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Singular Boundary Value Problem for the Integrodifferential Equation in an Insurance Model with Stochastic Premiums: Analysis and Numerical Solution

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Abstract—A singular boundary value problem for a second-order linear integrodifferential equation with Volterra and non-Volterra integral operators is formulated and analyzed. The equation is defined on $\mathbb{R}_+$, has a weak singularity at zero and a strong singularity at infinity, and depends on several positive parameters. Under natural constraints on the coefficients of the equation, existence and uniqueness theorems for this problem with given boundary conditions at singular points are proved, asymptotic representations of the solution are given, and an algorithm for its numerical determination is described. Numerical computations are performed and their interpretation is given. The problem arises in the study of the survival probability of an insurance company over infinite time (as a function of its initial surplus) in a dynamic insurance model that is a modification of the classical Cramer–Lundberg model with a stochastic process rate of premium under a certain investment strategy in the financial market. A comparative analysis of the results with those produced by the model with deterministic premiums is given.

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Keywords: dynamic insurance models; Cramer–Lundberg model with stochastic premiums; survival probability of an insurance company as a function of its initial surplus; second-order linear integrodifferential equation on a half-line; singular boundary value problem with constraints; related singular boundary value problems for ordinary differential equations; existence, uniqueness, and behavior of a solution; numerical solution algorithm.

1. INTRODUCTION: DESCRIPTION OF MATHEMATICAL INSURANCE MODELS INVOLVING RISKY INVESTMENTS

We consider a constrained singular boundary value problem (BVP) for a second-order linear integrodifferential equation (IDE). Overall, the problem is formulated and studied for the first time. It arises in the study of paying capacity in a dynamic insurance model assuming surplus investment in the financial market (see [1]). The model is based on a modification of the classical Cramer–Lundberg collective risk model (a description of collective risk dynamic processes can be found, for example, in [2, Chapters 7–9]).

In the classical Cramer–Lundberg model, the process describing surplus variations (risk process) consists of two ones, namely, a deterministic premium process and a compound Poisson process describing claims. If we discard the simplifying assumption that the premium process is deterministic, the most natural assumption is that it is also a compound Poisson process with parameters other than those of the claim process. Following [3], the corresponding model, which is briefly described below, is referred to as the Cramer–Lundberg model with stochastic premiums.

In the case of surplus investment in the financial market, the surplus variation is affected (in addition to two indicated factors) by at least another two, namely, by variations in the asset prices and by various investment possibilities. In what follows, we consider only investment strategies with a constant structure, when a fixed fraction of the surplus is invested in risky assets (stocks whose prices are simulated by a geometric Brownian motion) while the rest of the surplus is invested in a risk-free asset (bank account with a constant interest rate). If the original risk process is governed by the classical Cramer–Lundberg model, then the corresponding model with financial market immersion is referred to as *model I*. If the original risk process is governed by the Cramer–Lundberg model with stochastic premiums, then the corresponding model with investment is called *model II*.

A central issue in dynamic insurance models is the determination or estimation of the survival probability, which is a traditional deterministic characteristic of paying capacity. In most models, the evolution of the surplus of an insurance company is governed by a homogeneous Markov process with continuous time. If the surplus is invested in risky assets, this process is described by a stochastic differential equation (SDE). In such processes, for the survival probability regarded as a function of the initial surplus, an IDE can be obtained under certain assumptions concerning the properties of this function by applying the apparatus of generating operators (see, e.g., [4] and the references therein).

IDEs are defined on $\mathbb{R}_+$ ~~and are nonnegative on $\mathbb{R}_+$~~. Their solutions not exceeding unity and tending to unity at infinity with given conditions at the left endpoint (if they exist) determine the desired probability, which can be proved by applying probabilistic methods (for more detail, see [5] and the references therein). Specifically, problems for models I and II were justified in the above sense in [5].

This paper focuses primarily on model II. For comparison purposes, we also describe model I and outline previously obtained major results concerning this model.

Below, we use the following notation: $\mathbf{P}(A)$ is the probability of an event A and $\mathbf{E}X$ is the expectation of a random variable X. The other notation will be introduced when necessary.

1.1. Cramer–Lundberg Model with Stochastic Premiums and Singular IDE on a Half-Line for Model with Investments

The Cramer–Lundberg model with stochastic premiums can be briefly described as follows (for more detail, see [1; 3; 2, Section 9.5]).

Let a continuous-time risk process be given by

$$R_t = u + \sum_{i=1}^{N_1(t)} C_i - \sum_{j=1}^{N(t)} Z_j, \quad t \ge 0. \tag{1.1}$$

Here, R_t is the surplus of an insurance company at time t; u is the initial surplus; the first sum on the right-hand side represents the aggregate premiums up to time t; $N_1(t)$ is a Poisson process with intensity $\lambda_1 > 0$ ($\mathbf{E}N_1(t) = \lambda_1 t$, $N_1(0) = 0$) that, for any $t > 0$, determines the number of premiums charged over the time interval $(0, t]$, C_i are independent identically distributed random variables with a distribution function $G(y)$ ($G(0) = 0$, $\mathbf{E}C_1 = n < \infty$) that determine the premium sizes and are assumed to be independent of $N_1(t)$ (C_i is the premium with index i at the moment of the ith jump in $N_1(t)$); the second sum represents the aggregate claims; $N(t)$ is a Poisson process with intensity $\lambda > 0$ ($\mathbf{E}N(t) = \lambda t$, $N(0) = 0$; $\lambda \le \lambda_1$) that, for any $t > 0$, determines the number of claims received on the time interval $(0, t]$; and Z_j are independent identically distributed random variables with a distribution function $F(x)$ ($F(0) = 0$, $\mathbf{E}Z_1 = m < \infty$) that determine the claim sizes and are assumed to be independent of $N(t)$ (Z_j is the claim with index j at the moment of the jth jump in $N(t)$). Overall, the aggregate premium and aggregate claim processes are also assumed to be independent.

Remark 1. The independence of these processes is a simplifying assumption as compared to the more general situation described, for example, in [2, Section 9.5], where $N(t) \le N_1(t)$ for any $t \ge 0$; i.e., the total number of claims cannot be larger than the total number of premiums. However, assuming that the company can have a premium and claim history prior to the zero time, this condition is replaced by a weaker one imposed on the expected rate of claims, namely, $\lambda \le \lambda_1$.

In the subsequent presentation, we need a quantity (called the relative safety load) characterizing the expected specific profit of an insurance company per unit time (for the classical Cramer–Lundberg model, see Definition 2 below).

Definition 1. The safety load (coefficient) for risk process (1.1) is defined as

$$\rho_2 = (\lambda_1 n - \lambda m)/(m\lambda). \tag{1.2}$$

To describe model II, we consider a situation when an insurance company with risk process (1.1) continuously invests a constant fraction α of its surplus ($0 < \alpha \le 1$) in stocks with prices described by the SDE

$$dS_t = S_t(\mu dt + \sigma dw_t), \quad t \ge 0.$$

Here, S_t is the stock price at time t, μ is the expected stock return rate, $0 < \sigma$ is the variability, $\{w_t\}$ is a standard Wiener process, or a Brownian motion (S_t is known as a geometric Brownian motion). Assume that the remaining fraction of the surplus is invested in a risk-free asset, namely, in a bank account with an interest rate r, $0 < r < \mu$, whose evolution is governed by the ordinary differential equation (ODE)

$$dB_t = rB_t dt, \quad t \geq 0,$$

where B_t is the amount of money in the bank account at time t.

In this case, the dynamics of the surplus (resulting risk process) is described by the initial value problem for the SDE

$$dX_t = [(\alpha\mu + (1-\alpha)r)dt + \alpha\sigma dw_t]X_t + dR_t, \quad t \geq 0, \quad X_0 = u. \tag{1.3}$$

Here, X_t is the cost of the portfolio at time t (for more detail on problem (1.3), see, e.g., [4] and the references therein).

As a measure of the company's paying capacity, we use the survival probability $\varphi(u)$ (as a function of u) in infinite time:

$$\varphi(u) = \mathbf{P}\{X_t \geq 0, t > 0\},$$

where $X_0 = u$ for $u \geq 0$; for $u < 0$, $\varphi(u) \equiv 0$.

Note that, after making the substitutions

$$a = \alpha\mu + (1-\alpha)r > 0, \quad b = \alpha\sigma > 0, \tag{1.4}$$

the SDE in (1.3) can be treated as an equation for the evolution of the surplus completely invested in stocks with the expected stock return rate a and volatility b. In what follows, unless otherwise stated, we assume without loss of generality that the company follows this strategy in the financial market.

An equation for the survival probability $\varphi(u)$ in risk process (1.3) was derived in [1]. In view of the above remark and notation (1.4), it has the form

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) - \lambda\left[\varphi(u) - \int_0^u \varphi(u-x)dF(x)\right] - \lambda_1\left[\varphi(u) - \int_0^\infty \varphi(u+y)dG(y)\right] = 0, \tag{1.5}$$
$$0 < u < \infty.$$

In the case of exponential distributions of premium and claim sizes, i.e., when

$$F(x) = 1 - \exp(-x/m), \quad m > 0, \tag{1.6}$$

$$G(y) = 1 - \exp(-y/n), \quad n > 0, \tag{1.7}$$

IDE (1.5) becomes

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] - \lambda_1[\varphi(u) - (J_{1,n}\varphi)(u)] = 0, \quad u \in \mathbb{R}_+. \tag{1.8}$$

Here, J_m and $J_{1,n}$ are Volterra and non-Volterra integral operators, respectively:

$$(J_m\varphi)(u) = \frac{1}{m}\int_0^u \varphi(u-x)\exp(-x/m)dx = \frac{1}{m}\int_0^u \varphi(s)\exp(-(u-s)/m)ds, \tag{1.9}$$

$$(J_{1,n}\varphi)(u) = \frac{1}{n}\int_0^\infty \varphi(u+y)\exp(-y/n)dy = \frac{1}{n}\int_u^\infty \varphi(s)\exp(-(s-u)/n)ds, \tag{1.10}$$

where J_m, $J_{1,n} : C[0, \infty) \longrightarrow C[0, \infty)$ and $C[0, \infty)$ is the linear space of continuous bounded functions on $\mathbb{R}_+$. Formula (1.10) is convenient for proving some results and can be treated as a transformation of an advanced-argument non-Volterra operator into a singular Volterra operator (for the definitions of Volterra operators for classes of systems of functional differential equations (containing IDEs as a special case), including nonlinear and singular operators, see, for example, [6, 7] and the references therein).

This paper deals with a singular BVP on $\mathbb{R}_+$ for IDE (1.8) (with (1.6), (1.7)) involving Volterra and non-Volterra integral operators. A complete theoretical analysis of this BVP with constraints is provided, a numerical method for its solution is proposed, and numerical results are presented.

Additionally, the theoretical and numerical results are compared with those obtained earlier for a similar but simpler problem concerning model I with the original risk process governed by the classical Cramer–Lundberg model. In this case, the IDE involves only a Volterra integral operator (see [8–10] and the references therein). Below, a singular problem for a linear IDE for the survival probability corresponding to model I is formulated and the main results obtained for it in [8, 9] are briefly outlined.

1.2. Classical Cramer–Lundberg Model; Singular Problem on a Half-Line for a Model with Investments and Results of Its Analysis

In a classical risk process, the premium rate compound Poisson process is replaced by a deterministic process with a constant intensity $c > 0$; i.e., (1.1) is replaced by the process

$$R_t = u + ct - \sum_{j=1}^{N(t)} Z_j, \quad t \geq 0. \tag{1.11}$$

Following, for example, [11, p. 289] (see also [12]), we obtain the classical definition of the safety load for this risk process (cf. Definition 1).

Definition 2. The safety load for risk process (1.11) is

$$\rho_1 = (c - m\lambda)/(m\lambda) = c/(m\lambda) - 1. \tag{1.12}$$

Consider model I under the same assumptions about the investment portfolio structure as in model II and under assumption (1.6) about the exponential distribution of claim sizes. Then the survival probability $\varphi(u)$ satisfies the following singular problem for IDE (see [8, 9]):

$$(b^2/2)u^2\varphi''(u) + (au + c)\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \quad u \in \mathbb{R}_+, \tag{1.13}$$

$$\left|\lim_{u \to +0} \varphi(u)\right| < \infty, \quad \left|\lim_{u \to +0} \varphi'(u)\right| < \infty, \quad \lim_{u \to +0} [c\varphi'(u) - \lambda\varphi(u)] = 0, \tag{1.14}$$

$$0 \leq \varphi(u) \leq 1, \quad u \in \mathbb{R}_+, \tag{1.15}$$

$$\lim_{u \to \infty} \varphi(u) = 1, \quad \lim_{u \to \infty} \varphi'(u) = 0. \tag{1.16}$$

Here, we used notation (1.9) for the Volterra integral operator J_m, $J_m : C[0, \infty)$ ~~with~~ $[0, \infty)$, and all the parameters a, b^2, c, λ, and m are real and positive constants unless otherwise stated.

The third boundary condition in (1.14) is a consequence of the first two and IDE (1.13). For solutions of the latter, conditions (1.14) imply that $\lim_{u \to +0}[u^2\varphi''(u)] = 0$, which ensures that this IDE degenerates as $u \longrightarrow +0$: any solution of singular problem (1.13), (1.14) without initial data must satisfy Eq. (1.13) up to the singular point $u = 0$.

"Truncated" problem (1.13)–(1.15) always has the trivial solution $\varphi(u) \equiv 0$, while a nontrivial solution is singled out by condition (1.16).

In what follows, we use the equivalence of conditions (1.14) to the parametrized limiting conditions

$$\lim_{u \to +0} \varphi(u) = C_0, \quad \lim_{u \to +0} \varphi'(u) = \lambda C_0/c, \tag{1.17}$$

where C_0 is a parameter to be determined such that $0 < C_0 < 1$.

An important inequality for the study of models I and II is

$$2a/b^2 > 1, \tag{1.18}$$

which is known as the asset portfolio reliability condition.

Below are the basic consequences of the results of [8, 9].

Lemma (see [8, 9]). *Let the parameters* a, b, c, λ, *and* m *in IDE* (1.13) *be fixed constants such that* $c > 0$, $b \neq 0$, $\lambda \neq 0$, $m > 0$, *and* $a \in \mathbb{R}$.

Then the following assertions hold:

(i) *For any fixed value of the parameter* C_0 ($C_0 \in \mathbb{R}$), *the singular "integrodifferential" initial value problem* (1.13), (1.17) *is equivalent to a singular "differential" Cauchy problem of the form*

$$(b^2/2)u^2\varphi'''(u) + [c + (b^2 + a)u + b^2u^2/(2m)]\varphi''(u) + [a - \lambda + c/m + au/m]\varphi'(u) = 0, \quad 0 < u < \infty, \tag{1.19}$$

$$\lim_{u\to+0}\varphi(u) = C_0, \quad \lim_{u\to+0}\varphi'(u) = \lambda C_0/c, \quad \lim_{u\to+0}\varphi''(u) = [m(\lambda-a)-c]\lambda C_0/(mc^2). \tag{1.20}$$

(ii) *The singular Cauchy problem* (1.19), (1.20) (*equivalent to the initial value problem* (1.13), (1.17)) *has a unique solution* $\varphi(u, C_0)$ *and, for small u, this solution can be represented by the asymptotic series*

$$\varphi(u, C_0) \sim C_0\left[1+\frac{\lambda}{c}\left(u+\sum_{k=2}^{\infty}D_k u^k/k\right)\right], \quad u \sim +0, \tag{1.21}$$

where D_k are constant coefficients independent of C_0. They are determined by the formal substitution of series (1.21) *into ODE* (1.19), *which yields the recurrence formulas*

$$D_2 = -[(a-\lambda)/c+1/m], \tag{1.22}$$

$$D_3 = -[D_2(b^2+2a-\lambda+c/m)+a/m]/(2c), \tag{1.23}$$

$$D_k = -\{D_{k-1}[(k-1)(k-2)b^2/2+(k-1)a-\lambda+c/m]+D_{k-2}[(k-3)b^2/2+a]/m\}/[c(k-1)], \quad k = 4, 5, \ldots. \tag{1.24}$$

(iii) *All the solutions of ODE* (1.19) (*of initial value problem* (1.13), (1.17) *for the IDE*) *have finite limits as* $u \longrightarrow \infty$ *if and only if condition* (1.18) *holds.*

Theorem (see [8, 9]). *Let all the parameters* a, b^2, c, λ, *and* m *in IDE* (1.13) *be fixed positive constants, and let condition* (1.18) *be satisfied.*

Then the following assertions hold:

(i) *The singular linear problem* (1.13)–(1.16) *has a unique solution* $\varphi(u)$ *that is an infinitely differentiable monotonically increasing function on* $\mathbb{R}_+$.

(ii) *The solution* $\varphi(u)$ *can be obtained by solving the singular initial value problem* (1.13), (1.17) *with a parameter* (*equivalent to the singular Cauchy problem* (1.19), (1.20) *with a parameter*), *where the parameter* $C_0>0$ *is determined by requirements* (1.16) *as a condition for the normalization of the solution at infinity, while constraints* (1.15) *are satisfied for this solution automatically.*

(iii) *If*

$$m(a-\lambda)+c>0, \tag{1.25}$$

then the solution $\varphi(u)$ *is a concave function on* $\mathbb{R}_+$, *and if*

$$m(a-\lambda)+c\le 0, \tag{1.26}$$

then $\varphi(u)$ *is convex on the interval* $[0, \hat{u}]$, *where* $0<\hat{u}$ *is a point of inflection. Condition* (1.25) *holds, for example, if safety load* (1.12) *is positive, i.e., if*

$$c-\lambda m>0. \tag{1.27}$$

(iv) *For small* u, *the solution* $\varphi(u)$ *has asymptotic representation* (1.21), *where* C_0: $0<C_0<1$.

(v) *For large* u, *the solution* $\varphi(u)$ *has the representation*

$$\varphi(u) = 1-Ku^{1-2a/b^2}[1+o(1)], \quad u \longrightarrow \infty, \tag{1.28}$$

where $K = C_0\tilde{K}>0$ (*the values of* $C_0>0$ *and* $\tilde{K}>0$ *cannot be found by local analysis methods*).

Definition 3. If condition (1.27) does not hold, i.e., $\rho_1\le 0$, then the quantity

$$i_{r,\mathrm{I}} = ma+c-m\lambda \tag{1.29}$$

is called the risk factor (index) of model I: the most risky situation occurs when $i_{r,I}\le 0$ (i.e., inequality (1.26) holds) and the initial surplus values are small.

Remark 2 (algorithm for solving problem (1.13)–(1.16)). Based on the above assertions, the solution of problem (1.13)–(1.16) can be found by solving the auxiliary singular Cauchy problem (1.19), (1.20) with a parameter C_0 whose value is determined by requirements (1.16) as a condition for the normalization of the solution at infinity.

In practice, this can be implemented, for example, as follows. Setting $\psi(u) = \varphi'(u)$, we consider the auxiliary singular Cauchy problem

$$(b^2/2)u^2\psi''(u) + [c + (b^2 + a)u + b^2u^2/(2m)]\psi'(u) + [a - \lambda + c/m + au/m]\psi(u) = 0, \quad u \in \mathbb{R}_+, \tag{1.30}$$

$$\lim_{u \to +0} \psi(u) = 1, \quad \lim_{u \to +0} \psi'(u) = [m(\lambda - a) - c]/(mc). \tag{1.31}$$

This problem has a unique solution $\psi(u)$ and, for small u, this solution can be represented by the asymptotic series

$$\psi(u) \sim 1 + \sum_{k=2}^{\infty} D_k u^{k-1}, \quad \psi'(u) \sim \sum_{k=2}^{\infty} (k-1)D_k u^{k-2}, \quad u \sim +0, \tag{1.32}$$

where the coefficients D_k are determined in (1.22)–(1.24). Expansions (1.32) are used for approximate transfer of boundary conditions (1.31) to a finite point $u_0 > 0$. The solution $\varphi(u)$ of original problem (1.13)–(1.16) is found using the relation

$$\varphi(u) = \left[1 + \frac{\lambda}{c}\int_0^u \psi(s)ds\right]\left[1 + \frac{\lambda}{c}\int_0^\infty \psi(s)ds\right]^{-1}, \tag{1.33}$$

where $\psi(u)$ is the solution of problem (1.30), (1.31).

Numerical results produced by model I can be found in [8–10] and below in this paper.

Numerical results obtained with models I and II are compared in Section 6 under the following conditions.

Conditions for comparison of models I and II: (i) The values of the parameters a, b^2, λ, m in model II are the same as in model I; (ii) the values of c in model I are related to λ_1 and n in model II by the formula

$$\lambda_1 n = c; \tag{1.34}$$

i.e., the expected premium rates are identical in both models.

Preliminarily, we compare the exact solutions for the Cramer–Lundberg models under relation (1.34).

1.3. Comparison of Exact Solutions for Models without Investments

1.3.1. Exact solution of the classical Cramer–Lundberg model. For $a = b = 0$, problem (1.13)–(1.16) yields a "degenerate" singular problem for a first-order IDE:

$$c\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] = 0, \quad 0 \le u < \infty, \tag{1.35}$$

$$c\varphi'(0) - \lambda\varphi(0) = 0, \quad \lim_{u \to \infty} \varphi(u) = 1, \tag{1.36}$$

where all the parameters are positive and condition (1.27) holds. This problem corresponds to the classical Cramer–Lundberg model with an exponential distribution of claim sizes. It is well known for this model (see, e.g., [12, part 1; 11, pp. 229–230]) that the survival probability $\varphi_1(u)$ (as the solution of IDE (1.35)) is given by the exact formula

$$\varphi_1(u) = 1 - \frac{\lambda m}{c}\exp\left(-\frac{c - \lambda m}{mc}u\right), \quad 0 \le u < \infty. \tag{1.37}$$

It is easy to see that this function solves problem (1.35), (1.36), or an equivalent parametrized problem for a second-order ODE:

$$\varphi''(u) + [(c - m\lambda)/(mc)]\varphi'(u) = 0, \quad 0 < u < \infty, \tag{1.38}$$

$$\varphi(0) = C_0, \quad \varphi'(0) = \lambda C_0/c, \quad \lim_{u \to \infty} \varphi(u) = 1. \tag{1.39}$$

This implies $C_0 = (c - m\lambda)/c$ and $0 < C_0 < 1$ (see also the discussion of this case in [9]).

Remark 3. For problems (1.35), (1.36) and (1.38), (1.39), $c = \lambda m$ is a critical bifurcation parameter. Specifically, for $c \le \lambda m$, these problems have no solutions (for the survival probability in the classical

Cramer–Lundberg model, it is well known that $\varphi(u) \equiv 0$ for $c \le \lambda m$ (see, e.g., [12]), which corresponds to the trivial solution of IDE (1.35) (ODE (1.38)), while the normalization condition at infinity applies only to nontrivial solutions). For the corresponding model with investments (model I), the above results from [8, 9] imply that the inequality $\varphi(u) > 0$ on $\mathbb{R}_+$ holds even for a nonpositive risk factor, i.e., for $i_{r,\mathrm{I}} \le 0$, where $i_{r,\mathrm{I}}$ is given by (1.29).

1.3.2. Exact solution of the Cramer–Lundberg model with stochastic premiums. For $a = b = 0$, IDE (1.8) degenerates into the integral equation

$$(\lambda + \lambda_1)\varphi(u) = \lambda(J_m\varphi)(u) + \lambda_1(J_{1,n}\varphi)(u), \quad u \in \mathbb{R}_+. \tag{1.40}$$

This equation was derived and investigated in [3] (while IDE (1.8) was not studied in [3]). It was shown that, for positive safety load (1.2), i.e., for

$$\lambda_1 n - \lambda m > 0, \tag{1.41}$$

the survival probability $\varphi_2(u)$ (as a positive solution of Eq. (1.40) on $\mathbb{R}_+$ that does not exceed unity) is given by the exact formula (see also [2, Section 9.5])

$$\varphi_2(u) = 1 - \frac{\lambda(n+m)}{n(\lambda+\lambda_1)}\exp\left(-\frac{\lambda_1 n - \lambda m}{mn(\lambda+\lambda_1)}u\right), \quad 0 \le u < \infty. \tag{1.42}$$

Solution (1.42) also follows from the results of this paper.

1.3.3. Comparison of the exact solutions for Cramer–Lundberg models. A comparison of formulas (1.37) and (1.42) yields the following result.

Proposition 1. *Let the parameters* λ, m, c, λ_1, *and n be fixed positive numbers, and let*

$$c = \lambda_1 n > \lambda m, \tag{1.43}$$

*which, for safety loads (*1.2), (1.12), *implies that* $\rho_1 = \rho_2 > 0$.

Then

$$\varphi_2(u) < \varphi_1(u) \quad \text{for any finite} \quad u \ge 0. \tag{1.44}$$

Moreover, given a fixed $c > 0$, *if* $n \longrightarrow 0$, $\lambda_1 = c/n \longrightarrow \infty$ *(or* $n \longrightarrow \infty$, $\lambda_1 = c/n \longrightarrow 0$), *then* $\varphi_2(u) \longrightarrow \varphi_1(u)$ ($\varphi_2(u) \longrightarrow 0$, *respectively) for any* $u \in \mathbb{R}_+$.

This assertion is illustrated in Fig. 1, where the plots of solutions (1.37) and (1.42) are denoted by *1* and *2*, respectively. In all the cases, $\lambda = 0.09$, $m = 0.5$, and $c = 0.1$. For the other parameters and obtained solutions, the values are as follows: (1) in Fig. 1a, $\lambda_1 = 1.0$, $n = 0.1$; $\varphi_1(0) = 0.55$, $\varphi_1'(0) = 0.495$, $\varphi_2(0) \approx 0.50459$, $\varphi_2'(0) \approx 0.49996$; and the difference $\varphi_1(u) - \varphi_2(u)$ is maximal at the point $u = 0$ and is roughly equal to 0.045413; and (2) in Fig. 1b, $\lambda_1 = 0.25$, $n = 0.4$; $\varphi_1(0)$ and $\varphi_1'(0)$ are the same as in Fig. 1a, $\varphi_2(0) \approx 0.404412$, $\varphi_2'(0) \approx 0.481726$; and $\varphi_1(u) - \varphi_2(u)$ is maximal at the point $u \approx 0.09$ and is roughly equal to 0.146189.

Note once again that the main goal of this work is to formulate and investigate the singular problem for IDE (1.8) corresponding to model II. It should be expected that the numerical results produced by models I and II converge to each other under condition (1.34) (the expected premium rates are identical) and if $\lambda_1 \gg \lambda$ (the premiums are much more frequent than the claims) and $m \gg n$ (the claim sizes are much larger than the premium sizes). Note that, for models I and II, safety loads (1.12) and (1.2) in the corresponding Cramer–Lundberg models are generally not assumed to be positive.

Preliminarily, two remarks have to be made about related problems that are of interest on their own in risk theory, but are not addressed in this paper. They include (1) problems concerning the optimal control of investments (for model I, a review of results on this subject can be found, e.g., in [4]; for additional results, see [8]; for model II, see [13]) and (2) "degenerate" problems. For model II, degenerate problems arise when one or several parameters in IDE (1.8) vanish (~~except for~~ the case of no investments indicated in Section 3.2). For model I, degenerate problems were investigated in detail in [10]. Note that, if $c = 0$ in IDE (1.13) and $\lambda_1 = 0$ in IDE (1.8), then both models coincide (briefly outlined in [10], this case corresponds to a charity rather than insurance; a more detailed study of this case and a comparison of degenerate cases for models I and II will be addressed elsewhere).

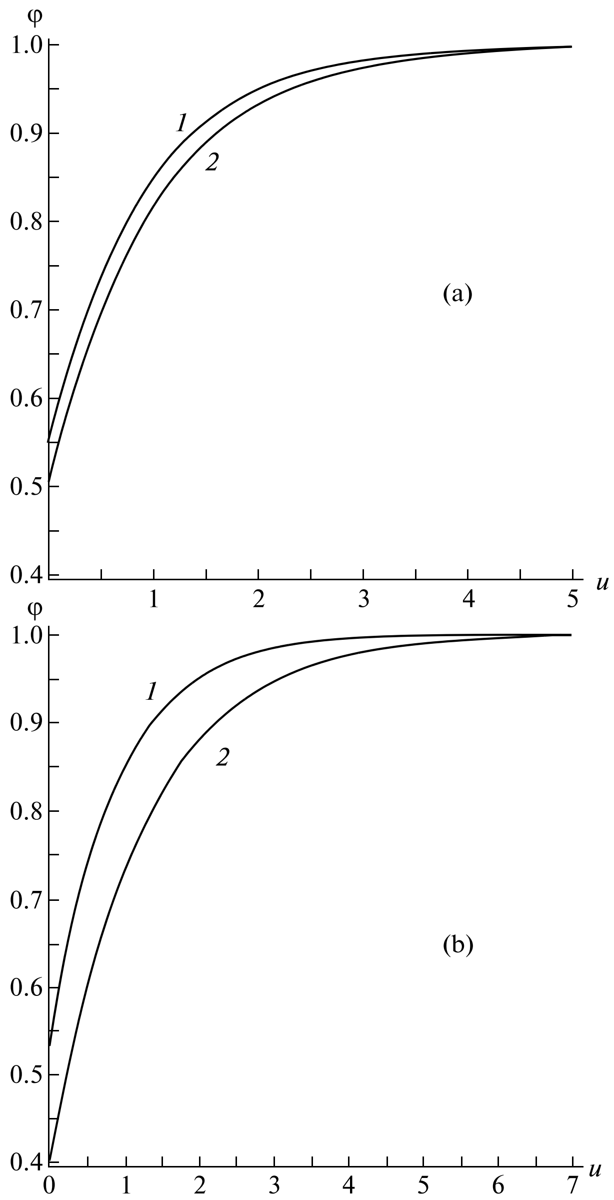

Fig. 1.

2. STATEMENT OF THE BASIC SINGULAR PROBLEM FOR A LINEAR INTEGRODIFFERENTIAL EQUATION AND UNIQUENESS OF ITS SOLUTION

2.1. Statement of the Problem

Consider the second-order linear IDE (1.8) defined on the nonnegative real half-line, i.e.,

$$(b^2/2)u^2\varphi''(u) + au\varphi'(u) - \lambda[\varphi(u) - (J_m\varphi)(u)] - \lambda_1[\varphi(u) - (J_{1,n}\varphi)(u)] = 0, \quad u \in \mathbb{R}_+. \tag{2.1}$$

Here, unless otherwise stated, a, b^2, λ, λ_1, m, and n are real positive numbers, and J_m and $J_{1,n}$ are the Volterra and non-Volterra integral operators defined by (1.9) and (1.10), respectively.

The goal is to find a nonnegative nondecreasing solution of IDE (2.1) on $\mathbb{R}_+$ that satisfies the conditions

$$\left|\lim_{u \to +0} \varphi(u)\right| < \infty, \quad \lim_{u \to +0}[u\varphi'(u)] = 0, \tag{2.2}$$

$$(\lambda+\lambda_1)\lim_{u\to+0}\varphi(u) = \lambda_1(J_{1,n}\varphi)(0) = \frac{\lambda_1}{n}\int_0^\infty\varphi(y)\exp(-y/n)dy, \tag{2.3}$$

$$0\le\varphi(u)\le 1 \quad \forall u\in\mathbb{R}_+, \tag{2.4}$$

$$\lim_{u\to+\infty}\varphi(u) = 1, \quad \lim_{u\to+\infty}\varphi'(u) = 0. \tag{2.5}$$

IDE (2.1) always has the trivial solution $\varphi(u)\equiv 0$, which satisfies conditions (2.2)–(2.4). A nontrivial solution is specified by condition (2.5).

Limiting conditions (2.2) and nonlocal relation (2.3) imply $\lim_{u\to+0}[u^2\varphi''(u)] = 0$, which ensures that IDE (2.1) degenerates as $u\longrightarrow +0$ (the solution $\varphi(u)$ of singular problem (2.1)–(2.5) (if it exists) must satisfy IDE (2.1) up to the singular point $u = 0$). Note that nonlocal condition (2.3) is a consequence of the presence of non-Volterra operator (1.10).

For the subsequent analysis, limiting conditions (2.2) are rewritten in the equivalent parametrized form

$$\lim_{u\to+0}\varphi(u) = C_0, \quad \lim_{u\to 0+}[u\varphi'(u)] = 0, \tag{2.6}$$

where C_0 is the parameter to be determined; moreover, $0\le C_0\le 1$ for the desired solution of problem (2.1)–(2.5).

2.2. Uniqueness of a Solution and Two-Sided Estimates

Some results concerning the solution of the problem can be obtained a priori.

Lemma 1. *Let* b^2, λ, λ_1, m, *and* n *in* (2.1) *be real positive constants and* a *be a number of any sign* ($a\in\mathbb{R}$). *Given fixed values of these parameters, assume that the constrained singular linear BVP* (2.1)–(2.5) *has a solution* $\varphi(u)$. *Then this solution is unique.*

Proof. Assume the opposite: let $\tilde{\varphi}(u)$ be another solution of problem (2.1)–(2.5). Then the following two cases are possible: variant I

$$\lim_{u\to+0}\varphi(u) = \lim_{u\to+0}\tilde{\varphi}(u) \tag{2.7}$$

and variant II

$$\lim_{u\to+0}\varphi(u)\ne\lim_{u\to+0}\tilde{\varphi}(u). \tag{2.8}$$

Consider the case of (2.7). Since the problem is linear, IDE (2.1) has a nontrivial solution $\hat{\varphi}(u)$ such that

$$\lim_{u\to+0}\hat{\varphi}(u) = \lim_{u\to\infty}\hat{\varphi}(u) = 0.$$

Let $0<\hat{u}$ be a maximizer of this solution: $\hat{\varphi}(\hat{u}) = \max_{u\in(0,\infty)}\hat{\varphi}(u) > 0$ (if $\hat{\varphi}(u)$ does not take positive values on $\mathbb{R}_+$, we consider $-\hat{\varphi}(u)$). Then $\hat{\varphi}'(\hat{u}) = 0$ and $\hat{\varphi}''(\hat{u})\le 0$. However, (2.1) yields a contradiction, namely,

$$(b^2/2)\hat{u}^2\hat{\varphi}''(\hat{u}) = \lambda[\hat{\varphi}(\hat{u})-(J_m\hat{\varphi})(\hat{u})]+\lambda_1[\hat{\varphi}(\hat{u})-(J_{1,n}\hat{\varphi})(\hat{u})]\ge\lambda\hat{\varphi}(\hat{u})\exp(-\hat{u}/m)>0. \tag{2.9}$$

Consider the case of (2.8). It is easy to see that there is a linear combination of solutions $\hat{\varphi}(u) = c_1\varphi(u) + c_2\tilde{\varphi}(u)$ such that $\hat{\varphi}(u)\not\equiv 1$ is a solution of IDE (2.1) satisfying the limiting conditions

$$\lim_{u\to+0}\hat{\varphi}(u) = \lim_{u\to\infty}\hat{\varphi}(u) = 1.$$

If there exists $u\in\mathbb{R}_+$ such that $\hat{\varphi}(u)>1$, then we proceed in the same manner as in variant I. Assuming that $\hat{\varphi}(u)\le 1$ $\forall u>0$, we obtain the inequality

$$(J_{1,n}\hat{\varphi})(0) = \frac{1}{n}\int_0^\infty\hat{\varphi}(y)\exp(-y/n)dy<1.$$

Taking into account $\lim_{u\to+0}\hat{\varphi}(u) = 1$, we obtain a contradiction: condition (2.3) is not satisfied.

Lemma 2. *Let* b^2, λ, λ_1, m, *and* n *in* (2.1) *be real positive numbers and a be a number of any sign* ($a \in \mathbb{R}$). *Given fixed values of these parameters, assume that the unconstrained singular linear BVP* (2.1)–(2.3), (2.5) *has a solution* $\varphi(u)$. *Then constraints* (2.4) *hold for this solution.*

Proof. (i) We prove that $\varphi(u) < 1$ for any finite $u \in \mathbb{R}_+$. For this purpose, we first show that $\varphi(u)$ does not take its largest positive value as $u \longrightarrow +0$. Assume the opposite: $\varphi(u) \le C_0$ for any $u \in \mathbb{R}_+$, where $\lim_{u\to+0}\varphi(u) = C_0 > 0$. However, (2.3) yields a contradiction: $(\lambda + \lambda_1)C_0 \le \lambda_1 C_0$, whence $C_0 \le 0$.

Now $\varphi(u) \ge 1$ at some finite point $u > 0$. Then $\varphi(u)$ must have a maximum on $\mathbb{R}_+$ that exceeds 1. Proceeding as in the proof of inequality (2.9) in Lemma 1, we obtain a contradiction at the maximizer.

(ii) In a similar fashion, we can prove that $\varphi(u)$ cannot take the least negative value as $u \longrightarrow +0$ and cannot have a negative minimum on $\mathbb{R}_+$.

Remark 4. Nonlocal condition (2.3) is not fixed in [1, 14]. This is associated with the fact that the entire problem (2.1)–(2.5) was not previously formulated or studied and, as a consequence, Lemmas 1 and 2 are stated for the first time.

2.3. *Preliminary Remarks*

For reasons to be explained later, we distinguish the following two cases:

Case I: $$0 < a < \lambda + \lambda_1, \tag{2.10}$$

Case II: $$a \ge \lambda + \lambda_1 > 0. \tag{2.11}$$

It will be shown in Section 3 that, in case I, limiting conditions (2.2) are equivalent to the following ones:

$$\lim_{u\to+0}\varphi(u) \quad \text{and} \quad \lim_{u\to+0}\varphi'(u), \quad \text{exists and are finite.} \tag{2.12}$$

These conditions are rewritten in the equivalent parametrized form

$$\lim_{u\to+0}\varphi(u) = C_0, \quad \lim_{u\to+0}\varphi'(u) = D_1, \tag{2.13}$$

where C_0 and D_1 are parameters to be determined; moreover, for the desired solution of problem (2.1)–(2.5) in case I (i.e., under condition (2.10)), they satisfy

$$0 \le C_0 \le 1, \quad D_1 \ge 0.$$

In case II (i.e., under (2.11)), we have to take into account the more general limiting boundary conditions (2.2), according to which the derivatives of solutions to IDE (2.1) can be unbounded as $u \longrightarrow +0$ and, importantly, the first derivative can have only an integrable singularity at zero.

Note that, according to the economic interpretation of the model under study (as an insurance model with investments), case I is more natural.

Remark 5. It is easy to see that, if $a = b = 0$ and inequality (1.41) holds (the safety load is positive), then exact solution (1.42) of integral equation (1.40) satisfies all conditions (2.2)–(2.5), where, in (2.13),

$$C_0 = 1 - \lambda(n+m)/[n(\lambda+\lambda_1)], \quad D_1 = \lambda(n+m)(\lambda_1 n - \lambda m)/[mn^2(\lambda+\lambda_1)^2];$$

moreover, by virtue of (1.41), $0 < C_0 < 1$ and $D_1 > 0$.

3. REDUCTION OF THE PROBLEM TO RELATED SINGULAR BOUNDARY VALUE PROBLEMS FOR ODE

3.1. *Reduction of IDE to a Fourth-Order ODE Defined on a Half-Line*

For the subsequent analysis, following [1, 14], IDE (2.1) is reduced to an ODE by applying additional differentiation and taking into account the equalities

$$(J_m\varphi)'(u) = [\varphi(u) - (J_m\varphi)(u)]/m, \quad (J_{1,n}\varphi)'(u) = [(J_{1,n}\varphi)(u) - \varphi(u)]/n, \tag{3.1}$$

which follow from (1.9) and (1.10):

$$(J_m\varphi)'(u) = \left(\frac{1}{m}\exp(-u/m)\int_0^u\varphi(s)\exp(s/m)ds\right)' = -(J_m\varphi)(u)/m + \varphi(u)/m;$$

$$(J_{1,n}\varphi)'(u) = \left(\frac{1}{n}\exp(u/n)\int_u^{\infty}\varphi(s)\exp(-s/m)ds\right)' = (J_{1,n}\varphi)(u)/n - \varphi(u)/n.$$

Differentiating IDE (2.1) two times and taking into account (3.1), we obtain

$$\begin{gathered}(b^2/2)u^2\varphi'''(u) + (b^2 + a)u\varphi''(u) + (a - \lambda - \lambda_1)\varphi'(u)\\ + \lambda[\varphi(u) - (J_m\varphi)(u)]/m + \lambda_1[(J_{1,n}\varphi)(u) - \varphi(u)]/n = 0, \quad 0 < u < \infty,\end{gathered} \tag{3.2}$$

$$\begin{gathered}(b^2/2)u^2\varphi''''(u) + (2b^2 + a)u\varphi'''(u) + (b^2 + 2a - \lambda - \lambda_1)\varphi''(u) + (\lambda/m - \lambda_1/n)\varphi'(u)\\ - \lambda[\varphi(u) - (J_m\varphi)(u)]/m^2 - \lambda_1[\varphi(u) - (J_{1,n}\varphi)(u)]/n^2 = 0, \quad 0 < u < \infty.\end{gathered} \tag{3.3}$$

For the integral terms to cancel out, the original IDE (2.1) times $-1/(mn)$ and IDE (3.3) times $(n-m)/(mn)$ are added to IDE (3.2). As a result, we obtain a fourth-order linear ODE of the form

$$u^2\varphi''''(u) + (a_1 + a_2u)u\varphi'''(u) + (a_3 + a_4u + a_5u^2)\varphi''(u) + (a_6 + a_7u)\varphi'(u) = 0, \quad u \in \mathbb{R}_+, \tag{3.4}$$

where

$$\begin{gathered}a_1 = 2(2 + a/b^2), \quad a_2 = (n-m)/(mn), \quad a_3 = 2[1 + (2a - \lambda - \lambda_1)/b^2],\\ a_4 = 2(1 + a/b^2)(n-m)/(mn), \quad a_5 = -1/(mn),\\ a_6 = 2[a(n-m) + \lambda m - \lambda_1 n]/(b^2mn), \quad a_7 = -2a/(b^2mn).\end{gathered} \tag{3.5}$$

Since ODE (3.4) must degenerate as $u \longrightarrow +0$, the following results hold:

(i) In case I, taking into account conditions (2.12), we obtain

$$\lim_{u \to +0}[a_3\varphi''(u) + a_6\varphi'(u)] = 0, \quad \lim_{u \to +0}[u\varphi'''(u)] = 0, \tag{3.6}$$

which implies the finiteness of $\lim_{u \to +0}\varphi''(u)$ and yields the limiting relation

$$\lim_{u \to +0}[mn(b^2 + 2a - \lambda - \lambda_1)\varphi''(u) + (a(n-m) + \lambda m - \lambda_1 n)\varphi'(u)] = 0. \tag{3.7}$$

(ii) In case II, taking into account the more general conditions (2.2), we obtain

$$\lim_{u \to +0}[u\varphi'(u)] = \lim_{u \to +0}[u^2\varphi''(u)] = \lim_{u \to +0}[u^3\varphi'''(u)] = 0. \tag{3.8}$$

3.2. Statement of Singular BVPs for ODEs and Necessary and Sufficient Conditions for Their Equivalence to the Original IDE Problem

Setting $\varphi'(u) = \psi(u)$, we reduce the order of ODE (3.4). The resulting third-order linear ODE for $\psi(u)$ with a regular singularity at zero is written in the canonical form

$$u^3\psi'''(u) + (a_1 + a_2u)u^2\psi''(u) + (a_3 + a_4u + a_5u^2)u\psi'(u) + (a_6u + a_7u^2)\psi(u) = 0, \quad u \in \mathbb{R}_+. \tag{3.9}$$

ODE (3.9) has a regular (weak) singularity at $u \longrightarrow +0$ and an irregular (strong) singularity of rank 1 at $u \longrightarrow \infty$ (for the classification of singular points of the pole type for systems of linear ODEs, see, for example, [15–19], which complement each other).

3.2.1. Case I. Under condition (2.10), taking into account (3.6), (2.5), and the form of ODE (3.9), we obtain the following limiting boundary conditions at singular boundary points:

$$\left|\lim_{u \to +0}\psi(u)\right| < \infty, \quad \lim_{u \to +0}[u\psi'(u)] = \lim_{u \to +0}[u^2\psi''(u)] = 0, \tag{3.10}$$

$$\lim_{u \to +\infty}\psi(u) = \lim_{u \to +\infty}\psi'(u) = \lim_{u \to +\infty}\psi''(u) = 0. \tag{3.11}$$

The singular BVP (3.9)–(3.11) always has the trivial solution $\psi \equiv 0$. We are interested in the existence of a nontrivial solution of this problem that is singled out by the normalization conditions (see Lemma 3 and Corollary 1 below).

Conditions (3.10) represent the most general boundary conditions at a regular singular point. They can be rewritten in the equivalent parametrized form

$$\lim_{u \to +0} \psi(u) = D_1, \quad \lim_{u \to +0} [u\psi'(u)] = \lim_{u \to +0} [u^2\psi''(u)] = 0. \tag{3.12}$$

Here, D_1 is a parameter to be determined such that $D_1 \geq 0$ for the desired solution.

Remark 6. In view of (3.6), limiting conditions (3.12) could be replaced by the stronger ones

$$\lim_{u \to +0} \psi(u) = D_1, \quad \lim_{u \to +0} \psi'(u) = -D_1 a_6/a_3, \quad \lim_{u \to +0} [u\psi''(u)] = 0. \tag{3.13}$$

However, weaker conditions (3.12) are sufficient, while conditions (3.13) are satisfied automatically.

Note that any solution of IDE (2.1) must satisfy ODE (3.4). The converse is not true: ODE (3.4) has a continuum of solutions $\varphi(u) \equiv \text{const} \neq 0$ that are not solutions of (2.1).

Lemma 3. *Let the parameters* b^2, λ, λ_1, m, n, *and* a *satisfy the assumptions of Lemma* 1, *and let* $\psi(u)$ *be a nontrivial solution of the singular linear BVP* (3.9)–(3.11) *such that*

$$\int_0^\infty |\psi(s)| ds < \infty.$$

Then the function $\varphi(u)$ *defined as*

$$\varphi(u) = C_0 + \int_0^u \psi(s) ds, \quad u \geq 0, \tag{3.14}$$

where C_0 *is a parameter, is a solution of IDE* (2.1) *if and only if condition* (2.3) *is satisfied or, equivalently,* C_0 *is given by*

$$C_0 = \frac{\lambda_1}{\lambda} \int_0^\infty \psi(y) \exp(-y/n) dy. \tag{3.15}$$

Proof. In one direction, the lemma is obvious and equality (3.15) follows from (2.3) in view of the relation

$$\int_0^\infty \left(\int_0^y \psi(s) ds \right) \exp(-y/n) dy = \int_0^\infty \psi(y) \exp(-y/n) dy.$$

Now assume that $\psi(u)$ is a solution of BVP (3.9)–(3.11), $\psi \in L_1(0, \infty)$. Let $\varphi(u)$ be defined by formulas (3.14) and (3.15). Then $\varphi(u)$ satisfies ODE (3.4), condition (2.3), and the limiting boundary conditions

$$\left| \lim_{u \to +0} \varphi(u) \right| < \infty, \quad \left| \lim_{u \to +0} \varphi'(u) \right| < \infty, \quad \lim_{u \to +0} [u\varphi''(u)] = \lim_{u \to +0} [u^2\varphi'''(u)] = 0, \tag{3.16}$$

$$\left| \lim_{u \to \infty} \varphi(u) \right| < \infty, \quad \lim_{u \to \infty} \varphi'(u) = \lim_{u \to \infty} \varphi''(u) = \lim_{u \to \infty} \varphi'''(u) = 0. \tag{3.17}$$

The left-hand side of (2.1) with this function $\varphi(u)$ is denoted by $g(u)$. Let us show that $g(u) \equiv 0$. Indeed, taking into account the method for deriving ODE (3.4), we write it as

$$g''(u) + \frac{(n-m)}{mn} g'(u) - \frac{1}{mn} g(u) = 0, \quad 0 < u < \infty. \tag{3.18}$$

The general solution of ODE (3.18) has the form

$$g(u) = c_1 \exp(-u/m) + c_2 \exp(u/n), \quad u \geq 0, \tag{3.19}$$

where c_1 and c_2 are arbitrary constants.

Taking into account the definition of $g(u)$ and limiting conditions (3.17) yields $\lim_{u\to\infty}[g(u)/u^2] = 0$, which implies that $c_2 = 0$ in (3.19). Finally, in view of (3.16) and (2.3), we obtain $g(0) = 0$, which implies that $c_1 = 0$ in (3.19).

Noting that the solution $\psi(u)$ of BVP (3.9)–(3.11) is determined up to a normalizing constant and taking into account condition (2.5), we finally derive the following result.

Corollary 1. Under the assumptions of Lemma 3, the functions $\varphi(u)$ defined in (3.14) is a solution of problem (2.1)–(2.3), (2.5) if and only if condition (3.15) holds and

$$C_0 + \int_0^\infty \psi(s)ds = 1,$$

which yields the following normalization condition for $\psi(u)$:

$$\int_0^\infty [1 + (\lambda_1/\lambda)\exp(-s/n)]\psi(s)ds = 1. \tag{3.20}$$

Remark 7. For $a = b = 0$, from (2.1)–(2.3), (2.5), we obtain a degenerate problem for Eq. (1.40) that is equivalent to the BVP on a half-line for a second-order ODE with a nonlocal condition at zero:

$$\varphi''(u) = \frac{\lambda m - \lambda_1 n}{mn(\lambda+\lambda_1)}\varphi'(u), \quad (\lambda+\lambda_1)\varphi(0) = \frac{\lambda_1}{n}\int_0^\infty \varphi(y)\exp(-y/n)dy, \quad \lim_{u\to\infty}\varphi(u) = 1.$$

It is easy to see that, under condition (1.41), this problem has a unique solution and that solution is given by exact formula (1.42) with constraints (2.4) satisfied automatically.

3.2.3. Case II. Under condition (2.11), relations (3.8) imply more general limiting boundary conditions for $\psi(u)$ as $u \longrightarrow +0$:

$$\lim_{u\to+0}[u\psi(u)] = \lim_{u\to+0}[u^2\psi'(u)] = \lim_{u\to+0}[u^3\psi''(u)] = 0. \tag{3.21}$$

It is easy to see that Lemma 3 and Corollary 1 hold for the solution $\psi(u)$ of BVP (3.9), (3.21), (3.11). More specifically, conditions (3.16) in the proof of Lemma 3 are replaced by

$$\left|\lim_{u\to+0}\varphi(u)\right| < \infty, \quad \lim_{u\to+0}[u\varphi'(u)] = \lim_{u\to+0}[u^2\varphi''(u)] = \lim_{u\to+0}[u^3\varphi'''(u)] = 0. \tag{3.22}$$

As a result, we study the linear singular BVPs (3.9)–(3.11) and (3.9), (3.21), (3.11) with normalization condition (3.20).

First, we examine the singular points of ODE (3.9) and reduce the singular BVPs to equivalent BVPs on a finite interval without singularities. The transfer of limiting boundary conditions from singular points is based on the theory of stable initial manifolds of solutions, or manifolds of conditional Lyapunov stability (see, e.g., [20–23]) taking into account the concept of admissible boundary conditions at singular points of pole type (see, e.g., [24, 25]).

4. ANALYSIS OF SINGULAR POINTS OF ODE AND REDUCTION OF SINGULAR BOUNDARY VALUE PROBLEMS TO REGULAR ONES

4.1. Transfer of Boundary Conditions from the Regular Singular Point $u = 0$

First, we explain why we distinguish between two cases (2.10) and (2.11). For this purpose, consider ODE (3.9). This ODE has a regular (weak) singularity at $u = 0$. Assuming, as usual, that $\psi(u) = u^\mu[1 + o(1)]$, $\psi'(u) = \mu u^{\mu-1}[1 + o(1)]$, $\psi''(u) = \mu(\mu-1)u^{\mu-2}[1 + o(1)]$, and $\psi'''(u) = \mu(\mu-1)(\mu-2)u^{\mu-3}[1 + o(1)]$, $u \longrightarrow +0$, the following characteristic equation is derived for μ:

$$\mu(\mu-1)(\mu-2) + a_1\mu(\mu-1) + a_3\mu = 0,$$

which implies that $\mu_1 = 0$, while μ_2 and μ_3 satisfy the equation

$$\mu^2 - \mu(3-a_1) + 2 + a_3 - a_1 = 0.$$

In view of notation (3.5), we obtain

$$\mu_2 + \mu_3 = -1 - 2a/b^2 < 0,$$
$$\mu_2\mu_3 = 2(a - \lambda - \lambda_1)/b^2 < 0 \ (\geq 0) \quad \text{for} \quad a < \lambda + \lambda_1 \quad (\text{for } a \geq \lambda + \lambda_1), \tag{4.1}$$
$$\mu_2 = -\frac{1}{2} - \frac{a}{b^2} + \sqrt{\left(\frac{1}{2} + \frac{a}{b^2}\right)^2 + \frac{2(\lambda + \lambda_1 - a)}{b^2}},$$

$$\mu_3 = -\frac{1}{2} - \frac{a}{b^2} - \sqrt{\left(\frac{1}{2} + \frac{a}{b^2}\right)^2 + \frac{2(\lambda + \lambda_1 - a)}{b^2}}. \tag{4.2}$$

The last formulas can be conveniently rewritten as

$$\mu_2 + 1 = 1/2 - a/b^2 + \sqrt{(1/2 - a/b^2)^2 + 2(\lambda + \lambda_1)/b^2},$$
$$\mu_3 + 1 = 1/2 - a/b^2 - \sqrt{(1/2 - a/b^2)^2 + 2(\lambda + \lambda_1)/b^2}.$$

Then the behavior of solutions to ODE (3.9) near the singular point $u = 0$ depends on whether condition (2.10) or (2.11) holds:

(i) Under constraint (2.10), the characteristic exponents μ_j ($j = 1, 2, 3$) satisfy the relations

$$\mu_1 = 0, \quad \mu_2 > 0, \quad \mu_3 < -1. \tag{4.3}$$

Moreover, if

$$\lambda + \lambda_1 > b^2 + 2a, \tag{4.4}$$

then

$$\mu_2 > 1, \tag{4.5}$$

and, if

$$0 < a < \lambda + \lambda_1 \leq b^2 + 2a, \tag{4.6}$$

then

$$0 < \mu_2 \leq 1, \tag{4.7}$$

where the equality holds if it holds in (4.6).

(ii) Under constraint (2.11), we obtain

$$\mu_1 = 0, \quad -1 < \mu_2 \leq 0, \quad \mu_3 < -1, \tag{4.8}$$

where μ_2 vanishes if an equality holds in (2.11), i.e., if

$$a = \lambda + \lambda_1, \tag{4.9}$$

when the characteristic exponent $\mu_2 = 0$ corresponds to a solution of ODE (3.9) that is unbounded at zero and is $O(\ln(u))$ as $u \longrightarrow +0$.

It can be seen that equality (4.9) determines a critical value of the bifurcation parameter a: if (2.10) holds, then ODE (3.9) has a two-parameter family of solutions that are bounded at zero, and if (2.11) holds, then it has a one-parameter family of solutions. In the case of (2.11), the necessary two-parameter family of solutions is singled out by the admissible boundary conditions (3.21).

Remark 8. In [1, 3] the ~~of~~ behavior of solutions to ODE (3.9) as $u \longrightarrow +0$ was not analyzed at all. The case of $a > \lambda + \lambda_1$ was partially covered in [14], which will be additionally discussed later.

4.1.1. Case I: singular problem (3.9), (3.10) without initial data and its two-parameter family of solutions. Under constraints (2.10), we consider ODE (3.9) with conditions (3.10). By virtue of (4.3), this singular problem (3.9), (3.10) without initial data has a two-parameter family of solutions. In the phase space of ODE (3.9), their values generate a two-dimensional linear subspace depending on the parameter u. This subspace is defined by a single linear relation in $\mathbb{R}^3$:

$$\Theta(u)z(u) = 0, \quad u > 0, \tag{4.10}$$

where $\Theta = (\Theta_1, \Theta_2, \Theta_3)$, $z = (z_1, z_2, z_3)^T$ (here and below, the superscript ^{T} denotes the transpose),

$$z_1(u) = \psi(u), \quad z_2(u) = u\psi'(u), \quad z_3(u) = u^2\psi''(u), \tag{4.11}$$

$$\Theta(u) = \Theta_0 + O(u), \quad u \longrightarrow +0. \tag{4.12}$$

To determine the elements of the principal row vector Θ_0, we pass from ODE (3.9) to a system of first-order ODEs in variables (4.11):

$$uz'(u) = [A_0 + A_1u + A_2u^2]z(u), \quad u > 0.$$

Here, A_j ($j = 0, 1, 2$) are 3×3 matrices given by

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -a_3 & 2-a_1 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_6 & -a_4 & -a_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_7 & -a_5 & 0 \end{pmatrix}, \tag{4.13}$$

where the principal matrix A_0 has the eigenvalues μ_k ($k = 1, 2, 3$), specifically, $\mu_1 = 0$ and $\mu_{2,3}$ are given by (4.1) and (4.2). Then, according to [20], Θ_0 is the left eigenvector of A_0 corresponding to $\mu_3 < -1$, which implies (up to a nonsingular transformation)

$$\Theta_0 = (0, 1 - \mu_2, 1). \tag{4.14}$$

As a result, from (4.10)–(4.12) and (4.14) in the first approximation, we obtain relation (4.10) for small u, namely,

$$u^2\psi''(u) + (1-\mu_2)u\psi'(u) \approx 0, \quad u \sim 0. \tag{4.15}$$

The final form of relation (4.10) is described in the following result.

Lemma 4. *Let condition* (2.10) *hold. Then, for sufficiently small u*, *the limiting boundary conditions* (3.10) *for solutions of ODE* (3.9) *are equivalent to the linear relation*

$$u^2\psi''(u) = \alpha(u)u\psi'(u) + \beta(u)\psi(u), \quad 0 < u \le u_0, \tag{4.16}$$

where the pair of functions $\{\alpha(u), \beta(u)\}$ *is a solution of the singular nonlinear Cauchy problem*

$$u\alpha' = (1 - a_1 - a_2u)\alpha - \alpha^2 - \beta - (a_3 + a_4u + a_5u^2), \quad u > 0, \tag{4.17}$$

$$u\beta' = (2 - a_1 - a_2u)\beta - \alpha\beta - (a_6u + a_7u^2), \quad u > 0, \tag{4.18}$$

$$\lim_{u\to+0}\alpha(u) = \alpha_0 = \mu_2 - 1, \quad \lim_{u\to+0}\beta(u) = 0. \tag{4.19}$$

For sufficiently small u, *problem* (4.17)–(4.19) *has a unique solution* $\{\alpha(u), \beta(u)\}$ *that is a function holomorphic at the point u* = 0:

$$\alpha(u) = \sum_{k=0}^{\infty}\alpha_ku^k, \quad \beta(u) = \sum_{k=1}^{\infty}\beta_ku^k, \quad |u| \le u_0, \quad u_0 > 0, \tag{4.20}$$

where α_0 *is given by* (4.19) *and the coefficients* α_k, β_k *for* $k \geq 1$ *are determined by* (4.17) *and* (4.18) *by the formal substitution of expansions* (4.20), *which yields the recurrence formulas*

$$\beta_1 = -\frac{a_6}{\alpha_0 + a_1 - 1}, \quad \alpha_1 = -\frac{\beta_1 + a_2\alpha_0 + a_4}{2\alpha_0 + a_1}, \tag{4.21}$$

$$\beta_2 = -\frac{a_7 + a_2\beta_1 + \alpha_1\beta_1}{\alpha_0 + a_1}, \quad \alpha_2 = -\frac{\beta_2 + \alpha_1^2 + a_2\alpha_1 + a_5}{2\alpha_0 + a_1 + 1}, \tag{4.22}$$

$$\beta_k = -\frac{a_2\beta_{k-1} + \Sigma_{l=1}^{k-1}\alpha_l\beta_{k-l}}{\alpha_0 + a_1 + k - 2}, \quad \alpha_k = -\frac{\beta_k + \Sigma_{l=1}^{k-1}\alpha_l\alpha_{k-l}}{2\alpha_0 + a_1 + k - 1}, \quad k = 3, 4, \ldots. \tag{4.23}$$

Proof. Differentiating (4.16), using ODE (3.9) and again (4.16), and combining the terms with $u\psi'(u)$ and $\psi(u)$, we obtain ODEs (4.17), (4.18). The limiting boundary conditions (4.19) follow from (4.15) and correspond to the fulfillment of ODEs (4.17), (4.18) up to the singular point $u = 0$.

The Jacobian matrix $Q(\alpha, \beta, u)$ on the right-hand side of system (4.17), (4.18) at the point $\{\alpha, \beta, u\} = \{\alpha_0, 0, 0\}$ has the form

$$Q(\alpha_0, 0, 0) = \begin{pmatrix} 1 - a_1 - 2\alpha_0 & -1 \\ 0 & 2 - a_1 - \alpha_0 \end{pmatrix}, \tag{4.24}$$

where the eigenvalues of matrix (4.24) satisfy

$$1 - a_1 - 2\alpha_0 = -2\mu_2 - 1 - 2a/b^2 = \mu_3 - \mu_2 < 0, \quad 2 - a_1 - \alpha_0 = -1 - \mu_2 - 2a/b^2 = \mu_3 < -1. \tag{4.25}$$

Then all solutions of the system of linear ODEs obtained by linearizing system (4.17), (4.18) about the function $\{\alpha, \beta, u\} = \{\alpha_0, 0, 0\}$ are not bounded as $u \longrightarrow +0$. Applying Theorem 5 from [26], we derive the assertion of the lemma for the nonlinear singular Cauchy problem (4.17)–(4.19).

It remains to prove relation (4.16) for solutions of ODE (3.9) that satisfy (3.10). Indeed, all the solutions of ODE (4.16), where $\{\alpha, \beta\}$ solves problem (4.17)–(4.19), satisfy conditions (3.10), since the characteristic exponents χ_j ($j = 1, 2$) of ODE (4.16) are obtained from the equation $\chi(\chi - 1) = \chi\alpha_0$, whence $\chi_1 = 0$ and $\chi_2 = \alpha_0 + 1 = \mu_2 > 0$.

On the other hand, let $\psi(u)$ be a solution of ODE (3.9) satisfying conditions (3.10). Define the function

$$\eta(u) = u^2\psi''(u) - \alpha(u)u\psi'(u) - \beta(u)\psi(u), \quad u > 0, \tag{4.26}$$

where $\{\alpha(u), \beta(u)\}$ is the above solution of problem (4.17)–(4.19). For $\eta(u)$ we derive a linear singular Cauchy problem of the form

$$u\eta'(u) = [2 - a_1 - a_2u - \alpha(u)]\eta(u), \quad u > 0, \quad \lim_{u \to +0}\eta(u) = 0. \tag{4.27}$$

On the right-hand side of the ODE in (4.27), at the point $u = 0$, we have

$$2 - a_1 - \alpha_0 = -1 - 2a/b^2 - \mu_2 = \mu_3 < -1.$$

Therefore, the singular Cauchy problem (4.27) has no solutions other than $\eta(u) \equiv 0$. Combining this with (4.26), we complete the proof of (4.16).

As a result, boundary conditions (3.10) are transferred from the singular point $u = 0$ to a close regular point $u = u_0 > 0$ with the help of (4.16): at the point $u = u_0$, we have the boundary condition

$$u_0^2\psi''(u_0) = \alpha(u_0)u_0\psi'(u_0) + \beta(u_0)\psi(u_0), \tag{4.28}$$

where $\alpha(u_0)$ and $\beta(u_0)$ can be found with a prescribed accuracy by applying expansions (4.20)–(4.23). By virtue of (4.24) and (4.25), near the singular point $u = 0$, condition (4.28) is stably transferred from left to right away from this singular point (for more detail, see the study of these problems in [27] for rather general systems of linear ODEs with pole-type singularities at boundary points).

Taking into account the results of the general theory of linear ODEs with a regular (weak) singular point (see, e.g., [16, Section 2]), we can state the following result.

Corollary 2. Let the assumptions of Lemma 4 be satisfied. Then the following representations hold for the two-parameter family of solutions $\psi(u, p_1, p_2)$ of singular problem (3.9), (3.10) without initial data:

(i) If $0 < \mu_2$ is not an integer, then

$$\psi(u, p_1, p_2) = p_1[1 + \psi_1(u)] + p_2 u^{\mu_2}[1 + \psi_2(u)], \quad \psi_j(0) = 0, \quad j = 1, 2, \tag{4.29}$$

where p_1 and p_2 are arbitrary constants and $\psi_1(u)$ and $\psi_2(u)$ are functions holomorphic at the point $u = 0$.

(ii) If $0 < \mu_2$ is an integer, then

$$\begin{gathered}\psi(u, p_1, p_2) = p_1\{1 + \psi_1(u) + Au^{\mu_2}\ln(u)[1 + \psi_2(u)]\} + p_2 u^{\mu_2}[1 + \psi_2(u)], \\ \psi_j(0) = 0, \quad j = 1, 2,\end{gathered} \tag{4.30}$$

where, as in (4.29), $\psi_1(u)$ and $\psi_2(u)$ are functions holomorphic at zero and A is a constant depending on the parameters of ODE (3.9).

In the phase space of system (3.9), the values of solutions (4.29) (respectively, (4.30)) generate a two-dimensional subspace (a plane in $\mathbb{R}^3$) depending on u as on a parameter. This subspace is specified by relation (4.16), where $\alpha(u)$ and $\beta(u)$ are described in Lemma 4. Then the expansions at zero of $\psi_1(u)$ and $\psi_2(u)$ from (4.29) and (4.30) can be obtained from ODE (4.16) taking into account expansions (4.20)–(4.23).

The following fact is known from the theory of transfer of boundary conditions from singular points for systems of linear ODEs: plane (4.16) as a whole is described by analytic functions. This fact is convenient when the plane is specified at a regular point as a boundary condition, while individual solutions generating it have a more complicated behavior, namely, they contain fractional and possibly irrational powers of u (see (4.29)) and/or logarithmic functions of u (see (4.30)).

Remark 9. If $0 < \mu_2$ is not an integer, then the unique (up to a multiplicative constant) solution $\hat{\psi}_1(u) = 1 + \psi_1(u)$ of ODE (3.9) that is holomorphic at zero (see (4.29)) has the expansion

$$\hat{\psi}_1(u) = 1 + \sum_{k=1}^{\infty} D_{k+1}u^k, \quad |u| \le u_0, \tag{4.31}$$

where the constant coefficients D_k ($k \ge 1$) are given by the recurrence formulas

$$D_2 = -a_6/a_3, \quad D_3 = -[D_2(a_4 + a_6) + a_7]/[2(a_1 + a_3)], \tag{4.32}$$

$$D_k = -\frac{D_{k-1}[a_2(k-2)(k-3) + a_4(k-2) + a_6] + D_{k-2}[a_5(k-3) + a_7]}{(k-1)[(k-2)(k-3) + a_1(k-2) + a_3]}, \quad k \ge 4. \tag{4.33}$$

For (4.29), we have the first-approximation representation

$$\psi(u, p_1, p_2) = p_1\left[1 + \sum_{k=1}^{K} D_{k+1}u^k + O(u^{K+1})\right] + p_2 u^{\mu_2}[1 + O(u)], \quad u \longrightarrow +0, \tag{4.34}$$

where $K = [\mu_2] + 1$ and $[\mu_2]$ is the integer part of μ_2.

If $\mu_2 = K \ge 1$, where K is an integer, then, from (4.30) in the first approximation, we obtain

$$\psi(u, p_1, p_2) = p_1\left[1 + \sum_{k=1}^{K} D_{k+1}u^k + Au^K\ln(u)[1 + O(u)]\right] + p_2 u^k[1 + O(u)], \quad u \longrightarrow +0. \tag{4.35}$$

Taking into account these formulas, we derive the following result.

Corollary 3. Let the assumptions of Lemma 4 hold and relation (4.4) be satisfied. Then, for the two-parameter family of solutions $\psi(u, p_1, p_2)$ to singular problem (3.9), (3.10) without initial data as defined in Corollary 2, there exists a finite limit

$$\lim_{u \to +0} \psi'(u, p_1, p_2) = p_1 D_2, \tag{4.36}$$

where

$$D_2 = -a_6/a_3 = [a(m-n) + \lambda_1 n - \lambda m]/[mn(b^2 + 2a - \lambda - \lambda_1)]. \tag{4.37}$$

The converse is also true: suppose that the assumptions of Corollary 2 hold and, for the two-parameter family of solutions $\psi(u, p_1, p_2)$ to singular problem (3.9), (3.10), without initial data defined in Corollary 2, there exists a finite limit (4.36), (4.37). Then either (4.4) holds or $p_2 = 0$ (or $p_1 = 0$) in (4.29) (in (4.30), respectively). Otherwise, $|\psi'(u, p_1, p_2)| \longrightarrow \infty$ as $u \longrightarrow +0$ and remains integrable at zero.

The sign of (4.37) (for positive p_1) is important in the computations (compare (4.36) with limiting relation (3.7)).

4.1.2. Case II. Let inequality (2.11) hold. Then the characteristic exponents of ODE (3.9) satisfy (4.8), where $\mu_2 = 0$ if equality (4.9) holds and, in this case, the multiple zero eigenvalue of matrix (4.13) is associated with a second-order Jordan block.

For further comparisons, we consider auxiliary problem (3.9), (3.10).

4.1.2.1. Auxiliary singular problem (3.9), (3.10) *without initial data and its one-parameter family of solutions.* Taking into account the characteristic exponents of ODE (3.9) and applying Theorem 5 from [26], we derive the following result.

Lemma 5. *Let inequality* (2.11) *hold. Then, for each fixed* D_1 ($D_1 \in \mathbb{R}$), *the singular Cauchy problem* (3.9), (3.12) *has a unique solution* $\psi(u, D_1)$. *This solution is a function holomorphic at the point* $u = 0$:

$$\psi(u, D_1) = D_1\left[1 + \sum_{k=1}^{\infty} D_{k+1}u^k\right], \quad |u| \le u_0, \quad u_0 > 0. \tag{4.38}$$

Here, the coefficients D_{k+1} ($k \ge 1$) *are independent of* D_1 *and are determined by the direct substitution of* (4.38) *into ODE* (3.9), *which yields recurrence formulas* (4.32) *and* (4.33).

Remark 10. Expansion (4.38), (4.32), (4.33) was obtained in (14) for $a > \lambda + \lambda_1$ with the use of the results from [26]. As was indicated in Remark 9, this expansion also holds for $0 < a < \lambda + \lambda_1$ and noninteger $\mu_2 > 0$ and defines the function $\psi_1(u)$ in (4.29) (see (4.31)–(4.33)). Note that, under the conditions of Lemma 5, the denominators in (4.32) and (4.33) do not vanish, since $a_1 > 0$ and $a_3 > 0$. For formulas (4.32) and (4.33), the same is true under the conditions of Lemma 4 if $0 < \mu_2$ is not an integer. We do not go into more detail about this issue.

The one-parameter family of solutions (4.38) of ODE (3.9) generates, in the phase space, a one-dimensional linear subspace. For small $u > 0$, the latter is defined by two relations

$$[\Theta_1 + \tilde{\Theta}_1(u)]z(u) = 0, \quad [\Theta_2 + \tilde{\Theta}_2(u)]z(u) = 0. \tag{4.39}$$

Here, $|\tilde{\Theta}_j(u)| = O(u)$, $u \longrightarrow +0, j = 1, 2$; for $a > \lambda + \lambda_1$, Θ_1 is the left eigenvector of A_0 corresponding to the eigenvalue $\mu_2 < 0$, while Θ_2 is the left eigenvector of A_0 corresponding to the eigenvalue $\mu_3 < -1$:

$$\Theta_1 = (0, 1-\mu_3, 1), \quad \Theta_2 = (0, 1-\mu_2, 1). \tag{4.40}$$

For $a = \lambda + \lambda_1$, when $\mu_2 = 0$, Θ_2 is the same as in (4.40) (for $\mu_2 = 0$) and Θ_1 is the left associated vector of A_0 corresponding to the zero eigenvalue, so that

$$\Theta_1 = (1 + 2a/b^2, 1, 0), \quad \Theta_2 = (0, 1, 1). \tag{4.41}$$

The form of relations (4.39) is not specified, since they are not needed in what follows. Importantly, under condition (2.11), for sufficiently small $u > 0$, the limiting boundary conditions (3.10) for solutions of ODE (3.9) are equivalent to two linear relations (4.39). By combining (4.40), (4.41), and (4.11), these relations are approximately represented as follows: for $a > \lambda + \lambda_1$,

$$u^2\psi''(u) + (1-\mu_3)u\psi'(u) \approx 0, \quad u^2\psi''(u) + (1-\mu_2)u\psi'(u) \approx 0, \quad u \sim +0; \tag{4.42}$$

and, for $a = \lambda + \lambda_1$, we have

$$u\psi'(u) + (1 + 2a/b^2)\psi(u) \approx 0, \quad u^2\psi''(u) + u\psi'(u) \approx 0, \quad u \sim +0. \tag{4.43}$$

4.1.2.2. Basic singular Cauchy problem (3.9), (3.21) *and its two-parameter family of solutions.* Consider the basic singular Cauchy problem indicated in the heading. It is easy to see that Lemma 4 remains valid with conditions (3.10) replaced by (3.21) and with the limiting condition at zero for solutions of Cauchy problem (4.27) replaced by the condition $\lim_{u \to +0}[u\eta(u)] = 0$.

Below is an analogue of Corollary 2.

Corollary 4. Let inequality (2.11) hold. Then, for the two-parameter family of solutions $\tilde{\psi}(u, p_1, p_2)$ to the singular Cauchy problem (3.9), (3.21) the following assertions hold:

(i) If $-1 < \mu_2 < 0$, then representation (4.29) holds, where p_1 and p_2 are arbitrary constants and $\psi_1(u)$ and $\psi_2(u)$ are functions holomorphic at the point $u = 0$. Representation (4.29) singles out a family of solutions of ODE (3.9) that grow no faster than $O(u^{\mu_2})$ as $u \longrightarrow +0$.

(ii) If $\mu_2 = 0$, then we have the representation

$$\psi(u, p_1, p_2) = p_1\{1 + \psi_1(u) + A\ln(u)[1 + \psi_2(u)]\} + p_2[1 + \psi_2(u)], \tag{4.44}$$

where $\psi_1(u)$ and $\psi_2(u)$ are functions holomorphic at zero and A is a constant depending on the original parameters. Representation (4.44) singles out a family of solutions of ODE (3.9) that grow as $u \longrightarrow +0$ no faster than $O(\ln(u))$.

4.2. *Transfer of Boundary Conditions from Infinity to a Finite Point*

For $u \longrightarrow \infty$, ODE (3.9) has an irregular (strong) singularity of rank 1. For large u, the behavior of solutions of these ODEs is independent of the relation between a and $\lambda + \lambda_1$: for any positive a, b^2, m, and n and for λ and λ_1 of any sign, there exists a two-parameter family of solutions vanishing at infinity. This circumstance also is closely related to the formulation of possible boundary conditions at the left end—as $u \longrightarrow +0$.

The singular Cauchy problem (3.9), (3.11) can be rewritten as

$$\psi'''(u) + \left[a_2 + \frac{a_1}{u}\right]\psi''(u) + \left[a_5 + \frac{a_4}{u} + \frac{a_3}{u^2}\right]\psi'(u) + \left[\frac{a_7}{u} + \frac{a_6}{u^2}\right]\psi(u) = 0, \quad 0 < u < \infty, \tag{4.45}$$

$$\lim_{u \to +\infty} \psi(u) = \lim_{u \to +\infty} \psi'(u) = \lim_{u \to +\infty} \psi''(u) = 0. \tag{4.46}$$

Setting, as usual, $\psi(u) = \exp(\nu u)[1 + o(1)]$, $\psi'(u) = \nu\exp(\nu u)[1 + o(1)]$, $\psi''(u) = \nu^2\exp(\nu u)[1 + o(1)]$, and $\psi'''(u) = \nu^3\exp(\nu u)[1 + o(1)]$, $u \longrightarrow \infty$, we derive the following characteristic equation for ν:

$$\nu^3 + a_2\nu^2 + a_5\nu = 0,$$

which yields, in view of notation (3.5), the characteristic exponents for ODE (4.45) responsible for the behavior of solutions for large u:

$$\nu_0 = 0, \quad \nu_1 = -1/m < 0, \quad \nu_2 = 1/n > 0. \tag{4.47}$$

To completely analyze the behavior of solutions to ODE (4.45) as $u \longrightarrow \infty$, we need to find the first perturbative correction (for large u) of $\nu_0 = 0$. For this purpose, we pass to a first order system by setting

$$y_1(u) = \psi(u), \quad y_2(u) = \psi'(u), \quad y_3(u) = \psi''(u), \quad y = (y_1, y_2, y_3)^{\mathrm{T}}. \tag{4.48}$$

Combining (4.45) and (4.46) produces the singular Cauchy problem

$$y'(u) = [B_0 + B_1/u + B_2/u^2]y(u), \quad 0 < u < \infty, \quad \lim_{u \to \infty} y(u) = 0. \tag{4.49}$$

Here, B_j ($j = 0, 1, 2$) are 3×3 matrices given by

$$B_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_5 & -a_2 \end{pmatrix}, \tag{4.50}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_7 & -a_4 & -a_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_6 & -a_3 & 0 \end{pmatrix}. \tag{4.51}$$

The matrix B_0 with eigenvalue (4.47) is reduced to diagonal form. Specifically, the equality $B_0T_0 = T_0\Lambda_0$, where

$$\Lambda_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/m & 0 \\ 0 & 0 & 1/n \end{pmatrix} \tag{4.52}$$

and B_0 is defined in (4.50) yields

$$T_0 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1/m & 1/n \\ 0 & 1/m^2 & 1/n^2 \end{pmatrix}, \quad T_0^{-1} = \begin{pmatrix} 1 & m-n & -mn \\ 0 & -m^2/(m+n) & m^2n/(m+n) \\ 0 & n^2/(m+n) & mn^2/(m+n) \end{pmatrix}. \tag{4.53}$$

Setting $y(u) = T_0v(u)$ in (4.49), we obtain for $v(u)$ the ODE system

$$v'(u) = [\Lambda_0 + \tilde{B}_1/u + \tilde{B}_2/u^2]v(u), \quad 0 < u < \infty, \tag{4.54}$$

where Λ_0 is given by (4.52), $\tilde{B}_1 = T_0^{-1}B_1T_0$, and $\tilde{B}_2 = T_0^{-1}B_2T_0$. In view of formulas (4.51) and (4.53), we then have

$$\tilde{B}_1 = 2\begin{pmatrix} -a/b^2 & (m+n)/m & (m+n)/n \\ am/[b^2(m+n)] & -1 & -m/n \\ an/[b^2(m+n)] & -n/m & -1 \end{pmatrix}, \tag{4.55}$$

$$\tilde{B}_2 = \begin{pmatrix} a_6mn & (a_6m-a_3)n & (a_6n+a_3)m \\ -a_6m^2n/(m+n) & (a_3-a_6m)mn/(m+n) & -(a_3+a_6n)m^2/(m+n) \\ -a_6mn^2/(m+n) & (a_3-a_6m)n^2/(m+n) & -(a_3+a_6n)mn/(m+n) \end{pmatrix}, \tag{4.56}$$

where a_3 and a_6 are defined in (3.5).

Then the correction to the zero eigenvalue of B_0 for large u is $-2a/(ub^2) < 0$, which is easy to show by applying the asymptotic (quasi) diagonalization of matrices (see [21, part II, 26] and the basic lemma in [28]). Namely, let in (4.54)

$$v(u) = (E + N/u)w(u), \tag{4.57}$$

where E is the 3×3 identity matrix, while N is a 3×3 matrix that is not yet defined. Let us show that it can be chosen so that, for large u, $w(u)$ satisfies the ODE

$$w'(u) = [\Lambda_0 + \Lambda_1/u + \Omega(u)/u^2]w(u), \quad u \ge u_\infty, \tag{4.58}$$

where Λ_0 is given by (4.52) and

$$\Lambda_1 = 2\begin{pmatrix} -a/b^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{4.59}$$

i.e., the diagonal elements of Λ_1 coincide with those of $\tilde{B}_1$ and the matrix $\Omega(u)$ has a finite limit as $u \longrightarrow \infty$.

Indeed, the substitution of (4.57) into (4.54) gives

$$w'(u) = (E + N/u)^{-1}[(\Lambda_0 + \tilde{B}_1/u + \tilde{B}_2/u^2)(E + N/u) + N/u^2]w(u), \quad u \ge u_\infty. \tag{4.60}$$

Comparing (4.58) with (4.60), we conclude that the matrix $N = (n_{ij})_{i,j=1,2,3}$ is constructed so that, for large u,

$$(E + N/u)^{-1}[(\Lambda_0 + \tilde{B}_1/u + \tilde{B}_2/u^2)(E + N/u) + N/u^2] = \Lambda_0 + \Lambda_1/u + \Omega(u)/u^2. \tag{4.61}$$

Equating the terms with $1/u$ in (4.61) yields

$$-\Lambda_0 N + N\Lambda_0 = \tilde{B}_1 - \Lambda_1.$$

From this and the form of matrices (4.52), (4.55), and (4.59), we find that n_{11}, n_{22}, and n_{33} are arbitrary numbers, so we can set

$$n_{11} = n_{22} = n_{33} = 0.$$

Then the other elements of N are uniquely determined as

$$n_{12} = -2(m+n), \quad n_{13} = 2(m+n), \quad n_{21} = 2am^2/[b^2(m+n)],$$

$$n_{23} = -2m^2/(m+n), \quad n_{31} = -2an^2/[b^2(m+n)], \quad n_{32} = -2n^2/(m+n).$$

After N has been found, for the matrix $\Omega(u)$, we obtain

$$\Omega(u) = (E + N/u)^{-1}[\tilde{B}_2 + \tilde{B}_1 N + N(E - \Lambda_1) + \tilde{B}_2 N/u].$$

Therefore, for large u, the matrix can be expanded in a converging series:

$$\Omega(u) = \sum_{k=0}^{\infty} \Omega^{(k)}/u^k, \quad u \ge u_\infty,$$

where $\Omega^{(0)} = \tilde{B}_1 N - N\Lambda_1 + \tilde{B}_2 + N$.

Returning to ODE system (4.58) and applying, for example, Theorem 8 from [15, Chapter II], we see that this system is "asymptotically equivalent" to the system

$$\tilde{w}'(u) = (\Lambda_0 + \Lambda_1/u)\tilde{w}(u), \quad u \gg 1, \tag{4.62}$$

whose solutions are easy to find, since matrices (4.52) and (4.59) are diagonal.

Taking into account the form of solutions of ODE system (4.62) and the above transformations and applying the results of the general theory of linear ODEs with irregular singular points, we arrive at the following result.

Lemma 6. *Let a, b^2, n, and m in system* (3.9) *be positive numbers, and let λ and λ_1 have an arbitrary sign. Then the singular Cauchy problem* (3.9), (3.11) *has a two-parameter family of solutions $\psi(u, q_1, q_2)$, and, for large u, we have the representation*

$$\psi(u, q_1, q_2) = q_1 u^{-2a/b^2}[1 + \xi_1(u)/u] + q_2 u^{-2}\exp(-u/m)[1 + \xi_2(u)/u]. \tag{4.63}$$

Here, q_1 and q_2 are arbitrary constants and the functions $\xi_j(u)$ ($j = 1, 2$) have finite limits as $u \longrightarrow \infty$ and, for large u, can be represented by the asymptotic series

$$\xi_j(u) \sim \sum_{k=0}^{\infty} \xi_k^{(j)}/u^k, \quad j = 1, 2, \tag{4.64}$$

where the coefficients of the series can be obtained from ODE (3.9) *by the formal substitution of all the expansions. If $q_1 \neq 0$, then $\psi(u, q_1, q_2)$ is integrable at infinity if and only if inequality* (1.18) *holds.*

Remark 11. Under condition (1.18) (reliability of the asset portfolio) the first-approximation asymptotic representation $\psi(u, q_1, q_2) = q_1 u^{-2a/b^2}[1 + o(1)] + q_2 u^{-2}\exp(-u/m)[1 + o(1)]$, $u \longrightarrow \infty$, was obtained in [1] in another way, namely, with the use of WKB approximations (see [16]). The same representation was obtained in [14] by applying the above method, i.e., using the results [26, 28]. A more detailed description of this approach is presented here to improve the asymptotic representations and for completeness' sake. However, no formulas for the coefficients in asymptotic series (4.64) are given, since we do not need them in what follows.

The values of solutions of the singular Cauchy problem (3.9), (3.11) generate, in the phase space, a two-dimensional linear subspace depending on u as on a parameter. This linear subspace is defined by a single linear relation in $\mathbb{R}^3$:

$$H(u)y(u) = 0, \quad u > 0. \tag{4.65}$$

Here, $H = (H_1, H_2, H_3)$, the vector y is defined in (4.48), and

$$H(u) = H_\infty + O(1/u), \quad u \longrightarrow \infty, \tag{4.66}$$

where H_∞ is the left eigenvector of B_0 (see (4.50)) in system (4.49) that corresponds to the positive eigenvalue $\nu_3 = 1/n$:

$$H_\infty = (0, 1/m, 1). \tag{4.67}$$

Then, for large u, in the first approximation, we obtain relation (4.65) in the form

$$\psi''(u) + \psi'(u)/m \approx 0, \quad u \gg 1. \tag{4.68}$$

The final form of relation (4.65) is established below.

Lemma 7. *Let the assumptions of Lemma* 6 *hold. Then, for sufficiently large u, the limiting boundary conditions* (3.11) *for solutions of ODE* (3.9) *are equivalent to the linear relation*

$$\psi''(u) = \gamma(u)\psi'(u) + \kappa(u)\psi(u), \quad u \ge u_\infty, \tag{4.69}$$

where the pair of functions $\{\gamma(u), \kappa(u)\}$ *is a solution of the singular nonlinear Cauchy problem*

$$\gamma' = -(a_2 + a_1/u)\gamma - \gamma^2 - \kappa - (a_5 + a_4/u + a_3/u^2), \tag{4.70}$$

$$\kappa' = -(a_2 + a_1/u)\kappa - \gamma\kappa - (a_7/u + a_6/u^2), \quad u_\infty \le u \le \infty, \tag{4.71}$$

$$\lim_{u \to +\infty} \gamma(u) = \gamma_0 = -1/m, \quad \lim_{u \to +\infty} \kappa(u) = 0. \tag{4.72}$$

For sufficiently large u, the singular Cauchy problem (4.70)–(4.72) *has a unique solution* $\{\gamma(u), \kappa(u)\}$ *and this solution can be represented as an asymptotic series:*

$$\gamma(u) \sim \gamma_0 + \sum_{k=1}^{\infty} \gamma_k/u^k, \quad \kappa(u) \sim \sum_{k=1}^{\infty} \kappa_k/u^k, \quad u \gg 1, \tag{4.73}$$

where γ_0 is given by (4.72) *and the coefficients γ_k and κ_k for $k \ge 1$ are determined from* (4.70) *and* (4.71) *by the formal substitution of expansions* (4.73), *which yields the recurrence formulas*

$$\kappa_1 = -a_7/(a_2 + \gamma_0), \quad \gamma_1 = -(a_1\gamma_0 + \kappa_1 + a_4)/(a_2 + 2\gamma_0), \tag{4.74}$$

$$\kappa_2 = [\kappa_1(1 - a_1 - \gamma_1) - a_6]/(a_2 + \gamma_0), \quad \gamma_2 = [\gamma_1(1 - a_1 - \gamma_1) - \kappa_2 - a_3]/(a_2 + 2\gamma_0), \tag{4.75}$$

$$\kappa_k = \left[(k - 1 - a_1)\kappa_{k-1} - \sum_{l=1}^{k-1} \gamma_l \kappa_{k-l}\right]/(a_2 + \gamma_0), \tag{4.76}$$

$$\gamma_k = \left[\gamma_{k-1}(k - 1 - a_1) - \kappa_k - \sum_{l=1}^{k-1} \gamma_l \gamma_{k-l}\right]/(a_2 + 2\gamma_0), \quad k = 3, 4, \ldots. \tag{4.77}$$

Proof. The proof of this lemma is completely similar to that of Lemma 4 with allowance for results concerning singular Cauchy problems for systems of nonlinear ODEs with an irregular (strong) singularity at infinity. Specifically, the Jacobian matrix $Q(\gamma, \kappa, u)$ of the right-hand side of system (4.70), (4.71) taken at the point $\{\gamma, \kappa, u\} = \{\gamma_0, 0, \infty\}$ ($\{\gamma_0, 0\}$ is a stationary point of the limiting autonomous system of nonlinear ODEs obtained from system (4.70), (4.71) by formal passage to the limit as $u \longrightarrow \infty$) has the form

$$Q(\gamma_0, 0, \infty) = \begin{pmatrix} -a_2 - 2\gamma_0 & -1 \\ 0 & -a_2 - \gamma_0 \end{pmatrix}, \tag{4.78}$$

where the eigenvalues of matrix (4.78) satisfy the relations

$$-a_2-2\gamma_0 = (m-n)/(mn)+2/m = (m+n)/(mn)>0, \quad -a_2-\gamma_0 = m/(m+n)>0. \tag{4.79}$$

Then all the solutions of the system of linear ODEs obtained by linearizing system (4.70), (4.71) about the function $\{\gamma, \kappa, u\} = \{\gamma_0, 0, \infty\}$ are not bounded as $u \longrightarrow \infty$. Taking into account the results from [21, part I, Section 1] (see also Theorem 3.7 in [28] with Corollary 7 taken into account), we obtain the assertion of the lemma for the nonlinear singular Cauchy problem (4.70)–(4.72).

The rest of the proof is the same as for Lemma 4.

As a result, boundary conditions (3.11) are transferred from infinity to the finite point $u = u_\infty \gg 1$ with the help of relation (4.69): at $u = u_\infty$, we have the boundary condition

$$\psi''(u_\infty) = \gamma(u_\infty)\psi'(u_\infty)+\kappa(u_\infty)\psi(u_\infty), \tag{4.80}$$

where approximate values of $\gamma(u_\infty)$ and $\kappa(u_\infty)$ can be found using expansions (4.74)–(4.77). Moreover, by virtue of (4.78) and (4.79), for large u, condition (4.80) is stably transferred from right to left away from the singular point $u = \infty$.

5. EXISTENCE OF NONTRIVIAL SOLUTIONS TO SINGULAR BVPS FOR ODES AND A UNIQUE SOLUTION OF THE BASIC CONSTRAINED SINGULAR BVP FOR IDE

Under conditions (2.10) (conditions (2.11)), the auxiliary singular linear BVP (3.9)–(3.11) (respectively, BVP (3.9), (3.21), (3.11)) defined on $\mathbb{R}_+$ is equivalent to the following homogeneous linear BVP on the finite interval $0 < u_0 \le u \le u_\infty$ without singularities:

$$u^3\psi'''(u)+[a_1+a_2u]u^2\psi''(u)+[a_3+a_4u+a_5u^2]u\psi'(u)+[a_6u+a_7u^2]\psi(u) = 0, \quad u_0 \le u \le u_\infty, \tag{5.1}$$

$$u_0^2\psi''(u_0) = \alpha(u_0)u_0\psi'(u_0)+\beta(u_0)\psi(u_0), \tag{5.2}$$

$$\psi''(u_\infty) = \gamma(u_\infty)\psi'(u_\infty)+\kappa(u_\infty)\psi(u_\infty). \tag{5.3}$$

Here, the functions $\alpha(u)$ and $\beta(u)$ are defined in Lemma 4, and $\gamma(u)$ and $\kappa(u)$, in Lemma 7.

As a result, the homogeneous BVP (5.1)–(5.3) is underdetermined with respect to the number of boundary conditions. This BVP, along with the trivial solution $\psi_0(u) \equiv 0$, always has a nontrivial solution. Indeed, in the phase space $\mathbb{R}^3$ of variables (ψ, ψ', ψ''), relation (5.2) for solutions of ODE (5.1) $\forall u \in [u_0, u_\infty]$ generates a two-dimensional subspace, which is a plane passing through the origin. The same is true of relation (5.3). Then $\forall u \in [u_0, u_\infty]$ these two planes either have a common line of intersection passing through the origin, which generates a one-parameter family of solutions to BVP (5.1)–(5.3) (i.e., a unique solution up to a normalizing constant) or these two planes coincide, which leads to the existence of a two-parameter family of solutions to BVP (5.1)–(5.3). Since the existence of solutions to BVP (5.1)–(5.3) is independent of the values of the positive parameters b^2, λ, λ_1, m, n, and a, the case of coincidence of the planes is only regarded as a possible exclusion. Moreover, under condition (1.18), this exclusion is not possible by Lemmas 1, 3, and 6.

Summarizing these arguments and the results obtained in Sections 3 and 4 proves the theorems stated below.

Theorem 1. *Let ODE* (3.9) *be such that the coefficients* a_j $(j = \overline{1, 7})$ *are given by formulas* (3.5), *where*

$$b^2>0, \quad m>0, \quad n>0, \quad \lambda>0, \quad \lambda_1>0, \quad 0<a<\lambda+\lambda_1, \quad 2a/b^2>1,$$

and, in the other respects, these parameters are arbitrary numbers.

Then, for fixed values of these parameters, singular BVP (3.9)–(3.11) *has a unique* (*up to a normalizing constant*) *nontrivial solution* $\psi(u)$, $\psi(u) \in L_1(0, \infty)$. *Moreover, the following assertions hold:*

(i) *Singular BVP* (3.9)–(3.11) *defined on* $\mathbb{R}_+$ *is equivalent to the regular homogeneous BVP* (5.1)–(5.3) *on the finite interval* $[u_0, u_\infty]$, *where* $\alpha(u)$ *and* $\beta(u)$ *are defined in Lemma* 4, $\gamma(u)$ *and* $\kappa(u)$, *in Lemma* 7, *and the values* u_0 *and* u_∞ $(0 < u_0 \ll 1,\ u_\infty \gg 1)$ *can generally vary in ranges depending on the parameters of the problem* (*moving boundaries*).

(ii) *The behavior of solutions to singular BVP* (3.9)–(3.11) *as* $u \longrightarrow +0$ *is determined in Corollaries* 2, 3 *and Remark* 9. *Specifically*, $|\lim_{u\to+0}\psi'(u)| < \infty$ *if and only if we have inequality* (4.4), *i.e., the inequality*

$$\lambda + \lambda_1 > b^2 + 2a. \tag{5.4}$$

More precisely, if $|\lim_{u\to+0}\psi(u)| = D_1 > 0$, *then*

$$\lim_{u\to+0}\psi'(u) = D_1D_2 = D_1[a(m-n)+\lambda_1 n - \lambda m]/[mn(b^2+2a-\lambda-\lambda_1)] \tag{5.5}$$

(*see formula* (4.37)), *whence* $D_2 < 0$ *for*

$$a(m-n)+\lambda_1 n - \lambda m > 0 \tag{5.6}$$

and $D_2 \ge 0$ *for*

$$a(m-n)+\lambda_1 n - \lambda m \le 0. \tag{5.7}$$

For $n > m$, *inequality* (5.6) *holds* (*in view of the constraints* $0 < a < \lambda + \lambda_1$) *if* $\lambda n < \lambda_1 m$, *while, for* $m \ge n$, *it holds if* $\lambda m < \lambda_1 n$. *For* $n \ge m$, *inequality* (5.7) *holds if* $\lambda m \ge \lambda_1 n$, *while, for* $m > n$, *it holds if* $\lambda n \ge \lambda_1 m$.

(iii) *If condition* (5.4) *is violated, i.e.,*

$$\lambda + \lambda_1 \le b^2 + 2a, \tag{5.8}$$

then the function $\psi'(u)$ *is unbounded as* $u \longrightarrow +0$, *but integrable at zero.*

(iv) *The behavior of solutions to singular BVP* (3.9)–(3.11) *as* $u \longrightarrow \infty$ *is determined in Lemma* 6. *Specifically, we have the first-approximation asymptotic representation*

$$\psi(u) = q_1 u^{-2a/b^2}[1+o(1)], \quad u \longrightarrow \infty, \tag{5.9}$$

where $q_1 \ne 0$.

Remark 12. Assertion (iv) in Theorem 1 is justified as follows: if we assume that $q_1 = 0$ in (5.9), then this is equivalent to the search for solutions of ODE (3.9) vanishing at infinity at least as an exponential function. Since such solutions comprise a one-parameter family, there are two homogeneous boundary conditions at the point $u = u_\infty$, which finally leads to the existence of only the trivial solution of BVP (3.9)–(3.11). We do not go into more detail about this issue (compare with Remark 13 below).

Theorem 2. *Let ODE* (3.9) *be such that the coefficients* a_j ($j = \overline{1,7}$) *are given by formulas* (3.5), *where*

$$b^2 > 0, \quad m > 0, \quad n > 0, \quad \lambda > 0, \quad \lambda_1 > 0, \quad a \ge \lambda + \lambda_1, \quad 2a/b^2 > 1$$

and, in the other respects, these parameters are arbitrary numbers.

Then, for fixed values of these parameters, singular BVP (3.9), (3.21), (3.11) *has a unique* (*up to a normalizing constant*) *nontrivial solution* $\psi(u)$, $\psi(u) \in L_1(0, \infty)$. *Moreover, the following assertions hold:*

(i) *Singular BVP* (3.9), (3.21), (3.11) *defined on* $\mathbb{R}_+$ *is equivalent to the regular homogeneous BVP* (5.1)–(5.3) *on the finite interval* $[u_0, u_\infty]$, *where the functions* $\alpha(u)$ *and* $\beta(u)$ *are defined in Lemma* 4, $\gamma(u)$ *and* $\kappa(u)$, *in Lemma* 7, *and the values* u_0 *and* u_∞ ($0 < u_0 \ll 1$, $u_\infty \gg 1$) *can generally vary in some ranges depending on the parameters of the problem* (*moving boundaries*).

(ii) *The behavior of solutions to singular BVP* (3.9), (3.21), (3.11) *as* $u \longrightarrow +0$ *is determined in Corollary* 4, *so the function* $\psi(u)$ *is not bounded as* $u \longrightarrow +0$, *but remains integrable at zero.*

(iii) *The behavior of solutions to singular BVP* (3.9), (3.21), (3.11) *as* $u \longrightarrow \infty$ *is determined in Lemma* 6. *Specifically, we have the first-approximation asymptotic representation* (5.9), *where* $q_1 \ne 0$.

Remark 13. Under the assumptions of Theorem 2, if we consider BVP (3.9)–(3.11), i.e., replace condition (3.21) at zero with condition (3.10), then, by virtue of the argument presented in Section 4.1.2.1, we have two homogeneous boundary conditions at the point $u = u_0$ (see formulas (4.39)–(4.43)). As a result, we conclude that BVP (3.9)–(3.11) has only the trivial solution.

Combining these theorems and Lemmas 1–3 and 6 and taking into account that the solution of singular problem (2.1)–(2.5) (if it exists) describes the survival probability (see [5]), we derive the following theorem, which is the main result of this paper.

Theorem 3. *Let the assumptions of Theorem* 1 (*or Theorem* 2) *hold, and let* $\psi(u)$ *be a nontrivial solution of the auxiliary singular BVP* (3.9)–(3.11) *for ODE* (*respectively, singular BVP* (3.9), (3.21), (3.11)) *with normalization condition* (3.20), *i.e., with*

$$\int_0^{\infty}[1+(\lambda_1/\lambda)\exp(-s/n)]\psi(s)ds = 1. \tag{5.10}$$

Then the following assertions hold:

(i) *The function* $\psi(u)$ *is positive for each finite value* $u \in \mathbb{R}_+$, *and the function* $\varphi(u)$ *defined by the equality*

$$\varphi(u) = (\lambda_1/\lambda)\int_0^{\infty}\psi(s)\exp(-s/n)ds + \int_0^{u}\psi(s)ds, \quad u \ge 0 \tag{5.11}$$

is a unique solution of the original constrained singular BVP (2.1)–(2.5) *for IDE and is an nondecreasing function on* $\mathbb{R}_+$.

(ii) *For small* $u > 0$, *the behavior of* $\varphi(u)$ *follow from the corresponding assertions in Theorem* 1 (*or Theorem* 2) *for* $\psi(u)$ *and from formula* (5.11). *Specifically, under the assumptions of Theorem* 1, *the derivative* $\varphi'(u)$ *has a finite limit as* $u \longrightarrow +0$, *and, under condition* (5.4), *the second derivatives* $\varphi''(u)$ *also has a finite limit as* $u \longrightarrow +0$, *and under condition* (5.8), *the second derivative* $\varphi''(u)$ *becomes unbounded but integrable at zero. Under the assumptions of Theorem* 2, *the derivative* $\varphi'(u)$ *becomes unbounded, but integrable at zero.*

(iii) *For large* u, *the solution* $\varphi(u)$ *has the representation*

$$\varphi(u) = 1 - Ku^{1-2a/b^2}[1+o(1)], \quad u \longrightarrow \infty, \tag{5.12}$$

where $K > 0$ (*the constant* K *cannot be found using local analysis methods*).

(iv) *Under the assumptions of Theorem* 1 *and conditions* (5.4) *and* (5.7), *the function* $\psi(u)$ *has a positive maximum on the interval* $[u_0, u_\infty]$, *while* $\varphi(u)$ *has a point of inflection on this interval.*

Note that, under the conditions of Theorem 1, if the safety load is positive; i.e., inequality (1.41) is satisfied, then condition (5.6) always holds for $m \ge n$. If inequality (1.41) is violated, when $\varphi(u) \equiv 0$ in the model without investments, we introduce the following definition.

Definition 4. If condition (1.41) (about the safety load for the Cramer–Lundberg model with stochastic premiums) is violated, but inequality (5.4) holds, then the quantity

$$i_{r,\,\mathrm{II}} = a(m-n) + \lambda_1 n - \lambda m \tag{5.13}$$

is called the risk factor (index) for model II.

In the limit as $n \longrightarrow 0$, $\lambda_1 n \longrightarrow c$, we have $i_{r,\,\mathrm{II}} \longrightarrow i_{r,\,\mathrm{I}}$, where $i_{r,\,\mathrm{I}}$ is the risk factor for model I defined by (1.29).

The most risky situation in model II occurs when $i_{r,\,\mathrm{II}} \le 0$ but the initial surplus is small. However, even in this situation, in the model with investments, we obtain $\varphi(u) > 0$ on $\mathbb{R}_+$.

6. ALGORITHMS FOR NUMERICAL ~~DETERMINATION~~ OF SOLUTIONS AND NUMERICAL RESULTS

6.1. Methods for Solving the Basic Singular Problem for Model II

It follows from the above study that the basic element of the solution to the original constrained singular BVP (2.1)–(2.5) for IDE is the ~~determination~~ of nontrivial solutions of auxiliary BVP (5.1)–(5.3) defined on the finite interval $[u_0, u_\infty]$ without singularities and with an insufficient number of boundary conditions.

It is well known that linear BVPs on a finite interval without singularities are effectively solved by differential ~~tridiagonal matrix~~ algorithms (T~~M~~A). Important results for the determination of nontrivial solutions of BVP (5.1)–(5.3) were obtained is [27], a brief overview of differential T~~M~~A methods was offered and the stability of the computations near singular points was analyzed as applied to BVPs obtained from singular BVPs by transferring boundary conditions from singular points, including eigenvalue problems. Specifically, BVP (5.1)–(5.3) is obtained from singular BVP (3.9)–(3.11) (or from singular BVP (3.9), (3.21), (3.11)) by the local transfer of boundary conditions from singular points as described in Section 4, while its nontrivial solutions are found by applying the techniques of [27], which are used for the stable

~~determination~~ of eigenfunctions. The difference is that these techniques were not previously applied to the solution of homogeneous BVPs with an insufficient number of boundary conditions.

We have proposed and implemented a T~~M~~A method for solving BVP (5.1)–(5.3) that is the most effective in the number of T~~M~~A equations and in the number of arithmetic operations. It consists of the~~ir~~ following steps.

Step 1. Fix u_0 and u_∞ ($0 < u_0 \ll 1$, $u_\infty \gg 1$). Find approximate values $\tilde{\alpha}_0 \approx \alpha(u_0)$ and $\tilde{\beta}_0 \approx \beta(u_0)$ in (5.2) with the help of converging series (4.20)–(4.23) and approximate values $\tilde{\gamma}_\infty \approx \gamma(u_\infty)$ and $\tilde{\kappa}_\infty \approx \kappa(u_\infty)$ in (5.3) with the help of asymptotic expansions (4.73)–(4.77) (see Lemmas 4 and 7). As a result, an approximate BVP with errors in the boundary conditions is obtained. Fix the points of the interval $[u_0, u_\infty]$ where the solution of BVP (5.1)–(5.3) is wanted.

Step 2. Fix $u = \hat{u} \in (u_0, u_\infty)$ as a point of the transfer of boundary conditions. It is chosen closer to a weaker singularity. Transfer boundary conditions (5.2) and (5.3) to this point.

Condition (5.2) is transferred from left to right (from the point $u = u_0$ to $u = \hat{u}$) by applying relation (4.16), where $\alpha(u)$ and $\beta(u)$ are found by numerically solving the Cauchy problem for the system of two ODEs (4.17), (4.18) with the initial conditions $\alpha(u_0) = \tilde{\alpha}_0$ and $\beta(u_0) = \tilde{\beta}_0$ ($\alpha(u)$ and $\beta(u)$ are stored at those points of $[u_0, \hat{u}]$, where we want to know the solution of BVP (5.1)–(5.3)). Condition (5.3) is transferred from right to left (from the point $u = u_\infty$ to $u = \hat{u}$) by applying relation (4.69), where $\gamma(u)$ and $\kappa(u)$ are found by numerically solving the Cauchy problem for the system of two ODEs (4.70), (4.71) with the initial conditions $\gamma(u_\infty) = \tilde{\gamma}_\infty$ and $\kappa(u_\infty) = \tilde{\kappa}_\infty$ ($\gamma(u)$ and $\kappa(u)$ are stored at those points of $[\hat{u}, u_\infty]$ where we want to know the solution of BVP (5.1)–(5.3).

The local numerical stability (near singular points) was discussed in Section 4. For comparison and monitoring the global numerical stability, we also used the orthogonal differential T~~M~~A method [29], which is less efficient in the number of equations and arithmetic operations, but more stable overall. The results produced by both versions were in good agreement in accuracy.

Step 3. After executing Step 2, we obtain two linear relations (4.16) and (4.69) at the point $u = \hat{u}$ with three unknowns $\hat{\psi} = \psi(\hat{u})$, $\hat{\psi}' = \psi'(\hat{u})$, and $\hat{\psi}'' = \psi''(\hat{u})$. After setting, for example, $\hat{\psi} = 1$ (an auxiliary preliminary normalization of the solution), this linear algebraic system yields the solution of BVP (5.1)–(5.3) and its derivatives at the point $u = \hat{u}$.

Step 4. At the point $u = \hat{u}$, the solution is additionally subject to an inhomogeneous boundary condition that is consistent with the found value and transversal to the result of transferring boundary condition (5.2) from the point $u = u_0$. More specifically, following [27], the row

$$\chi_l(\hat{u}) = (\beta(\hat{u}), \alpha(\hat{u}), -\hat{u}^2)$$

is supplemented with another two to obtain a basis in $\mathbb{R}^3$. If $S_l(\hat{u})$ is the matrix consisting of these two rows, then find the value (column vector of two components)

$$\hat{b}_l = S_l(\hat{u})(\hat{\psi}, \hat{\psi}', \hat{\psi}'')^{\mathrm{T}}. \tag{6.1}$$

Similarly, find $S_r(\hat{u})$ and $\hat{b}_r$ taking into account the transfer of boundary condition (5.3) from the point $u = u_\infty$.

Step 5. For solutions of ODE (5.1), the inhomogeneous boundary condition (6.1) yields the linear relation

$$b_l(u) = S_l(u)(\psi(u), \psi'(u), \psi''(u))^{\mathrm{T}}, \tag{6.2}$$

which is rewritten in the equivalent form

$$(\psi(u), \psi'(u), \psi''(u))^{\mathrm{T}} = Z_l(u)r_l(u) + w_l(u), \tag{6.3}$$

where $Z_l(u)$ and $w_l(u)$ are columns consisting of three components and $r_l(u)$ is a scalar function; moreover, at the point $u = \hat{u}$,

$$S_l(\hat{u})Z_l(\hat{u}) = 0, \quad w_l(\hat{u}) = (S_l^{\mathrm{T}}(\hat{u})S_l(\hat{u}))^{-1}S_l^{\mathrm{T}}(\hat{u})\hat{b}_l, \tag{6.4}$$

and $r_l(\hat{u}) = -(\chi_l(\hat{u})Z_l(\hat{u}))^{-1}\chi_l(\hat{u})w_l(\hat{u})$.

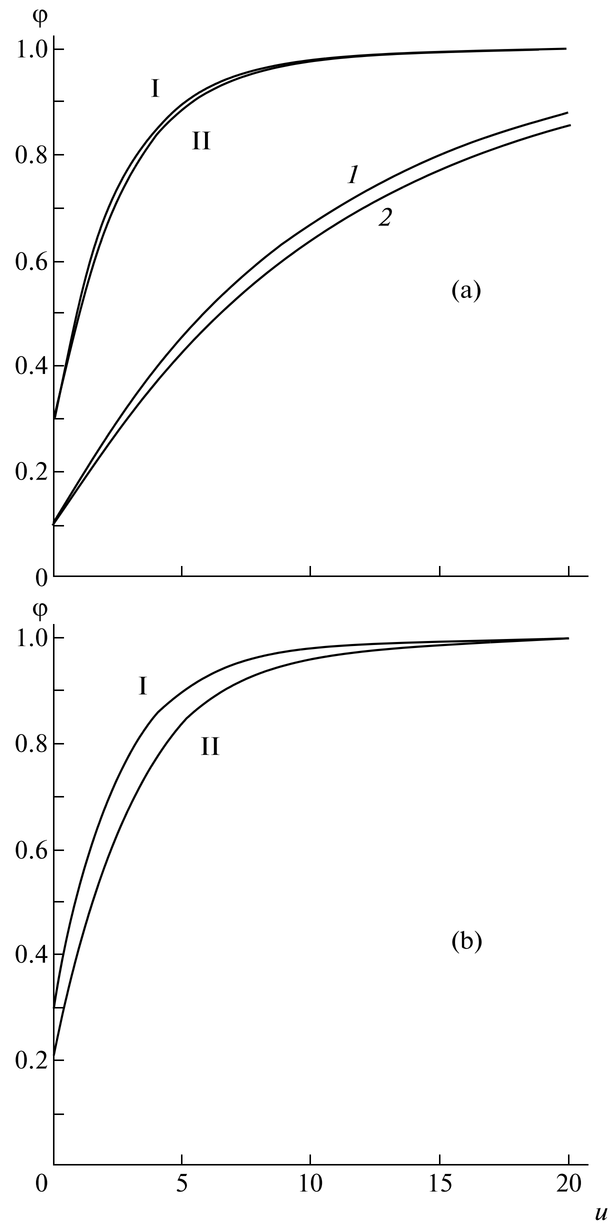

Fig. 2.

Then the following two approaches are possible, of which the second is more efficient in the number of TMA equations and was implemented in this paper. (1) Transfer relation (6.2) from right to left from the point $u = \hat{u}$ to $u = u_0$ by the orthogonal differential TMA method [29] with allowance for the initial values $S_l(\hat{u})$ and $\hat{b}_l$ determined at Step 4 (see (6.1)). At the fixed points of $[u_0, \hat{u}]$ where the direct TMA results are stored, use the TMA method in the opposite direction to determine the solutions of BVP (5.1)–(5.3) based on two relations (4.16) and (6.2) obtained at these points (three linear algebraic equations with three unknowns ψ, ψ', ψ'' at a given point). (2) Transfer relation (6.3) from right to left from the point $u = \hat{u}$ to $u = u_0$ by the orthogonal differential TMA method [30, Chapter IX, pp. 581–582] (also see the description in [27]) with allowance for the initial data determined in (6.4). At the fixed points of $[u_0, \hat{u}]$ where the direct TMA results are stored, use the TMA method in the opposite direction to determine the solutions of BVP (5.1)–(5.3) based on two relations (4.16) and (6.3) obtained at these points (four linear algebraic equations with four unknowns ψ, ψ', r_l at a given point).

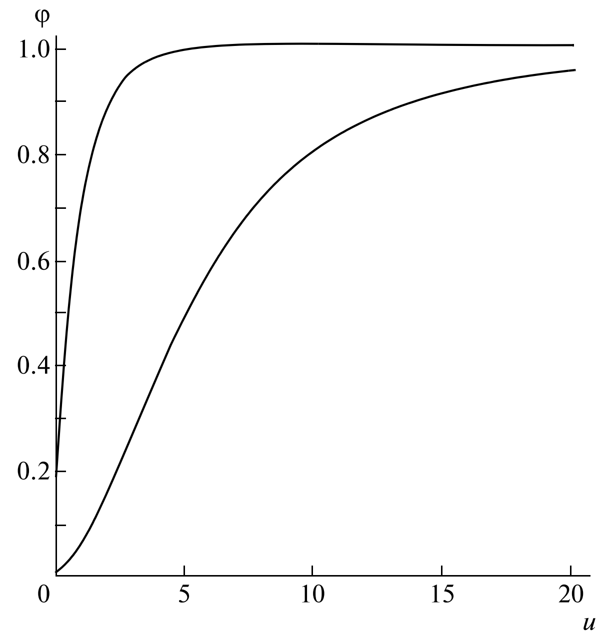

Fig. 3.

Similarly, execute the T~~M~~A in the opposite direction (according to the more economic second version) from the point $\hat{u}$ to u_∞ to find the solutions of BVP (5.1)–(5.3) at points of the interval $[\hat{u}, u_\infty]$.

The stability of solutions of the T~~M~~A equations ~~of~~ in the direction toward singular points was analyzed in [27].

Step 6. Find an approximate solution of the original constrained singular BVP (2.1)–(2.5) by numerical integration based on formula (5.11).

6.2. Numerical Results and Comparison with Model I

The goal of the computations was (1) to confirm the suitability of the developed methods and algorithms and to illustrate the theorems; (2) to compare the numerical results with exact solutions for the Cramer–Lundberg models and with numerical results produced by model I and to obtain results with a financial interpretation; and (3) to examine the behavior of the survival probability function for model II depending on the parameter values, including the convexity property.

The computations were performed in Maple 14.01[1] up to a prescribed accuracy with additional techniques used for monitoring the number of significant digits.

The survival probability was plotted as a function of the initial surplus under the conditions for comparison of models I and II (see Section 1.2) for certain sets of parameters.

1. Figure 2 compares models I and II for positive safety loads with inequality (2.10) satisfied for model II. Here and below, the plots and the resulting values for models I and II are marked with I and II respectively. The plots of exact solutions and related quantities calculated for the Cramer–Lundberg models are marked with *1* and *2*. For Fig. 2, we used the fixed values $a = 0.02$, $b = 0.1$, $m = 1$, $\lambda = 0.09$, and $c = 0.1$ for models I and II and the same values $m = 1$, $\lambda = 0.09$, and $c = 0.1$ with $a = b = 0$ for the Cramer–Lundberg models. For model I and the classical Cramer–Lundberg model, we obtained $\varphi_{\rm I}(0) \approx 0.29440$, $\varphi_{\rm I}'(0) \approx 0.26496$, $\varphi_{\rm I}''(0) \approx -0.07949$; $\varphi_1(0) = 0.1$, and $\varphi_1'(0) = 0.09$. For the other parameter values and obtained values, we have (1) $n = 0.1$, $\lambda_1 = 1.0$ ($n\lambda_1 = c = 0.1$), $\varphi_{\rm II}(0) \approx 0.279785$, $\varphi_{\rm II}'(0) \approx 0.25861$, $\varphi_{\rm II}''(0) \approx -0.06963$; $\varphi_2(0) \approx 0.09174$, and $\varphi_2'(0) \approx 0.08332$ for Fig. 2a; and (2) $n = 0.9$, $\lambda_1 = 1/9$ ($n\lambda_1 = c = 0.1$), $\varphi_{\rm II}(0) \approx 0.20535$, $\varphi_{\rm II}'(0) \approx 0.215336$, and $\varphi_{\rm II}''(0) \approx -0.01900$ for Fig. 2b.

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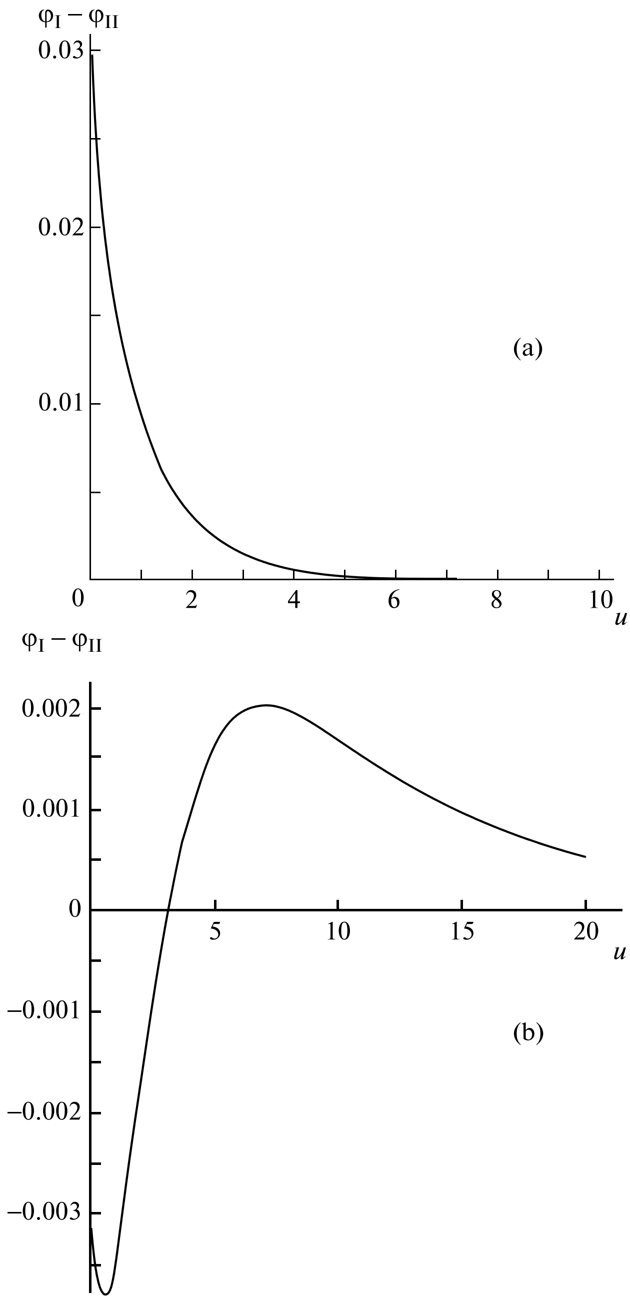

Fig. 4.

It can be seen that the plots for model I are higher than those for model II. In Fig. 2a, both plots are higher than those for the Cramer–Lundberg models. The difference $\varphi_{\mathrm{I}}(u) - \varphi_{\mathrm{II}}(u)$ in Fig. 2a (Fig. 2b) is maximal at $u \approx 1.0$ ($u \approx 1.2$) and is approximately equal to 0.017 (0.115, respectively).

2. Figures 3 and 4 compare models I and II for negative safety loads and positive (negative) risk factors in the case when condition (2.10) holds for model II. Figure 3 shows the plots for model II. The numerical results produced by models I and II coincide to within the graphic accuracy. For this reason, the difference $\varphi_{\mathrm{I}}(u) - \varphi_{\mathrm{II}}(u)$ in Fig. 4 is depicted on a different scale. In all the cases, we used the fixed values $b = 0.1$, $m = 1$, $\lambda = 0.09$, $c = 0.02$, $n = 0.2$, and $\lambda_1 = 0.1$, while a was varied.

(1) $a = 0.1$ was used in Fig. 3 (upper plot) and Fig. 4a. This means that the risk factor was positive for both models and condition (5.4) was violated for model II. The resulting values were $\varphi_{\mathrm{I}}(0) \approx 0.193696$, $\varphi_{\mathrm{I}}'(0) \approx 0.871632$, $\varphi_{\mathrm{I}}''(0) \approx -1.307447$; $\varphi_{\mathrm{II}}(0) \approx 0.163144$, $\varphi_{\mathrm{II}}'(0) \approx 0.965437$, and $\varphi_{\mathrm{II}}''(u) \longrightarrow \infty$ as $u \longrightarrow +0$.

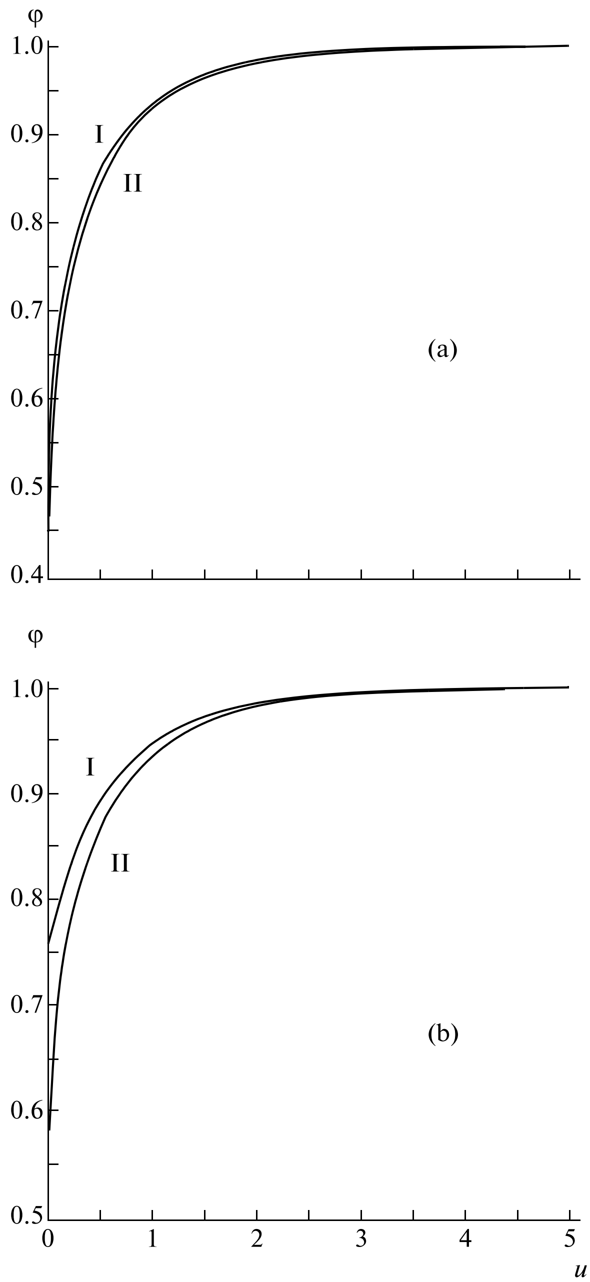

Fig. 5.

(2) $a = 0.02$ was used in Fig. 3 (lower plot) and Fig. 4b. This means that the risk factor was negative for both models and condition (5.4) was satisfied for model II. The resulting values were $\varphi_I(0) \approx 0.00527$, $\varphi_I'(0) \approx 0.02371$, $\varphi_I''(0) \approx 0.05927$; $\varphi_{II}(0) \approx 0.00838$, $\varphi_{II}'(0) \approx 0.02662$, and $\varphi_{II}''(0) \approx 0.051338$.

It can be seen that the following two cases are distinguished for nonpositive safety loads:

(a) For positive risk factors, the plot for model I lies higher than the plot for model II (see Fig. 4a).

(b) For nonpositive risk factors, the pattern is more complicated (see Fig. 4b): for small initial surplus (in the highest risk domain), the plot for model I is lower than that for model II. There is a point $u = u_{ins} > 0$ at which the function values for models I and II coincide. For all $u > u_{ins}$, the plot for model I is again higher than that for model II. At some point of the interval $[0, u_{ins}]$, the difference $\varphi_I(u) - \varphi_{II}(u)$ has a negative minimum, while, for $u > u_{ins}$, there is a point at which the difference $\varphi_I(u) - \varphi_{II}(u)$ has a positive maximum.

Thus, in the highest risk domain, i.e., for small values of the initial surplus, the survival probability is higher in the model with stochastic premiums (model II) than in model I. This means that, for small values of the initial surplus, the additional independent risk arising after making the premium process stochastic (with the expected premium size per unit time being preserved) more effectively compensates for claim risks. For moderate and large values of the initial surplus, i.e., in a more stable situation, the model with deterministic premiums yields higher survival probability values ~~than model I~~.

Note that, as is known, survival probability values can be small, while losses from decisions made as based on survival probability data can be significant.

3. Figure 5 compares models I and II in the case when condition (2.10) is violated for model II, i.e., inequality (2.11) holds. In all the cases, we used the fixed values $a = 0.2$, $b = 0.1$, $m = 1$, $\lambda = 0.05$, and $\lambda_1 = 0.1$. For Fig. 5a, the additional parameter values corresponded to negative safety loads: $c = 0.02$ and $n = 0.2$. The resulting values were $\varphi_{\mathrm{I}}(0) \approx 0.583001$, $\varphi'_{\mathrm{I}}(0) \approx 1.457501$, $\varphi''_{\mathrm{I}}(0) \approx -12.388762$; $\varphi_{\mathrm{II}}(0) \approx 0.455938$, $\varphi'_{\mathrm{II}}(0) \longrightarrow \infty$ as $u \longrightarrow +0$, and $\varphi''_{\mathrm{II}}(u) \longrightarrow -\infty$ as $u \longrightarrow +0$. For Fig. 5b, the values of the additional parameters corresponded to positive safety loads: $c = 0.08$ and $n = 0.8$. The resulting values were $\varphi_{\mathrm{I}}(0) \approx 0.756575$, $\varphi'_{\mathrm{I}}(0) \approx 0.472859$, $\varphi''_{\mathrm{I}}(0) \approx -1.359471$; $\varphi_{\mathrm{II}}(0) \approx 0.570519$, $\varphi'_{\mathrm{II}}(u) \longrightarrow \infty$ as $u \longrightarrow +0$, and $\varphi''_{\mathrm{II}}(u) \longrightarrow -\infty$ as $u \longrightarrow +0$.

It can be seen that the plots for model I are higher than those for model II: $\varphi_{\mathrm{I}}(u) > \varphi_{\mathrm{II}}(u)$ $\forall u \in [0, u_\infty]$.

7. CONCLUSIONS AND ECONOMIC INTERPRETATION OF THE RESULTS

The constrained singular BVP (2.1)–(2.5) was correctly stated and examined in order to compute the survival probability in an insurance model with stochastic premiums and investments in risky assets. An algorithm was constructed for computing the survival probability as a function of the initial surplus (IS), and numerical computations were performed. Importantly, the proof of the existence of a solution to the posed BVP was the necessary stage in the theoretical substantiation of the form of the survival probability function in the model.

The natural heuristic arguments from [1] (and the apparatus of generating operators for Markov processes) make it possible to derive an equation (specifically, a linear IDE) for the survival probability as an IS function under the assumption that the survival probability is a twice continuously differentiable function of the ~~initial surplus~~. Then, before using this IDE to analyze, for example, the asymptotic behavior of the survival probability for large IS values, it is necessary to prove that the survival probability is indeed a twice continuously differentiable function and, on the other hand, to substantiate the limiting condition for the solution of the IDE to tend to unity at infinity. This substantiation can be based, for example, on upper bounds for the ruin probability similar to Lundberg's estimate in the classical model, which shows that, for a positive safety load, the ruin probability vanishes as the initial surplus tends to infinity. However, in the study of this issue in the model with investment in risky assets in [1], these proofs were not presented. As a result, the function obtained in [1] as asymptotics of the solution to the IDE contained an undetermined additive constant. Eventually, the assertion that this function determines the asymptotic behavior of the survival probability (at least, for some value of this constant) remained unsubstantiated.

In this paper, we used an approach based on the correct statement and analysis of the problem for the survival probability on the entire nonnegative half-line with proving the existence of its solution and with the results of [5] taken into account. As a result, we were able to avoid the above difficulties. Specifically, there was no need to prove that the survival probability is twice continuously differentiable and to derive upper bounds at large IS values (for model I, ~~we used and improved~~ the estimates from [32] in order to substantiate the asymptotics of the survival probability for large IS values ~~obtained in [31]~~, but estimates of this type for model II were not obtained in [1]; they were obtained only for the Cramer–Lundberg model with stochastic premiums (see also [3]). Additionally, the approach taken made it possible to compute the survival probability, to compare the numerical results obtained for models I and II, and to give their economic interpretation.

The adequacy of the constructed solutions and computations is demonstrated by the fact that the plots of the survival probability functions in models I and II are close to each other for frequently received premiums of small sizes in model II if only the expected premium rates are equal in both models. This fact suggests that the premium process can be approximately treated as a deterministic process under the assumption that the premium rate is much higher than the claim rate, which underlies the classical Cramer–Lundberg model.

At the same time, the numerical results make it possible to analyze the cases in which the use of the classical Cramer–Lundberg model as an initial risk process can overestimate or underestimate the survival probability in comparison with the results produced by the model of a risk process with stochastic premiums. Specifically, for positive safety loads in the original model, the survival probability obtained with model I turns out to be overestimated on the entire nonnegative semiaxis of IS values. Moreover, for both models, the use of investments considerably increases the survival probability for small IS values in comparison with the corresponding models without investments (the classical Cramer–Lundberg model and the Cramer–Lundberg model with stochastic premiums). For negative safety loads, when ruin in the original risk models occurs with a probability of 1, the application of investments with a constant portfolio structure (provided that the portfolio is reliable) always leads to positive survival probability values. Thus, the application of investments effectively cancels out the insurer's risk at high values of this risk. Based on the study of solutions on the entire nonnegative half-line (performed for model II in this paper and for model I in [8–10]), this conclusion would not be possible as based on comparing only their asymptotic behavior at large IS values as in [31], where it was stated that in insurance investments in risky assets are dangerous. It turns out that, for small IS values, the conclusion is different. More specifically, risky investments at small IS values are not only safe, but also necessary for increasing the paying capacity. More accurate conclusions can be drawn from the study of optimal control of investments in the constrained Cramer–Lundberg model aimed at the minimization of the ruin probability. These results are based, in particular, on the study of model I (see [8]).

A comparison of the numerical results obtained for the survival probability in models I and II for nonpositive safety loads in the original risk models and for identical expected premiums has shown that the conclusions depend on the models' risk factors $i_{r,\,\mathrm{I}}$ and $i_{r,\,\mathrm{II}}$ defined in (1.29) and (5.13), respectively. The most risky is the case of nonpositive risk factors, when the plot of the survival probability has an inflection point (for more detail, see the discussion of this case in Section 6.2).

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