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SHORT COMMUNICATIONS

Topological Classification of Structurally Stable 3-Diffeomorphisms with Two-Dimensional Basis Sets

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This paper considers diffeomorphisms given on a smooth closed orientable 3-manifold M^3 and satisfying Smale's Axiom A (A-diffeomorphisms). According to Smale's spectral theorem [1], the nonwandering set $NW(f)$ of an A-diffeomorphism f can be represented as the finite union of pairwise disjoint closed invariant sets, called *basis* sets, each of which contains a dense orbit. The condition that a basis set of an A-diffeomorphism $f\colon M^3 \to M^3$ is zero- or one-dimensional imposes no constraints on the topology of the ambient manifold. In the case where $NW(f)$ contains a basis set $\mathcal{B}$ of dimension higher than 1, this is not so. If $\dim\mathcal{B} = 3$, then f is an Anosov diffeomorphism and the manifold M^3 is diffeomorphic to the 3-torus $\mathbb{T}^3$; a topological classification of such diffeomorphisms was given by Franks and Newhouse in [2], [3]. If $\dim\mathcal{B} = 2$, then, according to [4], $\mathcal{B}$ is either an attractor or a repeller. In this case, the topology of the ambient manifold depends on the topological structure of $\mathcal{B}$.

Recall that a basis set $\mathcal{B}$ of a diffeomorphism f is called an *attractor* if it has a closed neighborhood U for which

$$f(U) \subset \operatorname{int} U, \qquad \bigcap_{j\ge 0} f^j(U) = \mathcal{B}.$$

An attractor of f^{-1} is called a *repeller* of f. According to [5], an attractor $\mathcal{B}$ of a diffeomorphism f is said to be *expanding* if the topological dimension $\dim\mathcal{B}$ equals the dimension of the unstable manifold W^u_x for any point $x \in \mathcal{B}$. A *contracting repeller* of a diffeomorphism f is defined as an expanding attractor of f^{-1}. According to [6], a basis set $\mathcal{B}$ of a diffeomorphism $f\colon M^3 \to M^3$ is called a *surface set* if it is contained in an f-invariant closed surface $M^2_{\mathcal{B}}$ (not necessarily connected) topologically embedded in M^3 and called the *support* of the set $\mathcal{B}$.

It follows from results of [7] that any two-dimensional attractor (repeller) of an A-diffeomorphism $f\colon M^3 \to M^3$ is either an expanding attractor (a contracting repeller) or a surface attractor (a surface repeller). Results of [8] and [9] imply that any manifold M^3 admitting a structurally stable diffeomorphism $f\colon M^3 \to M^3$ with a two-dimensional expanding attractor (contracting repeller) is diffeomorphic to the 3-torus $\mathbb{T}^3$ and, moreover, f is topologically conjugate to a diffeomorphism obtained from an Anosov diffeomorphism by a generalized Smale surgery.

In this paper, we consider the class G of orientation-preserving A-diffeomorphisms $f\colon M^3 \to M^3$ whose nonwandering set consists of only two-dimensional surface basis sets and, for any basis set $\mathcal{B}$ of

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period k, the restriction of f^k to its periodic component preserves orientation. Recall that any basis set $\mathcal{B}$ can be represented as the union $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$, $k \geq 1$ of closed subsets for which

$$f^k(\mathcal{B}_i) = \mathcal{B}_i, \qquad f(\mathcal{B}_i) = \mathcal{B}_{i+1}$$

($\mathcal{B}_{k+1} = \mathcal{B}_1$). The sets $\mathcal{B}_1, \dots, \mathcal{B}_k$ are called the *periodic components* of the basis set $\mathcal{B}$, and the number k is the *period* of $\mathcal{B}$.

Let $\mathcal{A}$ ($\mathcal{R}$) denote the union of all attractors (repellers) belonging to $NW(f)$. According to [10], for any diffeomorphism $f \in G$, the following assertions hold:

(1) the sets $\mathcal{A}$ and $\mathcal{R}$ are nonempty and consist of the same number $n_f \geq 1$ of basis sets;

(2) all periodic components of basis sets have the same period $k_f \geq 1$;

(3) the set $M^3 \setminus (\mathcal{A} \cup \mathcal{R})$ comprises $2n_f k_f$ connected components, the boundary of each component consists of precisely one periodic component of an attractor and one periodic component of a repeller, and the closures of the components are homeomorphic to the manifold $\mathbb{T}^2 \times [0,1]$.

It follows from assertion (3) that the manifold M^3 is homeomorphic to the quotient space M_τ obtained from $\mathbb{T}^2 \times [0,1]$ by identifying the points $(z,1)$ and $(\tau(z),0)$, where $\tau\colon \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism.

Theorem 1. *If a manifold M^3 admits a diffeomorphism f of class G, then M^3 is diffeomorphic to the manifold $M_{\widehat{J}}$, where $\widehat{J}$ is the algebraic automorphism of the torus determined by a matrix J which either is hyperbolic, coincides with the identity matrix*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or coincides with the matrix

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us represent the manifold $M_{\widehat{J}}$ as the orbit space $M_{\widehat{J}} = (\mathbb{T}^2 \times \mathbb{R})/\Gamma$, where $\Gamma = \{\gamma^k,\ k \in \mathbb{Z}\}$ is the group of powers of the diffeomorphism $\gamma\colon \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ defined by

$$\gamma(z,r) = (\widehat{J}(z), r-1).$$

By $p_{\widehat{J}}\colon \mathbb{T}^2 \times \mathbb{R} \to M_{\widehat{J}}$ we denote the natural projection.

Let us introduce a class $\Phi \subset G$ of model diffeomorphisms. Recall that $\mathrm{SL}(2,\mathbb{Z})$ is the subset of $\mathrm{GL}(2,\mathbb{Z})$ consisting of hyperbolic matrices with determinant 1. Let $C \in \mathrm{SL}(2,\mathbb{Z})$ be a hyperbolic matrix for which $CJ = JC$. Given $n,k \in \mathbb{N}$, by $\psi_{n,k}\colon \mathbb{R} \to \mathbb{R}$ we denote the diffeomorphism which is the shift by unit time of the flow $\dot{r} = \sin 2\pi nkr$. For $k=1$, we set $l=0$, and for $k>1$, we let $l \in \{1,\dots,k-1\}$ be a positive integer coprime to k. By $\chi_{k,l}\colon \mathbb{R} \to \mathbb{R}$ we denote the diffeomorphism defined by

$$\chi_{k,l}(r) = r - \frac{l}{k}.$$

We set $\varphi_{n,k,l} = \psi_{n,k}\chi_{k,l}\colon \mathbb{R} \to \mathbb{R}$. Let $\widetilde{\phi}_{C,n,k,l}\colon \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ denote the diffeomorphism defined by

$$\widetilde{\phi}_{C,n,k,l}(z,r) = (\widehat{C}(z), \varphi_{n,k,l}(r)).$$

A direct verification shows that $\widetilde{\phi}_{C,n,k,l}\gamma = \gamma\widetilde{\phi}_{C,n,k,l}$; therefore, the map $\phi_{C,n,k,l}\colon M_{\widehat{J}} \to M_{\widehat{J}}$ defined by

$$\phi_{C,n,k,l} = p_{\widehat{J}}\widetilde{\phi}_{C,n,k,l}p_{\widehat{J}}^{-1},$$

where $p_{\widehat{J}}^{-1}(x)$ is the full preimage of x, is a diffeomorphism. Let Φ denote the set of such diffeomorphisms. By construction, any diffeomorphism in the class Φ is structurally stable.

Theorem 2. *Two diffeomorphisms* $\phi_{C,n,k,l}\colon M_{\widehat{J}} \to M_{\widehat{J}}$ *and* $\phi_{C',n',k',l'}\colon M_{\widehat{J'}} \to M_{\widehat{J'}}$ *of class* Φ *are topologically conjugate if and only if*

(1) *there exists a matrix* $H \in \mathrm{GL}(2,\mathbb{Z})$ *for which* $CH = HC'$;

(2) $k = k'$, $n = n'$, *and at least one of the following conditions holds*:

- $JH = HJ'$ *and* $l = l'$;
- $J^{-1}H = HJ'$ *and* $k - l = l'$;
- $J^{-1}H = HJ'$ *and* $l = l' = 0$.

Recall that two diffeomorphisms $f\colon M^n \to M^n$ and $f'\colon M'^n \to M'^n$ are said to be Ω-*conjugate* if there exists a homeomorphism $h\colon M^n \to M'^n$ such that

$$h(NW(f)) = NW(f'), \qquad hf|_{NW(f)} = f'h|_{NW(f)}.$$

Theorem 3. *Any diffeomorphism of class* G *is* Ω*-conjugate to a diffeomorphism of class* Φ.

Theorem 4. *Any structurally stable diffeomorphism of class* G *is topologically conjugate to a diffeomorphism of class* Φ.

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