Eigenvalue tunneling and decay of quenched random network

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We consider the canonical ensemble of N-vertex Erdős-Rényi (ER) random topological graphs with quenched vertex degree, and with fugacity μ for each closed triple of bonds. We claim complete defragmentation of large-N graphs into the collection of $[p^{-1}]$ almost full subgraphs (cliques) above critical fugacity, μ_c, where p is the ER bond formation probability. Evolution of the spectral density, $\rho(\lambda)$, of the adjacency matrix with increasing μ leads to the formation of a multizonal support for $\mu > \mu_c$. Eigenvalue tunneling from the central zone to the side one means formation of a new clique in the defragmentation process. The adjacency matrix of the network ground state has a block-diagonal form, where the number of vertices in blocks fluctuates around the mean value Np. The spectral density of the whole network in this regime has triangular shape. We interpret the phenomena from the viewpoint of the conventional random matrix model and speculate about possible physical applications.

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I. INTRODUCTION

Investigation of critical and collective effects in graphs and networks has become a new rapidly developing interdisciplinary area, with diverse applications and a variety of questions to be asked (see [1] for review). Ensembles of random Erdős-Rényi (ER) topological graphs (networks) provide an efficient laboratory for testing collective phenomena in statistical physics of complex systems, being also tightly linked to the conventional random matrix theory. Triadic interactions, being the simplest interactions beyond the free-field theory, play a crucial role in the network statistics. Presence of such interactions is responsible for emergence of phase transitions in complex distributed systems. The first example of a phase transition in random networks, known as Strauss clustering [2], has been treated by the random matrix theory (RMT) in [3]. It was argued that, when the fugacity, μ, having a sense of a chemical potential of closed triads of graph edges in the canonical ensemble, is increasing, the system develops two phases with essentially different triad concentrations. At large μ the system falls into the Strauss phase with the single clique (almost full subgraph) of nodes. The condensation of triads is a nonperturbative phenomenon identified in [4] with the first-order phase transition in the framework of the mean-field cavitylike approach.

Similar critical behavior has been found in [5] for the vertex-degree-conserved ER graphs. It was demonstrated in the framework of the mean-field approach that phase transition occurs here as well. The hysteresis for the dependence of the triad concentration on the fugacity of triads, μ, also has been observed in [5]. For bi-color vertex-degree-conserved networks a phenomenon of a wide plateau formation in the concentration of black-white bonds as a function of the fugacity of unicolor bond triples has been found in [6].

All these models are essentially athermic: in absence of an external field, the network partition function is a purely combinatorial object with no interactions and no temperature dependence; its evolution can be regarded as a Langevin dynamics in the stochastic quantization framework. The solution to the corresponding Fokker-Planck equation at the infinite stochastic time yields the exact ground state of the model.

Here we provide deeper insight into the phase transition structure of nondirected vertex-degree-conserved ER random graphs, with the number of fully connected triads of vertices, $n_\triangle$, controlled by the chemical potential μ. Numerical simulations possess a number of striking phenomena.

(1) The network splits above some critical value μ_c into the *maximally* possible number of clusters, identified as cliques.

(2) The number of cliques for large graphs is fixed *exclusively* by the average vertex degree at the network preparation, p.

(3) The distribution of the eigenvalues of adjacency matrices (spectral density) above μ_c has the triangular shape, usually observed for scale-free networks [7,8]. This behavior differs significantly from the unconstrained Strauss model, where above the transition point the single clique is formed.

Qualitatively the formation of the Strauss condensate in the unconstrained network can be understood as follows. For $\mu = 0$ the system lives in the largest entropic basin corresponding to some equilibrium distribution of triads. As μ is increasing, the triad distribution gets gradually more skewed. In the limit $\mu \to \infty$, the entropic effects become irrelevant, and the network approaches the state with the largest energy, $\mu n_\triangle$. Depending on the shape of the entropy, the function $n_\triangle(\mu)$ can be either a smooth function or can undergo an abrupt jump typical for the first-order phase transition. In contrast, in the vertex-degree-conserved model, constraints prevent the complete mixing of links, thus prohibiting the formation of a single large clique. The system does its best under specific "conservation laws" and splits into the maximally possible number of allowed cliques, which is again the true ground state for the network with the quenched vertex degree. The same decay we observe as well for the "regular random networks" (graphs, having one and the same degree Np in all vertices).

Any topological graph (collection of graphs) can be encoded by the adjacency matrix, A [9]. In the model with a quenched vertex degree, isolated eigenvalues of A form, within the transition region in μ, the second zone in the spectrum and correspond one by one to clusters in the large network (see [10–12] for general description). Above the transition point, μ_c, the spectral density (SD), $\rho(\lambda)$, of the adjacency matrix of each clique (almost fully connected subgraph) is the same as the spectral density of the sparse matrix, and has the Lifshitz tails typical for the one-dimensional Anderson localization, as discussed in [13]. Besides, the spectral density of the whole network has a trianglelike shape [7,8], typically seen in scale-free networks. Note, however, that in our system the set of vertex degrees is quenched at the preparation, having the Poisson distribution, and cannot be transformed in the entire graph in the course of redistribution of links.

II. THE MODEL

We consider the ensemble of topological Erdős-Rényi graphs with quenched vertex degree. Each closed triple of bonds (the closed triadic motif) is weighted with the fugacity μ. The partition function of the system can be written as

$$Z(\mu) = \sum_{\{\text{states}\}}{}' e^{-\mu n_\triangle} \tag{1}$$

where the prime in Eq. (1) means that the summation runs over all possible configurations of nodes, under the condition of fixed degree v_i in each vertex i ($i = 1,\ldots,N$) of the graph. The initial state of the ER network is prepared by connecting any randomly taken pair of vertices with the probability p (the double connections are excluded). When the initial pattern is prepared, one randomly chooses two arbitrary links, say, between vertices i and j, (ij), and between k and m, (km), and reconnect them, getting new links (ik) and (jm). Such reconnection conserves the vertex degree [14]. Now, one applies the standard Metropolis algorithm with the following rules: (i) if under the reconnection the number of closed triads is increased, a move is accepted; (ii) if the number of closed triads is decreased by $\Delta n_\triangle$, or remains unchanged, a move is accepted with the probability $e^{-\mu \Delta n_\triangle}$. Then the Metropolis algorithm runs repeatedly for a large set of randomly chosen pairs of links, until it converges. In [5] only the behavior of $\langle n_\triangle(\mu)\rangle$ was considered. Here we turn to a more detailed study of the constrained network ground state. In [15] it was proven that such Metropolis algorithm converges to the Gibbs measure $e^{\mu N_\triangle}$ in the equilibrium ensemble of random undirected Erdős-Renyi networks with fixed vertex degree.

III. THE RESULTS

We state that, given the bond formation probability, p, in the initial graph, the evolving network splits into the maximal number of clusters. The maximal number of clusters is given

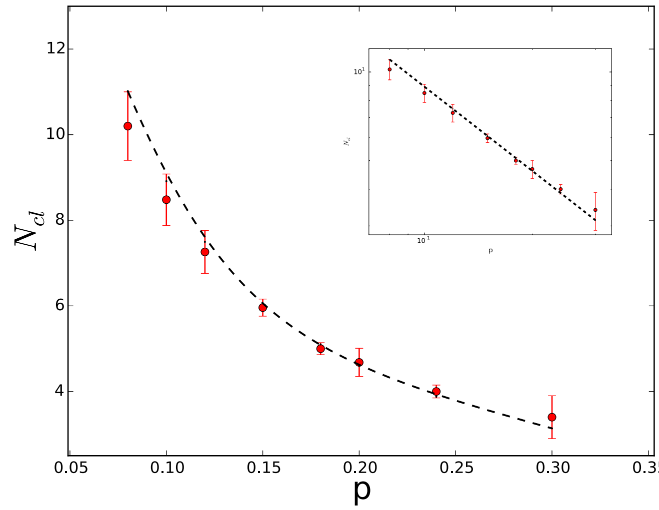

FIG. 1. The number of clusters $N_{\rm cl}$ as a function of the probability p in the ER graph. The numerical data are obtained by averaging over 100 randomly generated graphs of 512 vertices. For the chosen number of nodes $p^2N > 2$, which allows us to neglect the second term in Eq. (2). Numerical values are fitted by the curve $p^{-0.95}$; the behavior in doubly logarithmic scale is shown in the inset.

by

$$N_{\rm cl} = \left[\frac{N}{Np+1}\right] \approx \left[\frac{1}{p} - \frac{1}{p^2N}\right], \tag{2}$$

where $[\ldots]$ means the integer part and the denominator $(Np + 1)$ defines the minimal size of formed cliques. This expression leads to the asymptotic $\sim [p^{-1}]$, being independent on the particular set of corresponding vertex degrees, $\{v_1,\ldots,v_N\}$ (see Fig. 1). Thus, the system falls into the new phase different from the Strauss one. Below we confirm this by analyzing the spectral density of the adjacency matrix.

For any particular quenched network pattern and $\mu < \mu_c$, the spectral density has the shape typical for ER graphs with moderate connection probability, $p = O(1) < 1$, being the Wigner semicircle with a single isolated eigenvalue apart. At μ_c the eigenvalues decouple from the main core and a collection of isolated eigenvalues forms the second zone. The number of isolated eigenvalues exactly coincides with the number of clusters formed above μ_c. This perfectly fits the result of [11]. Averaging over an ensemble of graphs patterns smears the distribution of isolated eigenvalues in the second zone. Above μ_c the support of SD in the first (central) zone shrinks and the second zone becomes dense and connected—see Fig. 2.

We have investigated SD inside each cluster (C) aiming to prove that C is almost a full graph (clique). We see from Fig. 3 that the enveloping shape of the SD is drastically changed at μ_c, where SDs in the first (main) zone below and above μ_c are shown. One sees that as μ is increasing, the enveloping profile of SD is gradually changing from the semicircle to the triangle, typical for the scale-free networks [7,8]. However, in our case the vertex degree is conserved at the network preparation and can be only redistributed between cliques. The remarkable point is that the SD evaluated for each particular clique exhibits

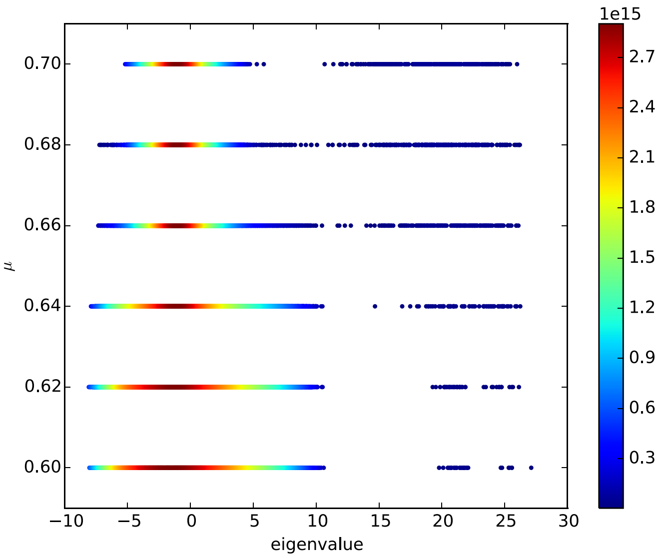

FIG. 2. Spectral density of ensembles of ER graphs for different μ. The numerical results are obtained for 50 ER graphs of 256 vertices and $p = 0.08$.

a hierarchical set of resonance peaks typical for sparse matrices (see [13] and references therein).

Using duality between the spectrum of sparse and almost full graphs [16], we see from Fig. 4 that SD in almost the complete graph fits perfectly the shifted SD of the sparse matrix ensemble, meaning that our identification of clusters with cliques and separated eigenvalues is true. The striking difference between the SD of the single clique and the whole network indicates that the triangle-shape SD of the whole networks occurs due to the interclique connections.

It is instructive to compare typical adjacency matrices in the ground state of ER networks with and without constraints at the same value of p. The corresponding matrices are shown in

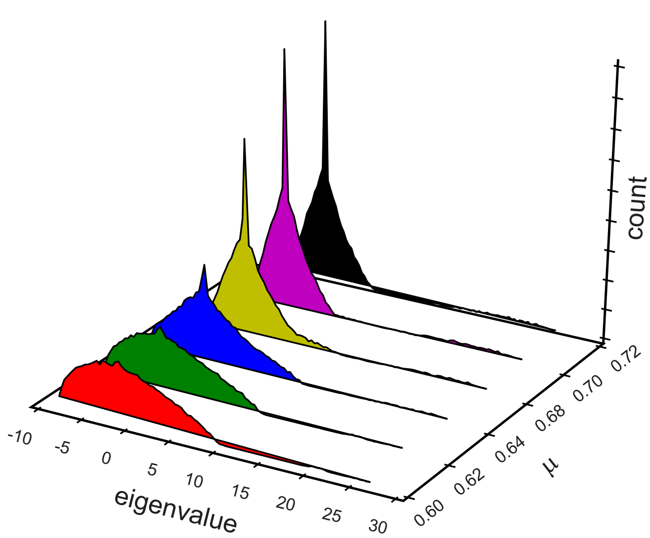

FIG. 3. The spectral density in the ensembles of ER graphs for different values μ. The numerical results are obtained for the ensembles of 50 random ER graphs of 256 vertices and the probability $p = 0.08$.

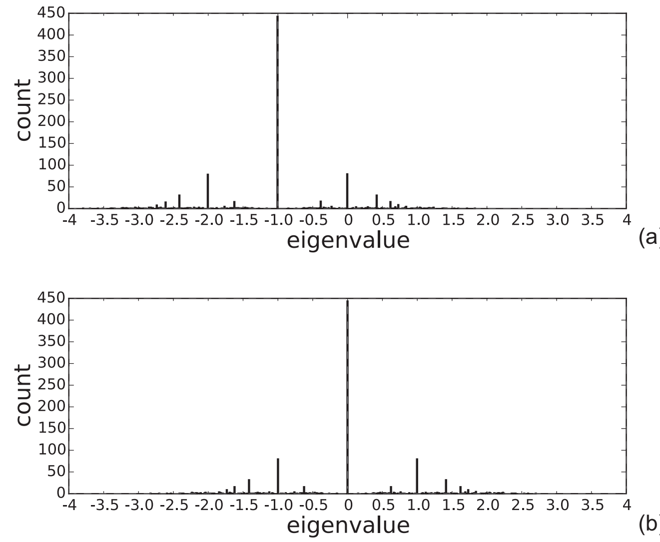

FIG. 4. The duality between spectral densities of ensembles of almost fully connected graphs and their sparse complements: (a) spectral density of almost full graphs and (b) spectral density of the dual system of sparse graphs.

Fig. 5, where different phases of the ground states are clearly seen. The ground state of the quenched network involves $[p^{-1}]$ almost complete graphs corresponding to blocks (cliques) of the adjacency matrix A with fluctuating sizes N_i ($\sum_i N_i = N$) and the mean value in the clique, $N_{\rm cl} = \langle N_i\rangle = N/[p^{-1}] \approx Np$. In contrast, the ground state in the Strauss phase consists of a single complete graph corresponding to only one block of some size, k, in A. To visualize the kinetics, we enumerate vertices at the preparation condition in arbitrary order and run the Metropolis stochastic dynamics. When the system is equilibrated and the cliques are formed, we re-enumerate vertices sequentially according to their belongings to cliques. Then we restore corresponding dynamic pathways back to the initial configuration.

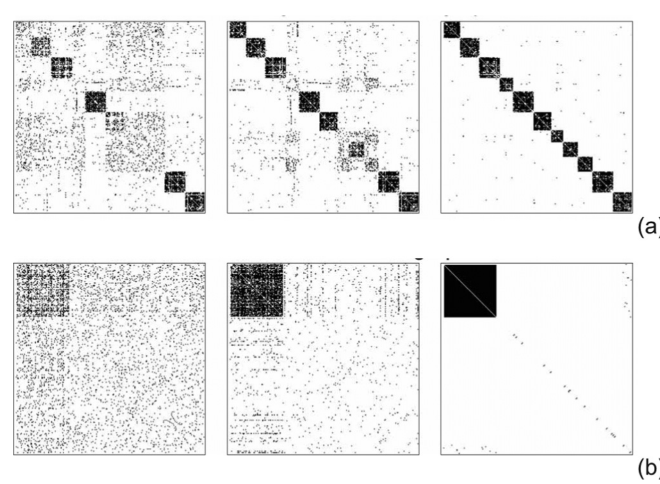

FIG. 5. Few typical samples of intermediate stages of network evolution: (a) networks with fixed vertex degree and (b) networks with nonfixed vertex degree.

In the sparse regime the percolation transition in the ER network occurs at $p_{\text{perc}} = N^{-1}$. Due to the duality, one could expect the dual percolation phase transition in the clique at $\bar{p}_{\text{perc}} = 1 - N_{\text{cl}}^{-1}$. The dual percolation corresponds to the creation of dislocations in the cliques extended through the whole droplet. The right tail of the spectral density in the first zone behaves as

$$\rho(\lambda)\big|_{\lambda\to\lambda_{\max}} \sim e^{-c/\sqrt{\lambda_{\max}-\lambda}} \tag{3}$$

(c is some positive constant) and is, as shown in [13] for sparse ensembles, a manifestation of a Lifshitz tail for the Andersen localization.

IV. COMPARISON WITH THE RANDOM MATRIX MODELS

Found behavior has many parallels in RMT. The effective potential of the Strauss model, proposed in [3], $V(M) = a\mathrm{Tr}M^2 + \mu\mathrm{Tr}M^3$, involves the quadratic term, which fixes the number of links, and the cubic term, encountering triangles [$\mathrm{Tr}M^3 = n_\triangle$ and $a = \log(p^{-1} - 1)$ for the unconstrained ER network]. The Strauss model exhibits the phase transition at the critical value of the chemical potential for triangles, being the "network counterpart" of the phase transition known in the matrix model description of the pure two-dimensional (2D) quantum gravity [17]. However, if the measure in the ensemble of adjacency matrices is unknown, the fermionic statistics of the eigenvalues coming from the Vandermonde factors in the standard matrix ensembles cannot be applied.

There is a Riemann surface associated with any matrix integral with the spectral density defined on the curve

$$y^2 = [V'(x)]^2 + f(x) \tag{4}$$

where y and x are complex variables, and the function $y(x)$ is related to the matrix model resolvent. For the potential $V = ax^2 + \mu x^3$ the Riemann surface for the conventional matrix model has genus 1. The coefficients of polynomial $f(x)$ are fixed by the filling fractions, N_i, around extrema of the potential $V(x)$ [$V'_{\text{eff}}(a_i) = 0$]. The distribution of eigenvalues between two zones in the RMT corresponds to the symmetry breaking $U(N) \to U(N-k) \times U(k)$, where k is the number of eigenvalues in the second zone, and is essentially nonperturbative. The eigenvalue tunneling between two zones is fairly general phenomena in the matrix model framework (see [18] for the review) meaning the formation of a kind of extended coherent object, like a baby Universe. In our case, such tunneling leads to the dense droplets (cliques) formation, being the peculiar example of the global symmetry breaking.

The seed for a second zone always exists for the cubic potential, however in the unconstrained case all, but one, eigenvalues belong to the first (central) zone with the Wigner semicircle distribution. The situation in the constraint-driven case is different since a fixed number of eigenvalues pass from the first zone to the second one through the gap. The eigenvalue tunneling was first discussed in the context of 2D gravity in [19]. The review of general description of eigenvalue tunneling in topological strings and the matrix models can be found in [18]. Depending on the physics described by the matrix model, it corresponds to the account of FZZT branes in noncritical strings, baby Universe creation in quantum gravity, or creation of the D-brane domain wall in supersymmetric gauge theories.

It is convenient to introduce the constraints into the matrix model via Lagrangian multiplies. They yield the linear term in the action $\mathrm{Tr}\Lambda X$ with $\Lambda_{ij} = \{z_1\delta_{1j}, \ldots, z_N\delta_{Nj}\}$ and one has to integrate over all z_i $(i = 1..N)$. The situation resembles the symmetry breaking by the Wilson loop observables in the matrix model framework. In our case the constraints provide the global symmetry breaking down to the block-diagonal form seen in Fig. 5. Therefore, qualitatively, the matrix model framework is consistent with two results of the numerical simulations: presence of multizonal support for the spectral density, and the formation of cliques, by the mechanism of interzone eigenvalue tunneling.

The transition from the Wigner semicircle to the trianglelike SD is known in the matrix model description of the Dirac operator spectrum in QCD [20,21] and admits deep physics behind. If we scan QCD at finite volume at energies, smaller then the Thouless energy, E_T, which divides the "ergodic" and "diffusive" regions in generic systems, the Dirac operator considered as the Hamiltonian in the 4+1 space time enjoys the Wigner semicircle spectral density. In this regime the spectrum is evaluated via the instanton liquid chiral matrix model and is saturated by the constant modes. On the other hand, at $E \gg E_T$, the nonconstant modes are important and the spectral density at small λ reads as

$$\rho(\lambda) = \rho(0) - c|\lambda| \tag{5}$$

where $c > 0$ is defined by the exchange of two soft Goldstone modes between two coherent states with scalar quantum numbers [22]. Such soft modes occur due to the spontaneous chiral symmetry breaking. This perfectly fits with our observation that the trianglelike SD in the multiclique phase appears due to the links connecting different cliques. The global symmetry is broken in our model above the phase transition as well, hence we could expect the presence of soft modes which would play the role of diffusons and could provide the required interclique interaction.

Nowadays the large-N matrix model is interpreted as the theory of an open string tachyon on the N unstable D0 or ZZ branes [23,24]. The final state of the evolution of the unstable system is the coherent state of closed string modes or stable D-branes. We have a clear counterpart of this phenomena in our network as a formation of highly coherent states—the set of cliques, the number of which is fixed from the very beginning. The system on the stable FZZT branes, identified as the Kontsevich-like matrix model [25,26], seems to be relevant for the description of the multiclique phase of our model.

V. CONCLUSION

In this paper we described a decay of the constrained random topological network into new phase above some critical value of the chemical potential for closed triads of bonds. The decay has been analyzed via evolution of the spectral density of the adjacency matrix. The ground state of the system above the transition point is identified with the interacting multiclique state. The eigenvalue tunneling is the key point in our problem. We believe that our model sheds some light on the formation of stable D-brane from unstable ones connected by strings. The imposed constraint is not exotic, being typical for chemical, biological, and social networks. The phenomena can be considered as the operational tool to split the network into the optimal droplets of almost full subgraphs (cliques) for generic random networks. Varying the constraints, the required design of the network ground state can be manufactured.

We conclude by mentioning the possible relation of a random network having quenched vertex degree with some known physical models. In the context of quantum gravity (see [27–29]) the model under consideration is topological and does not involve the metric structure. It is possible to interpret clustering as the appearance of an effective metric, as discussed in [30]. We conjecture that the transition discussed in our work corresponds to the transition from the topological $\langle g_{\mu\nu}\rangle = 0$ phase to the "geometric" phase $\langle g_{\mu\nu}\rangle \neq 0$ of the network, where $g_{\mu\nu}$ is the metric tensor. The geometric phase could be related with the polymer phase of 2D quantum gravity. Another application deals with the budding phenomena (formation of bubblelike vesicles due to spontaneous curvature) in lipid membranes [31]. If the membrane is liquid, the material can be redistributed over the whole tissue and only one vesicle is typically formed. However, in presence of quenched disorder in the membrane, the redistribution of the material over the whole sample is blocked and the formation of multivesicle phase seems plausible.

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