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MATHEMATICS

Pseudonormal Form and Finite-Smooth Equivalence of Real Autonomous Systems with Two Purely Imaginary Eigenvalues

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This paper studies real autonomous infinitely smooth systems of ordinary differential equations of class C^∞ in a neighborhood of a nondegenerate singular point. We consider systems whose matrix of linear part has two purely imaginary eigenvalues; all of the other eigenvalues are outside the imaginary axis. We are interested in the possibility of reducing such systems to a form similar to the well-known normal form. Since we allow transformations with singularities, we refer to the forms of transformed systems as pseudonormal forms. The reduction of systems to such forms makes it possible to solve the problem of local finitely smooth equivalence of the systems of equations under consideration and more deeply comprehend the notion of resonance. The problem of finitely smooth equivalence has been well studied for systems with linear part whose spectrum lies outside the imaginary axis (see the Sternberg–Chen theorem in [1, Chapter 9]), while even weakly degenerate systems have been studied very little (see, e.g., [2–9]). Results of [5–7] and of this paper allow us to assert that real autonomous infinitely smooth systems of class C^∞ with one zero or two purely imaginary eigenvalues (except systems from a certain exceptional set of infinite codimension) can be reduced by nondegenerate finitely smooth transformations to resonance polynomial normal forms.

Consider the real autonomous system

$$\dot{\xi} = \frac{d\xi}{dt} = Q(\xi), \tag{1}$$

where $\xi,\ Q(\xi) \in R^{n+2},\ n > 0,\ Q(\xi)$ is a function of class C^∞ in a neighborhood of the origin, $Q(0) = 0$, and the matrix $\tilde{A} = Q'(0)$ has n eigenvalues outside the imaginary axis and two purely imaginary (conjugate) eigenvalues. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the matrix $\tilde{A}$ with nonzero real part, and let $\pm i\omega$ be the pair of purely imaginary eigenvalues of this matrix, where $\omega > 0$, i and i is the imaginary unit. By means of a standard linear transformation, we reduce system (1) to the following form, where the matrix $\tilde{A}$ is Jordan:

$$\begin{aligned} \dot{x}_1 &= i\omega x_1 + f_1(x, y), \\ \dot{x}_2 &= -i\omega x_2 + f_2(x, y), \\ \dot{y}_j &= \varepsilon_i y_{j-1} + \lambda_j y_j + g_j(x, y), \quad j = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

Here, x and y are complex coordinates, $x = (x_1, x_2)$, $x_2 = \bar{x}_1$, $y = (y_1, y_2, \ldots, y_n)$, complex conjugate variables correspond to complex conjugate equations, and the Taylor expansions of the functions $f_1, f_2, g_j, j = 1, 2, \ldots, n$ contain no linear terms. We refer to the variables x as degenerate and to the variables y as nondegenerate. We observe the important principle that, under all transformations of complex systems, complex conjugate variables and the corresponding equations remain complex conjugate. To every such transformation, a real transformation of the initial system corresponds.

Without loss of generality, we assume that the Taylor expansions of the functions on the right-hand side of the system under consideration consist of resonance monomials.

Using formula (2.7) from [10], reality considerations, and results of [4], we apply a nondegenerate transformation of class C^∞ which reduce system (2) to a form in which one of the invariant central manifolds (corresponding to the imaginary part spectrum) is determined by the equation $y = 0$, and the linear (in the nondegenerate coordinates) part of system (2) has the form

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + i\varphi(\tilde{r}) + \tilde{b}(\tilde{r})), \\ \dot{x}_2 &= x_2(-i\omega - i\varphi(\tilde{r}) + \tilde{b}(\tilde{r})), \\ \dot{y} &= A(x)y, \end{aligned} \tag{3}$$

where $\tilde{r} = x_1x_2$, $\varphi(\tilde{r}) = \sum_{j=1}^{m} \varphi_j \tilde{r}^j$, $\tilde{b}(\tilde{r}) = b\tilde{r}^m + c\tilde{r}^{2m}$, $m \geq 1$ is an integer; $\varphi_1, \varphi_2, \ldots, \varphi_m$, b, c are real numbers; and

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$b \neq 0$. Thereby, we assume that the system has the form of a focus on the central manifold.

Remark 1. We do not consider systems for which $\tilde{b}(\tilde{r})$ is a flat function, which belong to the so-called exceptional set (in the terminology of [4]) of infinite codimension.

An important special feature of the systems under consideration is resonances, which arise in those cases in which $\frac{\lambda_{j_1} - \lambda_{j_2}}{i\omega}$ is integer for some $1 \leq j_1 \neq j_2 \leq n$. We refer to such eigenvalues, as well as to the corresponding variables and equations, as equivalent. In the absence of these resonances, system (1) can always be reduced to a polynomial resonance normal form by a transformation of finite smoothness [8, Theorem 3]. Among the resonances specified above, we distinguish those with $\lambda_{j_1} = \bar{\lambda}_{j_2}$. This means that the number $2\omega^{-1}\operatorname{Im}\lambda_{j_1}$ is integer. We refer to the variables corresponding to eigenvalues λ for which the number $2\omega^{-1}\operatorname{Im}\lambda$ is odd as special variables and to those for which this number is not odd (or not integer) as nonspecial variables. Applying a nondegenerate transformation of class C^∞ and appropriately renumbering the variables, we can make the matrix $A(x)$ in (3) to be of block-diagonal form such that separate blocks contain only groups of equivalent variables. The most important problem is transforming system (3) so that the matrix $A(x)$ becomes Jordan. This cannot be achieved by applying a smooth change variables; therefore, we use so-called shearing transformations ([5–7, 9]).

Definition 1. *Shearing transformations* are transformations of the form

$$y = S(\tilde{r})z, \quad y = S_1(x_1)z, \quad y = S_2(x_2)z, \tag{4}$$

where $S(\eta) = \operatorname{diag}(\eta^{\delta_1}, \eta^{\delta_2}, \ldots, \eta^{\delta_n})$, the δ_q are rationals, $S_j(\eta) = \operatorname{diag}(\eta^{h_{1j}}, \eta^{h_{2j}}, \ldots, \eta^{h_{nj}})$, and the h_{qj} are integers.

Definition 2. *A weakly degenerate transformation* is a change of variables of the form

$$\tilde{r} = du^l, \quad y = T^* z = BVTz. \tag{5}$$

Here, $B = S_1(x)S_2(x)$, $T = T(\tilde{r})$ is a finite product of transformations some of them are shearing transformations of the form $S(\tilde{r})$ and the others are transformations of class C^∞ close to the identity. The number l is a positive integer, and $d = \text{const} > 0$. The transformation V has block-diagonal structure coinciding with the structure of the matrix $A(x)$, $V = \operatorname{diag}(V_1, V_2, \ldots, V_K)$. Moreover, the transformations V_j corresponding to blocks not containing complex conjugate equations (i.e., corresponding $2\omega^{-1}\operatorname{Im}\lambda$ to noninteger numbers) are identity, and those corresponding to blocks containing complex conjugate equations have the form

$$V_j = \operatorname{diag}(e^{i\tilde{h}\alpha}E_1, e^{-i\tilde{h}\alpha}E_1, E_2), \quad \alpha = \arg x_1, \quad e^{i\alpha} = \frac{x_1}{\sqrt{\tilde{r}}},$$

$\tilde{h}$ is the least positive number among the numbers $h_q = \omega^{-1}\operatorname{Im}\lambda_q$ corresponding to the block under consideration. Here, E_1, E_2 are the identity matrices of sizes corresponding to the groups of complex and real variables in the block.

Remark 2. For blocks consisting of special variables, the number $2\tilde{h}$ is odd. In this case, transformation (5) is discontinuous at those points at which x_1 is a real positive number; at all other points at which $x \neq 0$, this transformation is infinitely smooth and nondegenerate of class C^∞.

Theorem 1. *There exists a weakly degenerate transformation* (5) *reducing system* (3) *to the pseudonormal form*

$$\begin{aligned}
\dot{u}_1 &= u_1(i\omega + i\psi(u) + b_1 u^p + c_1 u^{2p}), \\
\dot{u}_2 &= u_2(-i\omega - i\psi(u) + b_1 u^p + c_1 u^{2p}), \\
\dot{z} &= \left(A_0 + uA_1 + \ldots + u^{p-1}A_{p-1} + u^p A_p\right)z.
\end{aligned} \tag{6}$$

Here, $u = u_1 u_2, \psi(u) = \sum_{j=1}^{p}\psi_j u^j$, $\psi_1, \psi_2, \ldots, \psi_p, b_1, c_1$ *are real numbers;* $A_0, A_1, \ldots, A_{p-1}$ *are constant diagonal matrices;* A_p *is a constant Jordan matrix; and* $p = 2ml$.

The pseudonormal form (6) makes it possible to better comprehend the notion of resonance. Thus, consider a formal system of the form (2) whose linear part has pseudonormal form; on the central manifold, we introduce the polar coordinates $u_1 = re^{i\alpha}, u_2 = re^{-i\alpha}$; we have

$$\begin{aligned}
\dot{r} &= b_1 r^{2p+1} + c_1 r^{4p+1} + H_1(r, \alpha, z), \\
\dot{\alpha} &= \omega + \psi(r^2) + H_2(r, \alpha, z), \\
\dot{z} &= (A_0 + r^2 A_1 + \ldots + r^{2p}A_p)z + H(r, \alpha, z).
\end{aligned} \tag{7}$$

Here, the series $H(r, \alpha, z), H_j(r, \alpha, z)$ are sums of monomials of the form $r^L e^{iK\alpha} z^s$, where $L \geq 0, K$ are integers, s is a set of nonnegative integers (we use the notation of [5]), and the following resonance relations hold:

$Ki\omega + (s, \lambda) = 0$ if the monomial is contained in the first or the second equation of the system;

$Ki\omega + (s, \lambda) = \lambda_j$ if the monomial is contained in the jth equation, where $3 \leq j \leq n+2$.

Definition 3. The level of a resonance monomial $r^L e^{iK\alpha} z^s$ contained in the jth equation of system (7), where j, $3 \leq j \leq n+2$, is the number q such that $0 \leq q \leq p-2$, and the following conditions hold:

$$(\tilde{s}, \lambda^h) = \lambda^h_{j-2}, \quad 0 \leq h \leq q, \quad (\tilde{s}, \lambda^{q+1}) \neq \lambda^{q+1}_{j-2}, \tag{8}$$

$$\tilde{s} = (K, s_1, s_2, \ldots, s_n),$$

$$\lambda^h = (i\psi_h, \lambda^h_1, \lambda^h_2, \ldots, \lambda^h_n), \ 0 \leq h \leq p,$$

and $\psi_0 = \omega$, λ_l^h, $1 \le l \le n$, are the diagonal elements of the matrices A_h.

If, for some j such that $3 \le j \le n + 2$, the conditions

$$(\tilde{s}, \lambda^h) = \lambda_{j-2}^h, \quad 0 \le h \le p - 1 \tag{9}$$

hold, then the level of the monomial $r^L e^{iK\alpha} z^s$ in the jth equation is defined to be $p - 1$.

If a monomial $r^L e^{iK\alpha} z^s$ is contained in the equation number $j = 1$ or $j = 2$, then conditions (8) and (9) are assumed to hold for $\lambda_{-1}^h = \lambda_0^h = 0$.

Definition 4. We say that a monomial $r^L e^{iK\alpha} z^s$ is *removable* if either this monomial is of level q for $0 \le q \le p - 2$ and $L > 2(q + 1)$, or this monomial is of level $p - 1$, $L > 2p$, and one of the inequalities $(\tilde{s}, \lambda^p) + b_1(L - 2p) \ne \lambda_{j-2}^p$ (provided that the monomial is contained in an equation of the system whose number j satisfies the condition $3 \le j \le n + 2$), $(\tilde{s}, \lambda^p) + b_1(L - 4p - 1) \ne 0$ (if the monomial is contained in the first equation), and $(\tilde{s}, \lambda^p)$ + $b_1(L - 2p) \ne 0$ (if the monomial is contained in second equation) holds.

A monomial $r^L e^{iK\alpha} z^s$ is said to be *unremovable* if it is not a removable monomial.

Theorem 2. *For system* (7), *there exists a formal transformation which is close to the identity transformation and reduces* (7) *to the following formal system of the same form:*

$$\begin{gathered} \dot{r} = b_1 r^{2p+1} + c_1 r^{4p+1} + \tilde{H}_1(r, \alpha, z), \\ \dot{\alpha} = \omega + \psi(r^2) + \tilde{H}_2(r, \alpha, z), \\ \dot{z} = (A_0 + r^2 A_1 + \ldots + r^{2p} A_p) z + \tilde{H}(r, \alpha, z), \end{gathered} \tag{10}$$

where $\tilde{H}_1, \tilde{H}_2, \tilde{H}$ *are formal series whose summands are unremovable monomials. The summands of the formal series* $\tilde{H}_1, \tilde{H}_2, \tilde{H}$ *in system* (10) *can be represented as* $P(r, \alpha) z^s$, *where* $P(r, \alpha)$ *is a finite sum of monomials of the form* $r^L e^{iK\alpha}$ *with integer* $L \ge 0, K$, *whose maximal degree* L *linearly depends on* $|s|$.

The following theorem solves the problem of finitely smooth equivalence for systems with two purely imaginary eigenvalues.

Theorem 3. *For any integer* $k > 0$, *there exists an integer* N *with the following property: if the Taylor expansions of the right-hand sides of two systems of the form* (1) *having focus on the central manifold and not belonging to the exceptional set* (*see Remark* 1) *differ only in terms of order higher than* N (*i.e., the* N*-jets of these expansions coincide*), *then these systems are locally* C^k*-equivalent, i.e., there exists a nondegenerate (near identity) transformation of class* C^k *reducing one system to the other in a small neighborhood of the origin. The number* N *depends on* k, m, *and* n *and on the coefficients in an interval of the Taylor expansion of* $A(x)$ *whose length is equal to some number* $m_1(m, n)$.

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