Abstract. We present an infinite sequence of pairs \((A_n, B_n)\) of chess positions on an \(n \times n\) board such that (1) there is a legal sequence of chess moves leading from \(A_n\) to \(B_n\) and (2) any legal sequence leading from \(A_n\) to \(B_n\) contains at least \(\exp(n + o(n))\) moves.

This is a contribution to the complexity theory of puzzles or one-player games. Essentially, a puzzle is a directed graph whose vertices are called positions and arcs are called moves. A player is given a pair of positions and needs to transform one of them into the other with a sequence of moves. Well-known examples of puzzles include Rubik’s cube, Game of Fifteen, and computer simulations like Atomix and Sokoban. An important invariant of a puzzle is its diameter; that is, the greatest number of moves required to transform one of the positions into another. Computing the exact value of diameter is a hard problem that usually requires an extensive computer search [8].

If the diameter of a certain puzzle has a polynomial upper bound, then the player can decide whether a solution exists in nondeterministic polynomial time by executing every possible sequence of moves. This is the case for \(n \times n\) generalizations of Game of Fifteen [7], Rubik’s cube [3], and some other puzzles. On the other hand, the diameter of Sokoban has no polynomial upper bound [5], and this puzzle turns out to be PSPACE-complete.

The order of growth is still unknown for diameters of several classic puzzles, and this note aims to discuss the problem for \(n \times n\) chess. In the puzzle related to the game of chess, we are given two positions on an \(n \times n\) board, and we need to construct a legal series of moves that starts in one of them and ends in the other. Another related problem was mentioned in 1981 by Fraenkel and Lichtenstein [4], who noted that reachability of a given position from another one may be not quite infeasible. Chow gives the following formulation of this problem.

Problem. ([2]) Does there exist an infinite sequence \((A_n, B_n)\) of pairs of chess positions on an \(n \times n\) board such that the minimum number of legal moves required to get from \(A_n\) to \(B_n\) is exponential in \(n\)?

We give a positive solution of this problem, and our paper is structured as follows. In Section 1, we introduce and analyse a new puzzle in which we do not specify the color of pieces. We introduce the notion of flow, which turns out to be a useful invariant of this auxiliary puzzle. In Section 2, we introduce a class of chess positions related to the puzzle from Section 1. We describe each of these positions in terms of the corresponding positions of auxiliary puzzle and flow, and we prove the main result.

The following notation is used throughout our paper. A puzzle is a directed graph with vertices called positions and arcs called moves. A sequence \(\sigma = \sigma_1 \ldots \sigma_n\) (where \(\sigma_i\) is a move from \(\pi_i\) to \(\rho_i\)) is legal if the equality \(\rho_i = \pi_{i+1}\) holds for every \(i\). We call \(\pi_1\) the initial position and \(\rho_n\) the resulting position of \(\sigma\). Moreover, we say that \(\sigma\) is a repetition if \(\sigma\) is legal, \(n = 2, \) and \(\pi_1 = \rho_2\).

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1. AUXILIARY PUZZLE. Consider a graph $G$ that is a union of a cycle of length $3m$ and $m$ triangles; we denote the $j$th vertex of $i$th triangle by $(i, j)$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$. The vertices of the large cycle are labeled $(0, t)$ with $t \in \mathbb{Z}/3m\mathbb{Z}$. The puzzle is played with two types of pieces, one chip and $3m$ switches. We assume that every vertex of $G$ is either empty or occupied by the chip or occupied by exactly one switch. (In particular, it is not possible for a vertex to be occupied by more than one switch at the same time or by the chip and a switch at the same time.)

Also, we assume that either $(i, j)$ or $(0, 3i + j - 3)$ is occupied by a switch. There are two types of moves in this puzzle, and they can be described as follows.

Swap. If the chip is located at one of the vertices $(i, j)$ and $(0, 3i + j - 3)$, then it can be swapped with a switch from the other of these vertices.

Jump. If the chip is located at a triangle, then it can jump (that is, can be moved) to any vacant vertex of the same triangle. Also, the chip can jump from $(0, j)$ to $(0, j + 1)$ or $(0, j - 1)$ if any of these vertices is vacant.

In the initial position $P$, the vertices $(i, 1)$, $(i, 3)$, $(0, 3i - 1)$ are occupied by switches, and the chip is located at an arbitrary vertex of the large cycle. Figure 1 is an example of such a position; we denote the switches by dark squares and the chip by $C$.

![Figure 1. Initial position of an auxiliary puzzle](image)

Consider a move sequence $\sigma$ resulting in a position $P'$. Denote by $\#(\sigma, i_0, j_1, j_2)$ the number of single moves from $\sigma$ that are jumps from $(i_0, j_1)$ to $(i_0, j_2)$. (In particular, $\#(\sigma, 0, j_1, j_2)$ can be nonzero only if $j_1 - j_2$ is either 1 or $-1$.) Define the flow $\mathcal{F}(\sigma, i_0, j_0)$ through $(i_0, j_0)$ as the half of

$$\#(\sigma, i_0, j_0 - 1, j_0) + \#(\sigma, i_0, j_0, j_0 + 1) - \#(\sigma, i_0, j_0 + 1, j_0) - \#(\sigma, i_0, j_0, j_0 - 1).$$

Lemma. If $\sigma$ is legal and $P = P'$, then $\mathcal{F}(\sigma, i, 2) = \mathcal{F}(\sigma, 0, 3i - 1)$.

Proof. We say that a move $\pi$ from $\sigma$ is effective if one of the following conditions holds: (i) $\pi$ is a jump from either $(i, 2)$ or $(0, 3i - 1)$, (ii) $\pi$ is a jump from somewhere to either $(i, 2)$ or $(0, 3i - 1)$, (iii) $\pi$ is the swap between $(i, j)$ and $(0, 3i + j - 3)$, for some $j$. Otherwise, we say that $\pi$ is ineffective.

By $\sigma'$ we denote the subsequence of $\sigma$ consisting of all effective moves; note that $\sigma'$ need not be legal. By definition of flow, $\sigma$ and $\sigma'$ have the same flow through $(i, 2)$ and the same flow through $(0, 3i - 1)$. The combinatorial analysis shows that any effective move leads from a position indexed $t$ on Figure 2 to the position indexed either $t + 1$ or $t - 1$ (such a move is called a direct type $t$ move in the former case and a reversed type $t$ move in the latter case).

Note that a legal sequence of ineffective moves cannot lead from a position indexed $t_1$ to a position indexed $t_2$ unless $t_1 = t_2$ or $\{t_1, t_2\} = \{1, 12\}$. Therefore, a move of
direct type \( t \) (modulo 11) can be followed in \( \sigma' \) by a type \( t + 1 \) move only; similarly, a move of reversed type \( t \) can be followed by a type \( t - 1 \) move only. Since a direct type \( t \) move and a reversed type \( t + 1 \) move form a sequence with zero flow, we can remove such a pair from \( \sigma' \) without changing its flow. Therefore, it suffices to consider the case when \( \sigma' \) is a sequence of 11 consecutive direct type 1, \ldots, 11 moves (or reversed type 12, \ldots, 2 moves). In this case, the quantities \( F(\sigma', i, 2) \) and \( F(\sigma', 0, 3i - 1) \) equal \(-1\) (or 1, respectively).

Let us perform the legal sequence of effective moves leading from 1 to 12, then move the chip from \((0, 3i - 2)\) to \((0, 3i - 3)\). Then, we can make these moves again for switches between \((i - 1, j)\) and \((0, 3(i - 2) + j)\) instead of those between \((i, j)\) and \((0, 3i + j - 3)\). By induction, we construct a legal sequence \( \tau \) of moves that leads to the starting position and satisfies \( F(\tau, i, 2) = -1 \), for any \( i \). Then, the sequence \( \tau^k \) also leads to the starting position, and we have \( F(\tau^k, i, 2) = -k \).

2. CHESS POSITIONS. Define a short bishop graph on the \( n \times n \) chessboard by declaring a pair of squares adjacent if they have the same color and one of them can be reached from the other by a single king move. A set of squares is called a bishop cycle if the short bishop graph it induces is a cycle. Removing two nonadjacent squares splits a bishop cycle into two connected components that we call segments. We say that a pair of squares is touching if they have different colors and one of them can be reached from the other by a single king move. We enumerate the ranks and files of the
chessboard by consecutive integers, and we assume that the upper left corner belongs to the first rank and first file. We set \( p_1 = 7 \), and we denote by \( p_{i+1} \) the smallest prime exceeding \( p_i \). We fix an integer \( n \), and we denote by \( p_m \) the largest prime not exceeding \( n - 5 \).

Now we are ready to describe a pair of positions \( P \) and \( P' \) on the \( n \times n \) chessboard such that (1) there is a legal sequence \( \sigma \) of chess moves leading from \( P \) to \( P' \), and (2) the length of \( \sigma \) cannot be less than some function growing exponentially in \( n \). The positions \( P \) and \( P' \) will use the same multiset of pieces, and the pawns in \( P \) will be located at the same positions as in \( P' \). Therefore, \( \sigma \) can contain no captures and no pawn moves. Also, removing any repetition contained in \( \sigma \), we get a legal sequence of smaller length with the same initial and resulting positions; therefore, we can assume that \( \sigma \) contains no repetitions.

The only pieces actually used in our positions are pawns, bishops, and rooks. These pieces are located in a rectangular region \( R \) of size \( n \times (10m + 1) \) bounded by a pawn chain. The chain forbids pieces to leave \( R \) because pawns are untouchable. Since \( m = o(n) \) as \( n \to \infty \), the \( R \) region fits into the \( n \times n \) board for a sufficiently large \( n \). In what follows, we prove our main result for a chess-like game played in region \( R \) under the standard rules except that (1) white and black do not have to alternate their moves, (2) captures, pawn moves, and repetitions are forbidden, and (3) there are no kings. To prove the stated result under the standard rules of chess, we can place a pair of kings somewhere on the chessboard far from \( R \). From the sequence \( \sigma \) of moves within \( R \), which is constructed in the proof, we may form a sequence \( \tau \) that is in accordance with the standard rules, with white and black moving alternately, and that also leads from position \( P \) to \( P' \). Where \( \sigma \) contains several consecutive black moves, we may insert white moves outside \( R \), and it can easily be arranged that these moves ultimately leave the area outside \( R \) unchanged.

Now we describe the position within \( R \). There are \( m \) disconnected dark-squared bishop cycles of lengths \( 2(p_1 + 1) \), \ldots, \( 2(p_m + 1) \) and one light-squared bishop cycle; all other squares in \( R \) are occupied by pawns. There are \( 3m \) pairwise nonadjacent squares on light cycle that we call switch squares and identify with vertices \((0, t)\) from \( G \); we assume that one of the segments between \((0, t)\) and \((0, t + 1)\) contains no switch squares. There are also three nonadjacent switch squares on \( i \)th dark cycle, and we identify these squares with vertices \((i, 1), (i, 2), (i, 3)\) of \( G \). We say that a square \( x \) lies between \((i_0, j_0)\) and \((i_0, j_0 + 1)\) if \( x \) belongs to that segment with ends \((i_0, j_0)\) and \((i_0, j_0 + 1)\) that contains no switch squares. We assume that the \((i, j)\) and \((0, 3i - 3 + j)\) switch squares are touching, and no other pair of squares on cycles is touching.

In the starting position, the switch squares \((i, 1), (i, 3), (0, 3i - 1)\) are occupied by rooks; one of the squares on the light cycle is vacant, and all the other squares are occupied by bishops. One of the bishops on every dark cycle is white, and all the other pieces are black.

In order to construct the described position, we place the light cycle in six top rows of \( R \). The \( i \)th dark cycle can be located in the rectangle with rank coordinates from 7 to \( p_i + 4 \) and file coordinates from 10\( i \) - 8 to 10\( i \). An example of such a position, which corresponds to case \( n = 18 \) (then, we have \( p_1 = 7, p_2 = 11, p_3 = 13 \) and \( m = 3 \)) is provided on Figure 3; the dotted squares correspond to pawns, which are untouchable. Note that an empty square is unique, and it is marked with a cross.

Now we explain the relation of the constructed position to the puzzle described in the previous section. Note that every rook can only move between the \((i, j)\) and \((0, 3i - 3 + j)\) switch squares. Since repetitions are not allowed, we see that moving a bishop located between switch squares \((i_0, j_0)\) and \((i_0, j_0 + 1)\) to the switch square
(i₀, j₀ + 1) is to be followed by the sequence of consecutive bishop moves eventually leaving (i₀, j₀) vacant. We identify the rooks with switches and the vacant square with the chip, and we note that chess moves in constructed positions correspond to the moves of the puzzle from the previous section.

Now let σ be any legal sequence of moves such that the starting and resulting positions coincide, up to color of pieces. We can note that the flow \( F(\sigma, 0, j) \) is equal to the number of times bishops located between switch squares \((0, j)\) and \((0, j + 1)\) moved through \((0, j)\) to a position between \((0, j)\) and \((0, j - 1)\) minus the number of times bishops moved in an opposite direction. In particular, it is now clear that the flow \( F(\sigma, 0, t) \) does not depend on \( t \). Similarly, since there are one white and 2\( p_i - 1 \) black bishops moving in \( i \)th cycle, the remainder of \( F(\sigma, i, 2) \) modulo 2\( p_i \) determines the position of the white bishop. By Lemma, the flow \( F(\sigma, i, 2) \) is independent of \( i \), but it can take any value as \( \sigma \) varies.

Now consider the starting position \( P \), and construct a new position \( P' \) by swapping the positions of the white bishop from the first dark cycle and the black bishop located a distance two apart from this white bishop. (In particular, \( P' \) can be obtained from the position in Figure 3 by swapping the bishops contained in ovals; in general, we say that two bishops from the same dark cycle lie a distance two apart if they define a segment containing exactly one bishop.) Then, we conclude by the Chinese remainder theorem that there is a sequence \( \sigma' \) leading from \( P \) to \( P' \). Then, \( \sigma' \) is such that \( F(\sigma', i, 2) \) divides 2\( p_i \) if and only if \( i \neq 1 \). Now we see that \( F(\sigma', i, 2) \) is nonzero and divides \( \prod_{i=2}^{m} p_i \); the latter quantity is exponential in \( n \) since the product of primes not exceeding \( n - 5 \) is \( \exp(n + o(n)) \).

3. CONCLUDING REMARKS. The diameter problem has also been studied for some less natural generalizations of chess. In particular, there was an attempt of solving it for a chess-like game with \( O(n) \) kings, none of which can be left under attack; see a discussion in [2]. Also, it was noted that the diameter of chess has a polynomial upper bound if a game is assumed to terminate after 50 consecutive nonpawn moves without captures. Of course, it would be pointless to maintain this rule unchanged for generalized chess; at the very least, any natural generalization should involve replacing 50 by...
a larger number. Also, some authors [4] assume that the number of pieces is bounded from above by a fractional power of board size. In order to make our positions satisfy this restriction, we can add sufficiently many empty ranks and files to them, getting a lower bound for the diameter of \( n \times n \) chess exponential in a fractional power of \( n \).

Finally, let us mention a related decision problem known as retrograde chess. Given a pair of \( n \times n \) chess positions, is there a legal sequence of moves leading from one of them to the other? This problem is NP-hard [1] and belongs to PSPACE [2]. Hearn and Demaine ask the following question [5, 6]: Is retrograde chess PSPACE-complete?

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YAROSLAV SHITOV received his Ph.D. from the Moscow State University in 2012. Now he works as a senior lecturer at the Higher School of Economics.

*Higher School of Economics, National Research University, 20 Myasnitskaya Street, Moscow, Russia 101000 yaroslav-shitov@yandex.ru*
Monthly Gems

In a Monthly Note [1], Yoshio Matsuoka gave an elementary proof that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \). Here, we sketch a simplified version of Matsuoka’s proof.

For every integer \( n \geq 0 \), let \( I_n = \int_0^{\pi/2} \cos^{2n} x \, dx \) and \( J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx \). Clearly, \( I_0 = \pi/2 \) and \( J_0 = \pi^3/24 \). For \( n \geq 1 \), if we evaluate \( I_n \) using integration by parts, with \( dv = \cos x \, dx \), we get \( I_n = (2n - 1)(I_{n-1} - I_n) \), and therefore,

\[
I_n = \frac{2n - 1}{2n} \cdot I_{n-1}.
\]

(1)

Alternatively, we can apply integration by parts twice, first with \( dv = dx \) and then with \( dv = 2x \, dx \), to obtain

\[
I_n = n(2n - 1)J_{n-1} - 2n^2 J_n.
\]

(2)

Dividing (2) by \( n^2 I_n \) and then applying (1), we find that

\[
\frac{1}{n^2} = 2 \left( \frac{J_{n-1}}{I_{n-1}} - \frac{J_n}{I_n} \right).
\]

(3)

We now sum (3) for \( n \) from 1 to \( N \), and note that the right-hand side telescopes:

\[
\sum_{n=1}^{N} \frac{1}{n^2} = 2 \left( \frac{J_0}{I_0} - \frac{J_N}{I_N} \right) = \frac{\pi^2}{6} - 2 \cdot \frac{J_N}{I_N}.
\]

(4)

Finally, we use the inequality \( x \leq (\pi/2) \sin x \) for \( 0 \leq x \leq \pi/2 \) and (1) to estimate \( J_N \) as follows:

\[
0 \leq J_N \leq \frac{\pi^2}{4} \int_0^{\pi/2} \sin^2 x \cos^{2N} x \, dx = \frac{\pi^2}{4} (I_N - I_{N+1}) = \frac{\pi^2}{4} \cdot \frac{1}{2N + 2} \cdot I_N.
\]

Therefore, \( \lim_{N \to \infty} J_N/I_N = 0 \). Letting \( N \to \infty \) in (4), the conclusion follows.

REFERENCE


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