2$ iff x is an uncovered alternative, i.e., $SP_{(2)}$ is an uncovered set, $SP_{(2)} = UC$.

Theorem 2. $\exists m$: (1) $SP_{(k)} = \emptyset$, $\forall k$: $k > m$; (2) $SP_{(m)} \neq \emptyset$; (3) $MD = SP_{(1)} + SP_{(2)} + SP_{(3)} + \ldots + SP_{(m)}$.

By construction the classes of stable alternatives do not intersect, $i \neq j \Rightarrow SP_{(i)} \cap SP_{(j)} = \emptyset$. Since all alternatives that are generally stable belong to a minimal dominant set MD, and all alternatives from MD are generally stable, MD is a direct sum of all classes of k-stable alternatives, $MD = SP_{(1)} + SP_{(2)} + SP_{(3)} + \ldots + SP_{(k)} + \ldots$. Since A is finite, there is a generally stable alternative (at least one), the degree of stability of which is maximal $m = \max\limits_{x \in MD} l_{\max}(x)$.

Theorem 3. (Non-emptiness of classes of k-stable alternatives.) *If there is no Condorcet winner, each class of k-stable alternatives with the degree k equal or less than maximal is nonempty, except* $SP_{(1)}$, $\forall SP_{(k)} \neq \emptyset$, $2 \leq k \leq m = \max\limits_{x \in MD} l_{\max}(x)$.

Finally, let $P_{(k)}$ denote a set of those generally stable alternatives, from which it is possible to reach any given alternative in A in no more than k steps. By definition $P_{(k)} = SP_{(1)} + SP_{(2)} + \ldots + SP_{(k)}$. Therefore the following system of subsets emerges in a minimal dominant set.

(1) $P_{(1)} = \{\text{CW}\} = MD$; if $P_{(1)} = \emptyset$, then

(2) $P_{(2)} = UC \neq \emptyset$, an uncovered set;

(3) $P_{(1)} \subset P_{(2)} \subset P_{(3)} \subset \ldots \subset P_{(m-1)} \subset P_{(m)} = MD$, $m = \max\limits_{x \in MD} l_{\max}(x)$, all inclusions are strict according to Theorem 3.

In a similar to k-stable alternatives way it is possible to define concepts of generally stable sets, minimal k-stable sets and classes of minimal k-stable sets.

[2] The greater the order of stability of an alternative, the less it is stable.

ACKNOWLEDGMENTS

The work was partially supported by Scientific Foundation of SU-HSE (grant no. 08-04-0008).

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