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On the Pseudonormal Form of Real Autonomous Systems with Two Pure Imaginary Eigenvalues

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Abstract—The paper deals with real autonomous systems of ordinary differential equations in a neighborhood of a nondegenerate singular point such that the matrix of the linearized system has two pure imaginary eigenvalues, all other eigenvalues lying outside the imaginary axis. The reducibility of such systems to pseudonormal form is studied. The notion of resonance is refined, and the notions of removable and irremovable resonances are introduced.

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1. INTRODUCTION

The paper deals with issues of reducibility of a real autonomous system of ordinary differential equations to normal form in a neighborhood of a singular point. Namely, we consider systems for which the matrix of the linear part has two pure imaginary eigenvalues, all other eigenvalues lying outside the imaginary axis. Our approach is based on the methods presented in [1].

The normal form of a system of ordinary differential equations is sufficiently well studied. In particular, the problem of analytic reducibility to normal form has been analyzed in detail (e.g., see [2, Chaps. II, III] and [3]). The reduction to normal form is closely related to the local equivalence problem for systems of equations. Finitely smooth problems of this kind are well studied, but most of the relevant papers deal with systems with nondegenerate singular point (or invariant manifold), while even weakly degenerate systems remain largely unexplored yet. As to partially degenerate systems, see [4]. The problem of infinitely smooth equivalence for systems with one zero eigenvalue or a pair of pure imaginary eigenvalues was considered in [5]. We point out that the papers [1] and [5] use methods suggested in [6] and [7], where the relationship between formal and infinitely smooth solutions of linear systems of equations with one zero eigenvalue was studied, and involve transformations with singularities. Since these transformations are not invertible, we refer to the system obtained with the use of such transformations as a *pseudonormal form* rather than a normal form. This approach was developed in [1], where a new insight into the pseudonormal form of systems with one zero eigenvalue was suggested. (In that paper, we did not use the term pseudonormal form and referred to it as a normal form.) The present paper develops the cited methods further and uses them to provide new information on the pseudonormal form of systems with two pure imaginary eigenvalues.

Consider the real autonomous system

$$\dot{\xi} = \frac{d\xi}{dt} = Q(\xi), \tag{1}$$

where $\xi, Q(\xi) \in \mathbb{R}^{n+2}$, $n > 0$, $Q(\xi)$ is a function of class C^∞ in some neighborhood of the origin, $Q(0) = 0$, and the matrix $\widetilde{A} = Q'(0)$ has n eigenvalues outside the imaginary axis and a pair of pure imaginary eigenvalues. The aim of the present paper is to define what the pseudonormal form looks like

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and construct a transformation reducing system (1) to pseudonormal form in some neighborhood of the origin.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\widetilde{A}$ with nonzero real part, and let $\pm i\omega$ be the pair of pure imaginary eigenvalues of $\widetilde{A}$, where $\omega > 0$ and i is the imaginary unit.

By a standard linear transformation, we reduce system (1) to the following form, where $\widetilde{A}$ becomes a Jordan matrix:

$$\begin{aligned} \dot{x}_1 &= i\omega x_1 + f_1(x, y), \\ \dot{x}_2 &= -i\omega x_2 + f_2(x, y), \\ \dot{y}_j &= \epsilon_j y_{j+1} + \lambda_j y_j + g_j(x, y), \qquad j = 1, \dots, n. \end{aligned} \tag{2}$$

Here x and y are complex coordinates, $x = (x_1, x_2)$, $x_2 = \overline{x}_1$, $y = (y_1, \dots, y_n)$, complex conjugate variables satisfy complex conjugate equations (e.g., see [2, pp. 167–168]), and the Taylor series of the functions f_1, f_2, and g_j, $j = 1, \dots, n$, do not contain linear terms. We say that the variables x are *degenerate* and the variables y are *nondegenerate*.

Remark 1. The present paper deals with transformations of systems of equations for complex variables. At the same time, the original system (1) is real. When passing from system (1) to the complex system (2), all equations split into pairs of complex conjugate equations (corresponding to complex conjugate eigenvalues) and real equations (for real eigenvalues). It is important that all subsequent transformations of complex systems have the property that complex conjugate variables and the corresponding equations are taken to complex conjugate ones. To every transformation of this kind, there corresponds a real transformation of the original system. We refer to this principle (condition) as the reality principle (condition).

The following theorem, whose statement uses the well-known notion of resonant monomial (e.g., see [2]–[4]), is standard, and we omit the proof.

Theorem 1. *There exists a nondegenerate C^∞ transformation reducing system* (2) *to a form such that the Taylor series of the right-hand side is a sum of resonant monomials in all variables* (*degenerate as well as nondegenerate ones*).

In what follows, we assume that system (2) has already been reduced to the form indicated in Theorem 1.

For the systems in question, as well as for systems with one zero eigenvalue, the main challenge is to reduce the linear (in the nondegenerate coordinates) part of the system to a convenient form. Here the so-called unit weight resonances (see [8]) are the main obstruction; they occur if the number $(\lambda_{j_1} - \lambda_{j_2})/(i\omega)$ is an integer for some j_1 and j_2, $1 \le j_1 \ne j_2 \le n$. Such eigenvalues, as well as the corresponding variables and equations in the system, will be said to be *equivalent* to each other. If the above-mentioned resonances are absent, then system (1) can always be reduced to polynomial normal form by a transformation of finite smoothness [8, Theor. 3]. Of the above-mentioned resonances, we single out those for which $\lambda_{j_1} = \overline{\lambda_{j_2}}$. This means that $2\omega^{-1} \operatorname{Im} \lambda_{j_1}$ is an integer. The resonances for which this number is odd will be said to be *singular*. The variables corresponding to the eigenvalues λ for which $2\omega^{-1} \operatorname{Im} \lambda$ is odd will be referred to as *singular* variables as well, while if this number is not odd (or is noninteger), then we say that the corresponding variables are *nonsingular*.

The formal system corresponding to system (2) has a two-dimensional invariant center manifold, which corresponds to the imaginary part of the spectrum. It follows from [5] that system (2) has an invariant center manifold of class C^∞ (in general, nonunique) as well; on this manifold, the integral curves in a neighborhood of the singular point are either closed (the case of a center) or are spirals (the case of a focus). The present paper studies systems of the form (2) having a focus on a center manifold. On every center manifold, the system can be reduced by a formal transformation to the normal form

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + ig_2(\widetilde{r}) + g_1(\widetilde{r})), \\ \dot{x}_2 &= x_2(-i\omega - ig_2(\widetilde{r}) + g_1(\widetilde{r})), \end{aligned}$$

where $\widetilde{r} = x_1x_2$ and the $g_j(\widetilde{r})$, $j = 1, 2$, are real formal series without constant terms. We restrict ourselves to the main case in which $g_1(\widetilde{r}) \not\equiv 0$ (the case of a structurally unstable focus).

In view of Eq. (2.7) in [9] and reality considerations, we find that, in the case of a focus, the system can be reduced by a formal transformation to the following normal form on the center manifold:

$$\begin{aligned}\dot{x}_1 &= x_1(i\omega + i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})),\\ \dot{x}_2 &= x_2(-i\omega - i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})).\end{aligned} \tag{3}$$

Here and in what follows,

$$\widetilde{r} = x_1x_2, \qquad \varphi(\widetilde{r}) = \sum_{j=1}^{m} \varphi_j \widetilde{r}^j, \quad \widetilde{b}(\widetilde{r}) = b\widetilde{r}^m + c\widetilde{r}^{2m},$$

$m \ge 1$ is an integer, $\varphi_1, \dots, \varphi_m$, b, and c are real numbers, and $b \ne 0$. Note that the variable $\widetilde{r}$ satisfies the equation $\dot{\widetilde{r}} = 2\widetilde{r}\,\widetilde{b}(\widetilde{r})$. Let us fix one of the center manifolds. After an appropriate change of variables, one can assume that it is specified by the equation $y = 0$. The possibility of a formal transformation in the case of a focus on the center manifold means that there exists a C^∞ transformation reducing the system on the center manifold to the form (3) (e.g., see [5]). We assume that this transformation has already been made in system (2).

Consider the part of system (2) linear in the nondegenerate coordinates,

$$\begin{aligned}\dot{x}_1 &= x_1(i\omega + i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})),\\ \dot{x}_2 &= x_2(-i\omega - i\varphi(\widetilde{r}) + \widetilde{b}(\widetilde{r})),\\ \dot{y} &= A(x)y.\end{aligned} \tag{4}$$

Note that, by an appropriate renumbering of variables, one can ensure that the matrix $\widehat{A}(x)$, which is the formal counterpart of the matrix $A(x)$, is block-diagonal with each individual block containing only a group of equivalent variables. (By $\widehat{A}(x)$ we mean the Taylor series of $A(x)$.)

In the present paper, we often use the argument that the existence of formal transformations of the systems in question implies the existence of the corresponding C^∞ transformations (see [5]–[7]). Hence we replace the analysis of the system itself by the analysis of its formal counterpart in such cases. In view of this, we assume in what follows that the matrix $A(x)$ in (4) has the above-mentioned block-diagonal form as well.

Now note that if $2\omega^{-1} \operatorname{Im} \lambda_j$ is an integer, then the variables y_j and $\overline{y}_j$ are equivalent and the equations for these variables lie in one and the same block. Hence it is easily seen that, for each set of equivalent variables y_j, all numbers $2\omega^{-1} \operatorname{Im} \lambda_j$ are either simultaneously integer (and then all pairs of complex conjugate variables y_j and $\overline{y}_j$ lie in one and the same block) or simultaneously noninteger (and then complex conjugate variables lie in distinct blocks).

Our main goal is to reduce system (4) to a form such that the matrix $A(x)$ is in Jordan form. In view of the block-diagonal form of $A(x)$, it suffices to prove the corresponding assertion for the system corresponding to an individual block of $A(x)$. In each block of equivalent equations including complex conjugate equations (all $2\omega^{-1} \operatorname{Im} \lambda_j$ are integer), we renumber the nondegenerate variables so that the variables with positive imaginary parts of the eigenvalues come first, then the variables with negative imaginary parts of the eigenvalues follow, and the variables corresponding to real eigenvalues close the list. Note also that, in such a block, the numbers $2\omega^{-1} \operatorname{Im} \lambda_j$ are all even or all odd simultaneously; accordingly, all variables in a block are singular or nonsingular simultaneously. Indeed, if two numbers λ_{j_1} and λ_{j_2} lie in the block, then

$$2\omega^{-1} \operatorname{Im} \lambda_{j_1} = 2\omega^{-1} \operatorname{Im} \lambda_{j_2} + 2(i\omega)^{-1}(\lambda_{j_1} - \lambda_{j_2}),$$

the second term on the right-hand side being obviously even, which implies the desired property.

In [1], the author considered shearing and weakly degenerate transformations, which were widely used by Wasow when studying systems with one zero eigenvalue (see [6]). Here we also use similar transformations, albeit of a broader class. Let us give the corresponding definitions.

Definition 1. A *shearing transformation* is a transformation of the form

$$y = S(\widetilde{r})z, \qquad y = S_1(x_1)z, \qquad y = S_2(x_2)z, \tag{5}$$

where

$$S(\eta) = \operatorname{diag}(\eta^{\delta_1}, \eta^{\delta_2}, \dots, \eta^{\delta_n}),$$

the δ_q being rational numbers, and

$$S_j(\eta) = \operatorname{diag}(\eta^{h_{1j}}, \eta^{h_{2j}}, \dots, \eta^{h_{nj}}),$$

the h_{qj} being integers.

Definition 2. A *weakly degenerate transformation* is a change of variables of the form

$$\widetilde{r} = du^l, \qquad y = T^* z = BVTz. \tag{6}$$

Here $B = S_1(x)S_2(x)$ and $T = T(\widetilde{r})$ is the product of finitely many transformations, some of which are shearing transformations of the form $S(\widetilde{r})$ and the remaining ones are C^∞ transformations close to the identity. The number l is a positive integer, and $d = \text{const} > 0$. The transformation V has the same block-diagonal structure $V = \operatorname{diag}(V_1, \dots, V_K)$ as the matrix $A(x)$. Moreover, the V_j are the identity transformations for the blocks that do not include complex conjugate equations (i.e., the corresponding numbers $2\omega^{-1} \operatorname{Im} \lambda_q$ are noninteger), and the V_j have the form

$$V_j = \operatorname{diag}(e^{i\widetilde{h}\alpha}E_1, e^{-i\widetilde{h}\alpha}E_1, E_2), \qquad \alpha = \arg x_1, \quad e^{i\alpha} = \frac{x_1}{\sqrt{\widetilde{r}}}$$

for the blocks including complex conjugate equations, where $\widetilde{h}$ is the minimum of the positive numbers $h_q = \omega^{-1} \operatorname{Im} \lambda_q$ corresponding to the given block. Here E_1 and E_2 are the identity matrices of sizes corresponding to the groups of complex and real variables in the block in question.

Note that the transformation (6) does not affect the variable α, while the degenerate variables are transformed as follows:

$$x_j = \sqrt{du^{l-1}}\, u_j, \qquad u = u_1u_2, \quad j = 1, 2.$$

The following theorem (which is an analog of Theorem 2 in [1]) is our main result about system (4).

Theorem 2. *There exists a weakly degenerate transformation* (6) *reducing system* (4) *to the following pseudonormal form*:

$$\begin{aligned} \dot{u}_1 &= u_1(i\omega + i\psi(u) + b_1u^p + c_1u^{2p}), \\ \dot{u}_2 &= u_2(-i\omega - i\psi(u) + b_1u^p + c_1u^{2p}), \\ \dot{z} &= (A_0 + uA_1 + \dots + u^{p-1}A_{p-1} + u^pA_p)z. \end{aligned} \tag{7}$$

Here $u = u_1u_2$, $\psi(u) = \sum_{h=1}^p \psi_h u^h$, $\psi_1, \dots, \psi_p$, b_1, and c_1 are real numbers, $A_0, A_1, \dots, A_{p-1}$ are constant diagonal matrices, A_p is a constant Jordan matrix, and $p = 2ml$. The matrix A_p has the property that if two diagonal entries $\lambda_{j_1}^h$ and $\lambda_{j_2}^h$ of some matrix A_h, $0 \le h \le p-1$, are distinct, then the corresponding diagonal entries of the Jordan matrix A_p belong to distinct Jordan blocks. The variable $u = u_1u_2$ satisfies the equation

$$\dot{u} = 2(b_1u^{p+1} + c_1u^{2p+1}).$$

By an appropriate choice of d in the transformation (6)*, one can ensure that $|b_1|$ is an arbitrary given positive number.*

Remark 2. Any finite segments of the Taylor series of the C^∞ transformations occurring as factors in T are determined, according to Theorem 2, by finite segments of the Taylor series of the matrix $A(x)$ of system (4). The total number of factors forming T depends as well only on a finite segment of the Taylor series of $A(x)$, the length of this segment depending on m and n. The shearing transformations occurring as factors in T are determined by a finite segment of the Taylor series of $A(x)$, the length of this segment being some number $m_1(m,n)$. The number l depends on n alone. Note also that, for $x \neq 0$, the transformation T in (6) is a nondegenerate C^∞ transformation. The same is true for V, provided that none of the numbers $2\omega^{-1}\operatorname{Im}\lambda_q$ is odd (i.e., there are no singular variables). If some of the integers $2\omega^{-1}\operatorname{Im}\lambda_q$ are odd (i.e., there are singular variables), then the transformation V is discontinuous at the points where x_1 is a real positive number and nondegenerate of class C^∞ at all other points where $x \neq 0$.

Now consider the system obtained from system (2) by a transformation that reduces the linear part (4) of system (2) to the form (7). If there are no singular variables, then the system will have the form

$$\begin{aligned}\dot u_1 &= u_1(i\omega + i\psi(u) + b_1u^p + c_1u^{2p}) + G_1(u_1,u_2,z),\\ \dot u_2 &= u_2(-i\omega - i\psi(u) + b_1u^p + c_1u^{2p}) + G_2(u_1,u_2,z),\\ \dot z &= (A_0 + uA_1 + \dots + u^{p-1}A_{p-1} + u^pA_p)z + F(u_1,u_2,z),\end{aligned} \tag{8}$$

where

$$F(u_1,u_2,z) = u^{-h}F_1(u_1,u_2,z), \qquad G_j(u_1,u_2,z) = u^{-h}G_{1j}(u_1,u_2,z),$$

$F_1(u_1,u_2,z)$ and $G_{1j}(u_1,u_2,z)$ are C^∞ functions whose Taylor series are sums of resonant monomials nonlinear in z, and $h > 0$ is an integer. To get rid of the negative powers, we make the change of variables $z = u^{h+1}w$. As a result, the negative powers of the degenerate variable disappear from the equations in (8), and the Taylor series of the functions $F(u_1,u_2,z)$ and $G_j(u_1,u_2,z)$ become sums of resonant monomials nonlinear in z; moreover, $F(0,0,z) = 0$ and $G_j(0,0,z) = 0$. The last n equations in the linear (with respect to the nondegenerate variables) part of the system will now contain "superfluous" terms of the order of u^{2p}, but, by analogy with what is indicated in the proof of Theorem 2 in [1], one can remove these terms by a nondegenerate infinitely smooth transformation. In the presence of singular variables, these transformations should be supplemented by the change of variables $\alpha = 2\widetilde{\alpha}$, after which the system acquires the desired form. The above remarks concerning smoothness (see Remark 2) also pertain to this transformation. To avoid clumsiness in the exposition, we assume that the resulting system (8) already has the desired form. With regard for the above remarks, system (8) will be called the *pseudonormal form* of system (1).

The reduction of the linear part of the system in question to the form (7) provides new opportunities for refining the notion of resonance. From now on, it is expedient to pass to polar coordinates on the center manifold by the formulas

$$u_1 = re^{i\alpha}, \qquad u_2 = re^{-i\alpha}.$$

System (8) in the new coordinates becomes

$$\begin{aligned}\dot r &= b_1r^{2p+1} + c_1r^{4p+1} + H_1(r,\alpha,z),\\ \dot\alpha &= \omega + \psi(r^2) + H_2(r,\alpha,z),\\ \dot z &= (A_0 + r^2A_1 + \dots + r^{2p}A_p)z + H(r,\alpha,z).\end{aligned} \tag{9}$$

The terms of the Taylor series of the functions $F(u_1,u_2,z)$ and $G_j(u_1,u_2,z)$ in (8) become monomials of the form $r^Le^{iK\alpha}z^s$, where $L \ge 0$ and K are integers and s are tuples of nonnegative integers. We use the standard notation adopted in [1]: $z^s = z_1^{s_1}\cdots z_n^{s_n}$, $s = (s_1,\dots,s_n)$, where $s_1,\dots,s_n$ are nonnegative integers, and $|s|$ is the weight of a tuple s, $|s| = s_1 + \dots + s_n$. These monomials satisfy the following resonance relations:

- $Ki\omega + (s,\lambda) = 0$ if the monomial occurs in the first or second equation of the system.
- $Ki\omega + (s,\lambda) = \lambda_j$ if the monomial occurs in the jth equation, $3 \le j \le n+2$.

In particular, it follows that K is uniquely determined by the tuple s and the number j of the equation where the given monomial occurs.

Definition 3. The *level of a resonant monomial* $r^L e^{iK\alpha} z^s$ occurring in the jth equation, where $3 \le j \le n+2$, of system (9) is the integer q, $0 \le q \le p-2$, satisfying the conditions

$$(\widetilde{s}, \lambda^h) = \lambda^h_{j-2}, \quad 0 \le h \le q, \qquad (\widetilde{s}, \lambda^{q+1}) \ne \lambda^{q+1}_{j-2}. \tag{10}$$

Here and in what follows, we write

$$\widetilde{s} = (K, s_1, \dots, s_n), \qquad \lambda^h = (\lambda^h_0, \lambda^h_1, \dots, \lambda^h_n), \quad 0 \le h \le p, \quad \lambda^h_0 = i\psi_h, \quad \psi_0 = \omega.$$

Recall that the λ^h_j, $1 \le j \le n$, are the diagonal entries of the matrices A_h, $0 \le h \le p$.

If, for some j, $3 \le j \le n+2$, the conditions

$$(\widetilde{s}, \lambda^h) = \lambda^h_{j-2}, \qquad 0 \le h \le p-1, \tag{11}$$

are satisfied, then the level of the monomial $r^L e^{iK\alpha} z^s$ occurring in the jth equation is defined to be $p-1$.

If the monomial $r^L e^{iK\alpha} z^s$ occurs in the first or the second equation, then conditions (10) and (11) are replaced by the conditions

$$(\widetilde{s}, \lambda^h) = 0, \quad 0 \le h \le q, \qquad (\widetilde{s}, \lambda^{q+1}) \ne 0, \tag{12}$$

$$(\widetilde{s}, \lambda^h) = 0, \quad 0 \le h \le p-1, \tag{13}$$

respectively.

Definition 4. We say that a monomial $r^L e^{iK\alpha} z^s$ is *removable* if

- either this monomial is a monomial of level q, $0 \le q \le p-2$, and $L \ge 2(q+1)$
- or it is a monomial of level $p-1$, $L > 2p$, and one of the following inequalities is satisfied:

  $$(\widetilde{s}, \lambda^p) + b_1(L-2p) \ne \lambda^p_{j-2}$$

  if the monomial occurs in the jth equation for some j, where $3 \le j \le n+2$, or

  $$(\widetilde{s}, \lambda^p) + b_1(L-4p-1) \ne 0$$

  if the monomial occurs in the first equation, or

  $$(\widetilde{s}, \lambda^p) + b_1(L-2p) \ne 0$$

  if the monomial occurs in the second equation.

A monomial $r^L e^{iK\alpha} z^s$ is said to be *irremovable* if it is not removable.

It is clear from the last definition that all monomials z^s that are resonance (in the traditional sense) are irremovable.

Remark 3. It follows from the preceding definition that, for

$$L > 4p + 1 + |\widetilde{s}||b_1^{-1}| \max_{0 \le j \le n} |\lambda^p_j|,$$

all monomials $r^L e^{iK\alpha} z^s$ are removable.

Consider the following formal system similar to (9):

$$\begin{aligned} \dot{r} &= b_1 r^{2p+1} + c_1 r^{4p+1} + H_1^*(r, \alpha, z), \\ \dot{\alpha} &= \omega + \psi(r^2) + H_2^*(r, \alpha, z), \\ \dot{z} &= (A_0 + r^2 A_1 + \dots + r^{2p} A_p) z + H^*(r, \alpha, z). \end{aligned} \tag{14}$$

Here the matrices A_j, $1 \le j \le p$, are the same as in (9), and $H_1^*(r, \alpha, z)$, $H_2^*(r, \alpha, z)$, and $H^*(r, \alpha, z)$ are formal series whose terms are resonant monomials of the form $r^L e^{iK\alpha} z^s$ mentioned in the definition of system (9). Note that $|s| > 1$ in these monomials. The following theorem holds for this system.

Theorem 3. *There exists a formal transformation that is close to the identity and reduces system* (14) *to a formal system of the following form* (*for simplicity, we preserve the notation used in system* (14)):

$$\begin{aligned} \dot r &= b_1 r^{2p+1} + c_1 r^{4p+1} + \widetilde H_1(r,\alpha,z), \\ \dot\alpha &= \omega + \psi(r^2) + \widetilde H_2(r,\alpha,z), \\ \dot z &= (A_0 + r^2 A_1 + \dots + r^{2p} A_p) z + \widetilde H(r,\alpha,z), \end{aligned} \tag{15}$$

where $\widetilde H_1$, $\widetilde H_2$, *and* $\widetilde H$ *are formal series whose terms are irremovable monomials.*

In view of the definition of irremovable monomials, the terms in the formal series $\widetilde H_1$, $\widetilde H_2$, and $\widetilde H$ in system (15) can be represented in the form $P(r,\alpha)z^s$, where $P(r,\alpha)$ is a finite sum of monomials of the form $r^L e^{iK\alpha}$ with integer $L \ge 0$ and K, the maximum degree L of these monomials being a linear function of $|s|$.

2. EXAMPLES

Here we illustrate the proof of Theorem 2 on the reduction of a linear (in the nondegenerate coordinates) system (4) to a pseudonormal form. Consider the system of equations

$$\begin{aligned} \dot x_1 &= x_1(i\omega + x_1x_2) + f_1(x,y), \\ \dot x_2 &= x_2(-i\omega + x_1x_2) + f_2(x,y), \\ \dot y_1 &= \lambda_1 y_1 + g_{11}(x)y_1 + g_{12}(x)y_2 + g_1(x,y), \\ \dot y_2 &= \lambda_2 y_2 + g_{21}(x)y_1 + g_{22}(x)y_2 + g_2(x,y). \end{aligned} \tag{16}$$

Here the variables x_1 and x_2, as well as y_1 and y_2, are complex conjugate, the same is true for the pair (λ_1,λ_2), and complex conjugate variables satisfy complex conjugate equations. Moreover, $f_l(x,0) = g_{lj}(0) = 0$ and $|g_l(x,y)| = o(\|y\|)$, $1 \le l,j \le 2$. Let

$$\lambda_1 = a + \beta i, \quad \lambda_2 = a - \beta i, \qquad a \ne 0, \quad \beta > 0, \qquad \text{and} \qquad \omega > 0.$$

In contrast to the general case (see system (3)), we assume that $\varphi(\widetilde r) = 0$ and $\widetilde b(\widetilde r) = \widetilde r$. Note, however, that all the results remain valid in the general case. By Theorem 1, we can assume, without loss of generality, that the Taylor series of the functions on the right-hand sides in this system are sums of resonant monomials. However, the resonant monomials in the last two equations of the system can only be linear in the nondegenerate variables, while the first two equations contain no such resonant monomials related to the functions $f_j(x,y)$, $j = 1,2$, at all. Consequently, all functions $f_j(x,y)$ and $g_j(x,y)$ are planar. Since, according to [5], the formal equivalence of systems of the form (16) implies their C^∞ equivalence, we can assume that these functions are zero. Thus, without loss of generality, we assume that the system in question has the form

$$\begin{aligned} \dot x_1 &= x_1(i\omega + x_1x_2), \\ \dot x_2 &= x_2(-i\omega + x_1x_2), \\ \dot y_1 &= \lambda_1 y_1 + g_{11}(x)y_1 + g_{12}(x)y_2, \\ \dot y_2 &= \lambda_2 y_2 + g_{21}(x)y_1 + g_{22}(x)y_2. \end{aligned} \tag{17}$$

The monomials $x_1^p x_2^q$ occurring in the Taylor series of the function $g_{12}(x)$ satisfy the resonance equation $(p-q)i\omega + \lambda_2 = \lambda_1$. A similar equation holds for $g_{21}(x)$. Consequently, if $(\lambda_1 - \lambda_2)/(i\omega)$ is not an integer, then the functions $g_{12}(x)$ and $g_{21}(x)$ are planar and can be assumed to be zero (see the above reasoning). In this case, the reduction of system (17) to a system of the form (7) is rather elementary and can be achieved by a nondegenerate C^∞ transformation (e.g., see Lemma 2 in [8]).

Now consider the case of integer $h = (\lambda_1 - \lambda_2)/(i\omega) > 0$. The functions $g_{lj}(x)$, $1 \le l,j \le 2$, can be represented in the form

$$g_{ll}(x) = d_{ll}(\widetilde r) + \gamma_{ll}(x), \qquad g_{lj}(x) = x_l^h d_{lj}(\widetilde r) + \gamma_{lj}(x), \quad l \ne j,$$

where $\widetilde{r} = x_1x_2$, $\gamma_{ll}(x)$ and $\gamma_{lj}(x)$ are planar functions, and $d_{ll}(\widetilde{r})$ and $d_{lj}(\widetilde{r})$ are C^∞ functions. By [5], all functions $\gamma_{ll}(x)$ and $\gamma_{ij}(x)$ can be assumed to be zero. In addition, note that $d_{ll}(0) = 0$, $l = 1, 2$.

In what follows, we give a detailed consideration of the nonsingular case, where h is even. As to the singular case, we restrict ourselves to the special (but important) example of system (17); it will be considered at the end of this section.

We set $\widetilde{h} = h/2$ and make the following change of variables:

$$y_1 = e^{i\widetilde{h}\alpha}\nu_1, \qquad y_2 = e^{i\widetilde{h}\alpha}\nu_2, \qquad \alpha = \arg x_1, \quad e^{i\alpha} = \frac{x_1}{\sqrt{\widetilde{r}}}. \tag{18}$$

The variable α satisfies the equation $\dot{\alpha} = \omega$. Note that this transformation is nondegenerate for $\widetilde{r} \neq 0$.

The first two equations in system (17) remain the same, and the last two equations become

$$\begin{aligned} \dot{\nu}_1 &= a\nu_1 + d_{11}(\widetilde{r})\nu_1 + \widetilde{r}^{\widetilde{h}} d_{12}(\widetilde{r})\nu_2, \\ \dot{\nu}_2 &= a\nu_2 + \widetilde{r}^{\widetilde{h}} d_{21}(\widetilde{r})\nu_1 + d_{22}(\widetilde{r})\nu_2. \end{aligned} \tag{19}$$

Here $a = \operatorname{Re}\lambda_1$, $d_{22}(\widetilde{r}) = \overline{d_{11}(\widetilde{r})}$, and $d_{21}(\widetilde{r}) = \overline{d_{12}(\widetilde{r})}$. The variable $\widetilde{r}$ satisfies the equation $\dot{\widetilde{r}} = 2\widetilde{r}^2$. In the resulting system, consider two cases, first with $\widetilde{h} > 1$ and second with $\widetilde{h} = 1$.

In the first case, system (19) can be rewritten as

$$\dot{\nu} = (aE + \widetilde{r}B_1 + \widetilde{r}^2 B_2(\widetilde{r}))\nu, \qquad B_2(\widetilde{r}) \in C^\infty. \tag{20}$$

Here B_1 is a constant diagonal matrix whose diagonal entries are $d'_{11}(0)$ and $d'_{22}(0) = \overline{d'_{11}(0)}$. Consider the auxiliary system

$$\dot{\nu}^1 = (B_1 + r^1 B_2(\widetilde{r}^1))\nu^1, \qquad \dot{r}^1 = 2r^1. \tag{21}$$

Since the difference $d'_{11}(0) - d'_{22}(0)$ is not a nonzero integer, it follows that this system has no resonance terms. Consequently, there exists a nondegenerate transformation

$$\nu_1 = H(r^1)z^1, \qquad H(r^1) \in C^\infty, \tag{22}$$

reducing system (21) to the form

$$\dot{z}^1 = B_1 z^1, \qquad \dot{r}^1 = 2r^1.$$

Then the same transformation $\nu = H(\widetilde{r})z$ reduces system (20) to the normal form

$$\dot{z} = (aE + \widetilde{r}B_1)z. \tag{23}$$

This completes the analysis of the case where $\widetilde{h} > 1$.

If $\widetilde{h} = 1$, then system (19) acquires the form (20), where the eigenvalues of B_1 may differ by a nonzero integer. If this is the case, then the eigenvalues are real and hence so is the Jordan form of B_1. By passing to real coordinates in system (20) and by reducing the matrix B_1 to Jordan form, we obtain the real system

$$\dot{u} = (aE + \widetilde{r}\widetilde{B}_1 + \widetilde{r}^2 \widetilde{B}_2(\widetilde{r}))u, \qquad \widetilde{B}_2(\widetilde{r}) \in C^\infty, \tag{24}$$

where the matrix $\widetilde{B}_1$ is diagonal with the difference of diagonal entries being an integer. In this situation, it is expedient to consider shearing transformations of the form

$$u = P(\widetilde{r})u^1,$$

where

$$P(\widetilde{r}) = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{r} \end{pmatrix} \qquad \text{or} \qquad P(\widetilde{r}) = \begin{pmatrix} \widetilde{r} & 0 \\ 0 & 1 \end{pmatrix},$$

which can be used to ensure that the matrix $\widetilde{B}_1$ in system (24) has no eigenvalues differing by an integer (see [6]). Assuming that this condition is satisfied, we can use a nondegenerate transformation of the form $u = H(\widetilde{r})u^1$, $H(\widetilde{r}) \in C^\infty$, with real matrix $H(\widetilde{r})$ to reduce system (24) to the form

$$\dot{u}^1 = (aE + \widetilde{r}B_1)u^1,$$

which completes the analysis of the case where $\widetilde{h} = 1$. Thus, we have proved the assertion of Theorem 2 for system (16).

Now consider the exceptional case in which the number $h = 2\omega^{-1}\operatorname{Im}\lambda_1$ is odd. The following system provides the simplest example of this kind, where $h = 1$:

$$\begin{aligned} \dot{x}_1 &= x_1(i\omega + x_1x_2), \\ \dot{x}_2 &= x_2(-i\omega + x_1x_2), \\ \dot{y}_1 &= \lambda y_1 + x_1y_2, \\ \dot{y}_2 &= \overline{\lambda}y_2 + x_2y_1, \end{aligned}$$

where $\omega = 2\operatorname{Im}\lambda$.

The change of variables $y_1 = \sqrt{e^{ih\alpha}}\,\nu_1$, $y_2 = \sqrt{e^{-ih\alpha}}\,\nu_2$ reduces the equations for the nondegenerate variables to the form

$$\begin{aligned} \dot{\nu}_1 &= a\nu_1 + \sqrt{\widetilde{r}}\nu_2, \\ \dot{\nu}_2 &= a\nu_2 + \sqrt{\widetilde{r}}\nu_1, \qquad a = \operatorname{Re}\lambda. \end{aligned}$$

The main trouble is that this transformation is discontinuous at the points where x_1 is real and positive.

Obviously, the passage to real coordinates reduces the resulting system to diagonal form, and hence the original system acquires a pseudonormal form.

3. PROOF OF THE MAIN THEOREM

Proof of Theorem 2. First, consider the formal system corresponding to system (4). For simplicity, we preserve the notation used in system (4) and so far make no distinction between system (4) and its formal analog. Let us give a detailed description of the equivalence relation, mentioned in the Introduction, for the variables and the corresponding equations. Unless stipulated otherwise, we speak of equations for the nondegenerate variables. We split these variables and the corresponding equations in system (4) into separate groups according to the equivalence principle mentioned in the Introduction; namely, two variables y_j and y_l (and the corresponding equations) go into one and the same group if $(\lambda_j - \lambda_l)/(i\omega)$ is an integer. We say that the variables (and the corresponding spectral points λ_j) occurring in the same group are *equivalent* and indicate this by the symbol $\approx$. (This equivalence relation was also used in [5].) Note that if $y_j \approx y_l \approx \overline{y}_j$, then $\overline{y}_l \approx y_l$. In this case, the numbers $2\omega^{-1}\operatorname{Im}\lambda_j$ and $2\omega^{-1}\operatorname{Im}\lambda_l$ are integers. If $y_j \approx y_l$ and y_l is real ($y_l = \overline{y}_j$), the $y_j \approx \overline{y}_j$. In this case, $\omega^{-1}\operatorname{Im}\lambda_j$ is an integer. Obviously, the numbers $\operatorname{Re}\lambda_j$ are the same for the group of equivalent variables y_j (but are in general distinct for distinct groups). Also, note that all variables in a group of equivalent variables are singular or nonsingular simultaneously (see the Introduction).

Consider the equations for some group of equivalent variables. They form an independent subsystem of system (4); i.e., the right-hand sides of the equations for these variables contain only the variables from the same group. (Here we speak of the formal system.) By using an appropriate numbering of the variables, we can assume that the Jordan matrix $A(0)$ in (4) has the block-diagonal form $A(0) = \operatorname{diag}(J_1, J_2, J_3)$, where $J_2 = \overline{J}_1$ is a Jordan matrix of size L, $J_3 = \overline{J}_3$ is a Jordan matrix of size M,

$$\lambda_{j+L} = \overline{\lambda}_j, \qquad 1 \le j \le L, \quad \operatorname{Re}\lambda_j = a, \quad 1 \le j \le 2L + M,$$

and, moreover, we assume that

$$\operatorname{Im}\lambda_{j+1} \ge \operatorname{Im}\lambda_j > 0, \qquad 1 \le j \le L - 1.$$

Let $a_{jl(x)}$ be the entries of the matrix $A(x) - A(0)$, and let $\widehat{a}_{jl}(x)$ be the entries of the corresponding formal matrix. We represent the terms $x_1^{h_1}x_2^{h_2}$, which are elements of the series $\widehat{a}_{jl}(x)$, $1 \le j, l \le L$, in the form

$$x_1^{h_1}x_2^{h_2} = (x_1x_2)^{h_1}x_2^{h_2-h_1}.$$

In view of the resonance equation

$$\lambda_j = (h_1 - h_2)i\omega + \lambda_l,$$

we obtain

$$h_2 - h_1 = \frac{\lambda_l - \lambda_j}{i\omega}.$$

Consider the integers

$$\alpha_j = \frac{\lambda_j - \lambda_1}{i\omega}, \qquad 1 \le j \le L. \tag{25}$$

It follows that the functions $\widehat{a}_{jl}(x)$, $1 \le j, l \le L$, can be represented in the form

$$\widehat{a}_{jl}(x) = \widetilde{b}_{jl}(\widetilde{r})x_2^{\alpha_l-\alpha_j}, \qquad 1 \le j, l \le L, \quad \widetilde{r} = x_1x_2.$$

By a similar argument, we obtain the representation

$$\widehat{a}_{jL+l}(x) = \widehat{b}_{jL+l}(\widetilde{r})x_1^{\alpha_l+\alpha_j+2\widetilde{h}}, \qquad \widetilde{h} = \omega^{-1}\operatorname{Im}\lambda_1$$

for the functions $\widehat{a}_{jL+l}(x)$, $1 \le j, l \le L$, and the representation

$$\widehat{a}_{j2L+l}(x) = \widehat{b}_{j2L+l}(\widetilde{r})x_1^{\alpha_j+\widetilde{h}}$$

for the functions $\widehat{a}_{j2L+l}(x)$, $1 \le j \le L$, $1 \le l \le M$. Note that if there are real variables in the system ($M > 0$), then $\widetilde{h}$ is an integer. Since the corresponding variables are complex conjugate, we have

$$\begin{aligned}
\widehat{a}_{L+jl}(x) &= \overline{\widehat{b}_{jL+l}(\widetilde{r})}x_2^{\alpha_l+\alpha_j+2\widetilde{h}}, && 1 \le j, l \le L,\\
\widehat{a}_{L+jL+l}(x) &= \overline{\widetilde{b}_{jl}(\widetilde{r})}x_1^{\alpha_l-\alpha_j}, && 1 \le j, l \le L,\\
\widehat{a}_{L+j2L+l}(x) &= \overline{\widehat{b}_{j2L+l}(\widetilde{r})}x_2^{\alpha_j+\widetilde{h}}, && 1 \le j \le L, \quad 1 \le l \le M.
\end{aligned}$$

A similar representation for the functions $\widehat{a}_{2L+jl}(x)$, $1 \le j \le M$, has the form

$$\begin{aligned}
\widehat{a}_{2L+jl}(x) &= \widehat{b}_{2L+jl}(\widetilde{r})x_2^{\alpha_l+\widetilde{h}}, && 1 \le l \le L,\\
\widehat{a}_{2L+jL+l}(x) &= \overline{\widehat{b}_{2L+jl}(\widetilde{r})}x_1^{\alpha_l+\widetilde{h}}, && 1 \le l \le L,\\
\widehat{a}_{2L+j2L+l}(x) &= \overline{\widehat{a}_{2L+j2L+l}(x)} = \widehat{b}_{2L+j2L+l}(\widetilde{r}), && 1 \le l \le M.
\end{aligned}$$

In view of the remark made in the Introduction concerning the passage from formal functions to their actual counterparts, we assume, without loss of generality, that similar representations hold for the functions $a_{jl}(x)$.

Let us make the change of variables

$$\begin{aligned}
y_j = x_2^{-\alpha_j}\nu_j, \quad y_{L+j} &= x_1^{-\alpha_j}\nu_{L+j}, && 1 \le j \le L,\\
y_{2L+j} &= \nu_{2L+j}, && 1 \le j \le M.
\end{aligned} \tag{26}$$

The system acquires the form

$$\begin{aligned}
\dot{\nu}^1 &= \Lambda_1\nu^1 + B_{11}(\widetilde{r})\nu^1 + x_1^{2\widetilde{h}}B_{12}(\widetilde{r})\nu^2 + x_1^{\widetilde{h}}B_{13}(\widetilde{r})\nu^3,\\
\dot{\nu}^2 &= \Lambda_2\nu^2 + x_2^{2\widetilde{h}}\overline{B_{12}(\widetilde{r})}\nu^1 + \overline{B_{11}(\widetilde{r})}\nu^2 + x_2^{\widetilde{h}}\overline{B_{13}(\widetilde{r})}\nu^3,\\
\dot{\nu}^3 &= \Lambda_3\nu^3 + x_2^{\widetilde{h}}B_{31}(\widetilde{r})\nu^1 + x_1^{\widetilde{h}}\overline{B_{31}(\widetilde{r})}\nu^2 + B_{33}(\widetilde{r})\nu^3.
\end{aligned} \tag{27}$$

Here ν^1 is the vector whose components are the nondegenerate variables corresponding to the eigenvalues with positive real part, $\nu^2 = \overline{\nu}^1$, ν^3 is a vector with real coordinates, Λ_1 is the Jordan matrix whose structure coincides with that of J_1 and whose diagonal entries are λ_1, $\Lambda_2 = \overline{\Lambda_1}$, $\Lambda_3 = J_3$, and $B_{jl}(\widetilde{r}) \in C^\infty$. Also, note that the diagonal and subdiagonal entries of the matrices $B_{11}(0)$ and $B_{33}(0)$ are zero.

Note that similar transformations were used in [5], where, in contrast to the present paper, there was no need to observe the reality principle.

In the resulting system, we make yet another change of variables

$$\nu^1 = e^{i\widetilde{h}\alpha} w^1, \quad \nu^2 = e^{-i\widetilde{h}\alpha} w^2, \quad \nu^3 = w^3, \qquad \alpha = \arg x_1, \quad e^{i\alpha} = \frac{x_1}{\sqrt{\widetilde{r}}}. \tag{28}$$

Note that if $\widetilde{h}$ is an integer, then this transformation is nondegenerate and belongs to the class C^∞ for $\widetilde{r} \neq 0$. As a result, system (27) acquires the form

$$\begin{aligned}
\dot{w}^1 &= \widetilde{\Lambda}_1 w^1 + \widetilde{B}_{11}(\widetilde{r}) w^1 + \widetilde{r}^{\widetilde{h}} B_{12}(\widetilde{r}) w^2 + (\sqrt{\widetilde{r}})^{\widetilde{h}} B_{13}(\widetilde{r}) w^3, \\
\dot{w}^2 &= \widetilde{\Lambda}_2 w^2 + \widetilde{r}^{\widetilde{h}} \overline{B_{12}(\widetilde{r})} w^1 + \overline{\widetilde{B}_{11}(\widetilde{r})} w^2 + (\sqrt{\widetilde{r}})^{\widetilde{h}} \overline{B_{13}(\widetilde{r})} w^3, \\
\dot{w}^3 &= \Lambda_3 \nu^3 + (\sqrt{\widetilde{r}})^{\widetilde{h}} B_{31}(\widetilde{r}) w^1 + (\sqrt{\widetilde{r}})^{\widetilde{h}} \overline{B_{31}(\widetilde{r})} w^2 + B_{33}(\widetilde{r}) w^3.
\end{aligned}$$

Here $\widetilde{\Lambda}_1$ is the Jordan matrix whose structure coincides with that of J_1 and whose diagonal entries are equal to $a = \operatorname{Re}\lambda_1$, $\widetilde{\Lambda}_2 = \widetilde{\Lambda}_1$, $\Lambda_3 = J_3$, and $\widetilde{B}_{11}(\widetilde{r}) \in C^\infty$. The diagonal and subdiagonal entries of the matrix $\widetilde{B}_{11}(0)$ are zero.

Thus, the resulting system can be represented in the form

$$\dot{w} = Jw + C(\sqrt{\widetilde{r}})w, \tag{29}$$

where J is a Jordan matrix of the same structure as $A(0)$ in system (4), with the only difference that the diagonal entries of J are equal to $a = \operatorname{Re}\lambda_1$. The matrix $C = C(\sqrt{\widetilde{r}})$ has the same block structure as $A(x)$,

$$\begin{gathered}
C = (C_{jl}), \qquad C_{jl} = C_{jl}(\sqrt{\widetilde{r}}), \quad 1 \le j, l \le 3, \qquad C_{21} = \overline{C}_{12}, \\
C_{22} = \overline{C}_{11}, \qquad C_{23} = \overline{C}_{13}, \qquad C_{32} = \overline{C}_{31}, \qquad C_{33} = \overline{C}_{33}.
\end{gathered}$$

All entries of $C(\eta)$ are C^∞. The diagonal and subdiagonal entries of $C(0)$ are zero, so that all eigenvalues of the matrix $J + C(0)$ are equal to $a = \operatorname{Re}\lambda_1$. Thus, the transformation of the independent system for equivalent variables is nearly complete. It remains to note that the Jordan matrix J in (29) is not quite typical. It is real, while the variables w_j, $1 \le j \le 2L$, are complex. Thus, it is expedient to pass from system (29) to the corresponding real system.

Note that if the complex conjugate variables are not equivalent, then they are transformed according to formulas similar to (26) with $M = 0$, whence we see that the right-hand side of system (27) depends only on $\widetilde{r}$ in this case, and the transformation (28) is not needed, because we have already obtained a system of the form (29).

Now the resulting system can be treated by methods presented in [1] (see Theorem 2 in [1]); namely, one can construct a weakly degenerate transformation

$$\widetilde{r} = du^l, \qquad w = T(\widetilde{r})z \tag{30}$$

that reduces system (29) to a pseudonormal form (7). (The description of the transformation (30) is given in the Introduction.) The proof of Theorem 2 is complete. $\square$

4. FORMAL PSEUDONORMAL FORM CONSISTING OF IRREMOVABLE MONOMIALS

Proof of Theorem 3. First, note that the series $H^*(r,\alpha,z)$, $H_1^*(r,\alpha,z)$, and $H_2^*(r,\alpha,z)$ in system (14) consist of monomials of the form $r^L e^{iK\alpha} z^s$, $r^L e^{iK\alpha} z^s$, and $r^{L-1} e^{iK\alpha} z^s$, respectively, where $L \geq 1$ and K are integers. It was mentioned in the Introduction that the number K is uniquely determined by the tuple s and the number of the equation where the monomial in question occurs.

The subsequent argument largely reproduces that used in the proof of Theorem 3 in [1], and so we only present the main points.

First, note that, after the passage to polar coordinates, the monomials $cu_1^p u_2^q z^s$ that are terms of the Taylor series of the functions $F(u_1,u_2,z)$ in system (8) acquire the form $cr^L e^{iK\alpha} z^s$, where $L = p+q$ and $K = p-q$. The similar monomials in the first and second equations in (8) are transformed as follows:

$$r^L(Ae^{i(K-1)\alpha} z^s + \overline{A} e^{-i(K-1)\alpha} \overline{z}^s), \qquad A = \frac{c}{2}, \tag{31}$$

in the equation for r, and

$$r^{L-1}(Be^{i(K-1)\alpha} z^s - \overline{B} e^{-i(K-1)\alpha} \overline{z}^s), \qquad B = \frac{c}{2i}, \tag{32}$$

in the equation for α. These monomials are terms of the corresponding formal sums in system (14). As was mentioned in the Introduction, the number K is uniquely determined by the tuple s and the number of the equation where the monomial in question occurs.

For the reducible monomials $r^L e^{iK\alpha} z^s$ in the jth equation, $3 \leq j \leq n+2$, with level $q \leq p-2$ and $L > 2(q+1)$, we make the transformation

$$z_j = w_j + d_j r^{L-2(q+1)} e^{iK\alpha} w^s, \tag{33}$$

where

$$d_j = \frac{c_j}{(\widetilde{s}, \lambda^{q+1}) - \lambda_{j-2}^{q+1}}$$

and c_j is the coefficient of this monomial in the corresponding equation. For the similar monomials in the first two equations of the system, the transformations have the form

$$\begin{gathered} r = r_1 + r_1^{L-2(q+1)}(d_1 e^{iK\beta} w^s + \overline{d}_1 e^{-iK\beta} \overline{w}^s), \\ \alpha = \beta + r_1^{L-1-2(q+1)}(d_2 e^{iK\beta} w^s - \overline{d}_2 e^{-iK\beta} \overline{w}^s), \\ d_1 = \frac{A}{(\widetilde{s}, \lambda^{q+1})}, \qquad d_2 = \frac{B}{(\widetilde{s}, \lambda^{q+1})}. \end{gathered}$$

If the level q of these removable monomials is $p-1$, then the changes of variables have the same form, but the coefficients d_j are somewhat different,

$$d_j = \frac{c_j}{(\widetilde{s}, \lambda^p) - \lambda_{j-2}^p + b_1(L-2p)}, \quad 3 \leq j \leq n+2,$$

$$d_1 = \frac{A}{(\widetilde{s}, \lambda^p) + b_1(L-4p-1)}, \qquad d_2 = \frac{B}{(\widetilde{s}, \lambda^p) + b_1(L-2p-1)}.$$

Obviously, the reality condition is satisfied for these transformations (see Remark 1 in the Introduction).

It remains to reproduce the argument used in the proof of the similar Theorem 3 in [1]. The proof of the theorem is complete. □

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