

A BERNSTEIN-TYPE INEQUALITY FOR RATIONAL FUNCTIONS IN WEIGHTED BERGMAN SPACES

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ABSTRACT. Given $n \geq 1$ and $r \in [0, 1)$, we consider the set $\mathcal{R}_{n,r}$ of rational functions having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, where \mathbb{D} is the unit disc of the complex plane. We give an asymptotically sharp Bernstein-type inequality for functions in $\mathcal{R}_{n,r}$ in weighted Bergman spaces with “polynomially” decreasing weights. We also prove that this result can not be extended to weighted Bergman spaces with “super-polynomially” decreasing weights.

1. INTRODUCTION

Estimates of the norms of derivatives for polynomials and rational functions (in different functional spaces) is a classical topic of complex analysis (see surveys by A.A. Gonchar [10], V.N. Rusak [16], and P. Borwein and T. Erdélyi [3, Chapter 7]). Such inequalities have applications in many domains of analysis; to mention just some of them: 1) matrix analysis and in operator theory (see “Kreiss Matrix Theorem” [12, 17] or [19, 18] for resolvent estimates of power bounded matrices), 2) inverse theorems of rational approximation (see [4, 15, 14]), 3) effective Nevanlinna–Pick interpolation problems (see [23, 22]).

Here, we present Bernstein-type inequalities for rational functions f of degree n with poles in $\{z : |z| > 1\}$, involving Hardy norms and weighted Bergman norms. Let \mathcal{P}_n be the complex space of polynomials of degree less or equal to $n \geq 1$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of the complex plane and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ its closure. Given $r \in [0, 1)$, we define

$$\mathcal{R}_{n,r} = \left\{ \frac{p}{q} : p, q \in \mathcal{P}_n, d^\circ p < d^\circ q, q(\zeta) \neq 0 \quad |\zeta| < \frac{1}{r} \right\},$$

(where $d^\circ p$ denotes the degree of $p \in \mathcal{P}_n$), the set of all rational functions in \mathbb{D} of degree less or equal than $n \geq 1$, having at most n poles all outside of $\frac{1}{r}\mathbb{D}$. Notice that for $r = 0$, we get $\mathcal{R}_{n,0} = \mathcal{P}_{n-1}$.

1.1. Definitions of Hardy spaces and radial weighted Bergman spaces.

We denote by $\text{Hol}(\mathbb{D})$ the space of all holomorphic functions on \mathbb{D} . From now on, if $f \in \text{Hol}(\mathbb{D})$ then for every $\rho \in (0, 1)$ we define

$$f_\rho : \xi \mapsto f(\rho\xi), \quad \xi \in \frac{1}{\rho}\mathbb{D}.$$

We consider the two following scales of Banach spaces $X \subset \text{Hol}(\mathbb{D})$:

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a. The Hardy spaces $H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$:

$$H^p = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{H^p}^p = \sup_{0 \leq \rho < 1} \int_{\mathbb{T}} |f_\rho(\xi)|^p dm(\xi) < \infty \right\},$$

where m stands for the normalized Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. As usual, we denote by H^∞ the space of all bounded analytic functions in \mathbb{D} .

b. The radial weighted Bergman spaces $L_a^p(w)$, $1 \leq p < \infty$ (where "a" means analytic),

$$L_a^p(w) = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{L_a^p(w)}^p = \int_0^1 \rho w(\rho) \int_{\mathbb{T}} |f_\rho(\zeta)|^p dm(\zeta) d\rho < \infty \right\},$$

where the weight w satisfies $w \geq 0$ and $\int_0^1 w(\rho) d\rho < \infty$. For the classical power weights $w(\rho) = w_\beta(\rho) = (1 - \rho)^\beta$, $\beta > -1$, we have $L_a^p(w_\beta) = L_a^p((1 - |z|)^\beta dA(z))$, A being the normalized area measure on \mathbb{D} .

For general properties of these spaces we refer to [11, 24].

From now on, for two positive functions a and b , we say that a is dominated by b , denoted by $a \lesssim b$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a \lesssim b$ and $b \lesssim a$.

1.2. Statement of the problem and known results. By Bernstein-type inequalities for rational functions one usually understands the inequalities of the form

$$(1.1) \quad \|f'\|_X \leq \phi_{X,Y}(n) \|f\|_Y, \quad f \in \mathcal{R}_n,$$

where \mathcal{R}_n is the set of all proper rational functions of degree at most n with the poles in $\{|z| > 1\}$, X and Y are some normed spaces of functions analytic in the unit disc, and ϕ is some increasing (often polynomially growing) function. Thus, for a given pair of the function spaces X and Y , the question is to determine the dependence on n for the norm of the differentiation operator $(\mathcal{R}_n, \|\cdot\|_X)$ to Y . Bernstein-type inequalities of E.P. Dolzhenko [5] and A.A. Pekarskii [14] are of this form; e.g., it is shown in [5] that

$$\|f\|_{H_1^1} \leq c_1 n \|f\|_\infty, \quad \|f\|_{B_{2,2}^{1/2}} \leq c_2 n^{1/2} \|f\|_\infty, \quad f \in \mathcal{R}_n,$$

where H_1^1 is the Hardy–Sobolev space, and $B_{2,2}^{1/2}$ is the Besov (or Dirichlet) space, see the definition in Section 3. Let us also mention that this problem is a part of a more general one given by G. Lorentz in a letter sent to T. Erdélyi in 1988 (see [9]).

Looking at (1.1), we notice that for some choices of X and Y , we have $\phi_{X,Y}(n) = +\infty$ for every $n = 1, 2, \dots$. Indeed, it may happen for instance when the poles of our function f are allowed to be arbitrary close to the torus \mathbb{T} : we can observe this phenomenon for example in the special case $X = Y = H^p$, $1 \leq p \leq +\infty$ but also when $X = Y = L_a^p(w)$, $1 \leq p \leq +\infty$. This observation leads us to come back on the problem in (1.1) and to state it more generally : that is replacing \mathcal{R}_n by $\mathcal{R}_{n,r}$ (for any fixed $r \in [0, 1)$) and $\phi_{X,Y}(n)$ by $\phi_{X,Y}(n, r)$ so that to focus on this phenomenon of "natural dependence on the parameter r ". For most of the classical cases already studied by others (for instance E. P. Dolzhenko [5], A. A. Pekarskii [14], V.V. Peller [15]) the spaces X and Y are such that $\sup_{r \in (0, 1)} \phi_{X,Y}(n, r) < +\infty$: in this case we can set $\phi_{X,Y}(n) = \sup_{r \in (0, 1)} \phi_{X,Y}(n, r)$. As a consequence,

if $\sup_{r \in (0, 1)} \phi_{X, Y}(n, r) = +\infty$, it may be of interest to search (as a continuation of the investigations of the second author [20, 21]) for the “best possible” $\phi_{X, Y}(n, r)$ in an asymptotically sense, that is to say as $n \rightarrow \infty$ and $r \rightarrow 1^-$. This question has already been answered for the case $X = Y = H^p$, $1 \leq p \leq +\infty$ by K. M. Dyakonov [6] see (1.2) below. In this paper, we answer the same question for the case $X = Y = L_a^p(w)$, $1 \leq p \leq +\infty$. Let us give a general formulation of our problem for the special case $X = Y$ for which we set $\mathcal{C}_{n, r}(X) = \phi_{X, Y}(n, r)$: given a Banach space X of holomorphic functions in \mathbb{D} , we are searching for the best possible constant $\mathcal{C}_{n, r}(X)$ such that

$$\|f'\|_X \leq \mathcal{C}_{n, r}(X) \|f\|_X, \quad f \in \mathcal{R}_{n, r}.$$

For the case where $X = H^p$ is a Hardy space, an estimate which gives a correct order of growth for $\mathcal{C}_{n, r}(X)$ was obtained by K.M. Dyakonov [6] (as a very special case of more general results): for any $p \in [1, \infty]$ there exist positive constants A_p and B_p such that

$$(1.2) \quad A_p \frac{n}{1-r} \leq \mathcal{C}_{n, r}(H^p) \leq B_p \frac{n}{1-r}$$

for all $n \geq 1$ and $r \in [0, 1)$. More precisely, the upper estimate for $p \in (1, +\infty)$ is treated in [6, Theorem 1], the case $p = 1$, in [6, Corollary 1], and the case $p = +\infty$ (known much earlier) is given in [3, Theorem 7.1.7]. The below estimate follows trivially when applying the differentiation operator to the test function $f(z) = (1 - rz)^{-n}$.

For the case $p = 2$ an asymptotically sharp result was obtained later in [20]: for any $r \in (0, 1)$ there exists the limit

$$\lim_{n \rightarrow \infty} \frac{\mathcal{C}_{n, r}(H^2)}{n} = \frac{1+r}{1-r}.$$

Related results about Bernstein-type inequalities in a more general setting of the so-called model or star invariant subspaces may be found in [8, Theorems 10,11], [7, Theorem 1], and [1, 2].

1.3. Main results. We obtain estimates for the derivatives of rational functions with respect to weighted Bergman norms. It turns out that there is an essential difference between slowly (polynomially) decreasing weights and fast (super-polynomially) decreasing weights. In the first case we have a two-sided estimate analogous to (1.2), while in the second case only the above estimate remains true. Let us give the precise definitions. Recall that w is always an integrable nonnegative function on $(0, 1)$.

Definition 1.1. (*Polynomially decreasing weights*) The weight w is said to be γ -polynomially decreasing if there exists $\gamma > 0$ such that

$$\rho \mapsto (1 - \rho)^{-\gamma} w(\rho),$$

is increasing on $[r_0, 1)$ for some $0 \leq r_0 < 1$. We say that w is *polynomially decreasing* if it is γ -polynomially decreasing for some $\gamma > 0$.

Definition 1.2. (*Super-polynomially decreasing weights*) The weight w is said to be *super-polynomially decreasing* if for any $\gamma > 0$ there exists $r(\gamma) \in (0, 1)$ such that the function

$$\rho \mapsto (1 - \rho)^{-\gamma} w(\rho),$$

decreases on the interval $[r(\gamma), 1)$.

Typical example of the weights from the first class are given by $w(r) = (1-r)^\beta$, $\beta > -1$, or $w(r) = (1-r)^\beta (|\log(1-r)| + 1)^\gamma$, $\beta > -1$, $\gamma \in \mathbb{R}$. The weights $w(r) = \exp(-c(1-r)^{-\gamma})$, $c > 0$, $\gamma > 0$ are super-polynomially decreasing.

Our first result may be considered as an analogue of Dyakonov's theorem for the radial weighted Bergman spaces.

Theorem 1.3. *Let $1 \leq p < \infty$ and let w be an integrable nonnegative function on $[0, 1)$. Then there exists a positive constant K depending only on p (but not on the weight w) such that*

$$(1.3) \quad C_{n,r}(L_a^p(w)) \leq K \frac{n}{1-r}$$

for all $r \in [0, 1)$ and $n \geq 1$. Moreover, if we fix $r \in (0, 1)$ and let n tend to infinity, then we have

$$(1.4) \quad \frac{\tilde{K}r}{1-r} \leq \liminf_{n \rightarrow \infty} \frac{C_{n,r}(L_a^p(w))}{n} \leq \limsup_{n \rightarrow \infty} \frac{C_{n,r}(L_a^p(w))}{n} \leq \frac{K}{1-r},$$

where \tilde{K} is, as K , a positive constant depending only on p .

The next theorem shows that for the polynomially decreasing weights the quantity $C_{n,r}(L_a^p(w))$ admits a below estimate of the same form.

Theorem 1.4. *If w is γ -polynomially decreasing, then there exists a positive constant K' depending only on w and p such that*

$$(1.5) \quad K' \frac{n}{1-r} \leq C_{n,r}(L_a^p(w)) \leq K \frac{n}{1-r},$$

where K is defined in (1.3) and where the left-hand side inequality of (1.5) holds for all $r \in [0, 1)$ and $n \geq \frac{\gamma+3}{p} + 1$. In particular, (1.5) holds for the classical weights $w(\rho) = w_\beta(\rho) = (1-\rho)^\beta$, $\beta > -1$.

The polynomial decrease is essential and provides a sharp bound for the validity of the uniform estimate (1.5) for all possible values of n and r . Namely, if the weight is super-polynomially decreasing, then (1.5) will fail along some sequence of radii.

Theorem 1.5. *Suppose that w is super-polynomially decreasing. Then there exists a sequence $r_n \rightarrow 1^-$ such that for any p ,*

$$\frac{C_{r_n, n}(L_a^p(w))}{n} = o\left(\frac{1}{1-r_n}\right), \quad n \rightarrow \infty.$$

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2. PROOFS OF THEOREMS 1.3 AND 1.4

Proof of Theorem 1.3. First, we notice that for any $0 \leq \alpha < 1$,

$$(2.1) \quad \|f\|_{L_a^p(w)}^p \asymp \iint_{u \in C_\alpha} \rho |f(\rho\xi)|^p w(\rho) dm(\zeta) d\rho$$

for all $f \in L_a^p(w)$, where $C_\alpha = \{z : \alpha < |z| < 1\}$. Let $f \in \mathcal{R}_{n,r}$ with $r \in [0, 1)$ and $n \geq 1$. Using (2.1) with $\alpha = \frac{1}{2}$ we get

$$\begin{aligned} \|f'\|_{L_a^p(w)}^p &\asymp \iint_{\rho\xi \in C_{1/2}} |f'(\rho\xi)|^p w(\rho) dm(\zeta) d\rho \\ &= \int_{\frac{1}{2}}^1 \rho w(\rho) \frac{1}{\rho^p} \left(\|(f_\rho)'\|_{H^p}^p \right) d\rho. \end{aligned}$$

Now using the fact that $f_\rho \in \mathcal{R}_{n,\rho r} \subset \mathcal{R}_{n,r}$ for every $\rho \in (0, 1)$, we get

$$\begin{aligned} \int_{\frac{1}{2}}^1 \rho w(\rho) \frac{1}{\rho^p} \left(\|(f_\rho)'\|_{H^p}^p \right) d\rho &\leq (2\mathcal{C}_{n,r}(H^p))^p \int_{\frac{1}{2}}^1 \rho w(\rho) (\|f_\rho\|_{H^p})^p d\rho \\ &\asymp (\mathcal{C}_{n,r}(H^p))^p \|f\|_{L_a^p(w)}^p. \end{aligned}$$

In particular, using the right-hand side inequality of (1.2), we get

$$\mathcal{C}_{n,r}(L_a^p(w)) \leq K_p \frac{n}{1-r},$$

for all $p \in [1, \infty)$, and $\beta \in (-1, \infty)$, where K_p is a constant depending on p only.

Now, let us prove (1.4). Let

$$f_n(z) = \frac{1}{(1-rz)^n} \in \mathcal{R}_{n,r},$$

and $D = \{z \in \mathbb{D} : |1-rz| \leq 2|1-r|\}$. We claim that

$$\|f_n\|_{L_a^p(w)}^p \sim \int_D |f_n(z)|^p w(z) dA(z), \quad n \rightarrow \infty,$$

and, analogously,

$$\|f_n'\|_{L_a^p(w)}^p \sim \int_D |f_n'(z)|^p w(z) dA(z), \quad n \rightarrow \infty.$$

Indeed, by a very rough estimate

$$\int_{\mathbb{D} \setminus D} |f_n(z)|^p w(z) dA(z) \leq \frac{C_1}{2^{pn}(1-r)^{pn}},$$

where $C_1 > 0$ depends only on w . On the other hand, if we put $\tilde{D} = \{z \in \mathbb{D} : |1-rz| \leq \frac{3}{2}|1-r|\}$, then

$$\int_D |f_n(z)|^p w(z) dA(z) \geq \frac{1}{(3/2)^{pn}(1-r)^{pn}} \int_{\tilde{D}} w(z) dA(z).$$

Since r (thus D and \tilde{D}) are fixed we see that

$$\frac{1}{2^{pn}(1-r)^{pn}} = o\left(\frac{1}{(3/2)^{pn}(1-r)^{pn}} \int_{\tilde{D}} w(z) dA(z)\right), \quad n \rightarrow \infty.$$

Thus,

$$\frac{\|f_n'\|_{L_a^p(w)}^p}{\|f_n\|_{L_a^p(w)}^p} \sim \int_D |f_n'(z)|^p w(z) dA(z) / \int_D |f_n(z)|^p w(z) dA(z).$$

Obviously,

$$\begin{aligned} \int_D |f'_n(z)|^p w(z) dA(z) &= \int_D \frac{n^p r^p}{|1-rz|^{pn+p}} w(z) dA(z) \\ &\geq \frac{n^p r^p}{2^p (1-r)^p} \int_D \frac{1}{|1-rz|^{pn}} w(z) dA(z) \\ &= \frac{n^p r^p}{2^p (1-r)^p} \int_D |f_n(z)|^p w(z) dA(z). \end{aligned}$$

Thus,

$$\liminf_{n \rightarrow \infty} \frac{\|f'_n\|_{L^p_a(w)}}{n \|f_n\|_{L^p_a(w)}} \geq \frac{r}{2(1-r)}.$$

□

For the proof of Theorem 1.4 we will need two lemmas.

Lemma 2.1. *Let $r \in [0, 1)$ and $t \geq 0$. We set*

$$I(t, r) = \int_{\mathbb{T}} |1 - r\xi|^{-t} dm(\xi) \quad \text{and} \quad \varphi_r(t) = \int_{\mathbb{T}} |1 + r\xi|^t dm(\xi).$$

Then,

$$I(t, r) = \frac{1}{(1-r^2)^{t-1}} \varphi_r(t-2)$$

for every $t \geq 2$, and $t \mapsto \varphi_r(t)$ is an increasing function on $[0, +\infty)$ for every $r \in [0, 1)$. Moreover, both

$$r \mapsto \varphi_r(t-2) \quad \text{and} \quad r \mapsto I(t, r),$$

are increasing on $[0, 1)$, for all $t \geq 0$.

Proof. Indeed, supposing that $t \geq 2$, we can write

$$I(t, r) = \frac{1}{1-r^2} \int_{\mathbb{T}} |b'_r(\xi)| \frac{1}{|1-r\xi|^{t-2}} dm(\xi),$$

where $b_r(z) = \frac{r-z}{1-rz}$. Using the fact that $b_r \circ b_r(z) = z$ and changing the variable in the above integral we get

$$\begin{aligned} I(t, r) &= \frac{1}{1-r^2} \int_{\mathbb{T}} |b'_r(\xi)| \frac{1}{|1-rb_r \circ b_r(\xi)|^{t-2}} dm(\xi) \\ &= \frac{1}{1-r^2} \int_{\mathbb{T}} \frac{1}{|1-rb_r(\xi)|^{t-2}} dm(\xi) \\ &= \frac{1}{(1-r^2)^{t-1}} \varphi_r(t-2), \end{aligned}$$

since $1 - rb_r(z) = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$. Now,

$$\varphi_r(t) = \int_0^{2\pi} \exp\left(\frac{t}{2} \ln(1+r^2-2r \cos s)\right) ds,$$

$$\varphi'_r(t) = \frac{1}{4} \int_0^{2\pi} \ln(1+r^2+2r \cos s) \exp\left(\frac{t}{2} \ln(1+r^2+2r \cos s)\right) ds,$$

and

$$\varphi''_r(t) = \frac{1}{4} \int_0^{2\pi} [\ln(1+r^2-2r \cos s)]^2 \exp\left(\frac{t}{2} \ln(1+r^2-2r \cos s)\right) ds \geq 0,$$

for every $t \geq 0$, $r \in [0, 1)$. Thus, φ_r is a convex function on $[0, \infty)$ and φ'_r is increasing on $[0, \infty)$ for all $r \in [0, 1)$. Moreover,

$$\varphi'_r(0) = \frac{1}{4} \int_0^{2\pi} \ln(1 + r^2 - 2r \cos s) ds = 0.$$

Thus,

$$\varphi'_r(t) \geq \varphi'_r(0) = 0, \quad \forall t \in [0, \infty), r \in [0, 1),$$

and so φ_r is increasing on $[0, \infty)$. The fact that

$$r \mapsto I(t, r),$$

is increasing on $[0, 1)$ for all $t \geq 0$ is obvious since

$$I(t, r) = \left\| \frac{1}{(1 - rz)^{t/2}} \right\|_{H^2}^2 = \sum_{k \geq 0} a_k^2(t) r^{2k},$$

where $a_k(t)$ is the k th Taylor coefficient of $(1 - z)^{-t/2}$. The same reasoning gives that $r \mapsto \varphi_r(t)$ is increasing on $[0, 1)$. \square

Lemma 2.2. *If for some $r_0 \in [0, 1)$ and $\gamma > 0$ the function $\frac{w(\rho)}{(1 - \rho^2)^\gamma}$ is increasing on $[r_0, 1)$, then*

$$\int_r^1 \rho w(\rho) I(t, r\rho) d\rho \asymp \int_{r_0}^1 \rho w(\rho) I(t, r\rho) d\rho,$$

for all t such that $t \geq \gamma + 3$ and for all $r \geq r_0$, with constants independent on t .

Proof. Clearly,

$$\int_{r_0}^1 \rho w(\rho) I(t, r\rho) d\rho \geq \int_r^1 \rho w(\rho) I(t, r\rho) d\rho, \quad r \in [r_0, 1).$$

Moreover,

$$\int_{r_0}^1 \rho w(\rho) I(t, r\rho) d\rho = \int_r^1 \rho w(\rho) I(t, r\rho) d\rho + \int_{r_0}^r \rho w(\rho) I(t, r\rho) d\rho,$$

and applying Lemma 2.1,

$$\begin{aligned} \int_{r_0}^r \rho w(\rho) I(t, r\rho) d\rho &= \int_{r_0}^r \frac{\rho w(\rho)}{(1 - \rho^2)^\gamma} \frac{(1 - \rho^2)^\gamma}{(1 - (r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho \\ &\leq \frac{w(r)}{(1 - r^2)^\gamma} \int_{r_0}^r \frac{\rho (1 - \rho^2)^\gamma}{(1 - (r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho \\ &\leq \frac{w(r)}{(1 - r^2)^\gamma} \varphi_{r^2}(t) \int_{r_0}^1 \frac{\rho (1 - \rho^2)^\gamma}{(1 - (r\rho)^2)^{t-1}} d\rho, \end{aligned}$$

because $u \mapsto \varphi_u(t)$ is increasing for all $t > 0$. For the same reason,

$$\begin{aligned} \int_r^1 \rho w(\rho) \frac{1}{(1 - (r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho &= \int_r^1 \frac{w(\rho)}{(1 - \rho^2)^\gamma} \frac{\rho (1 - \rho^2)^\gamma}{(1 - (r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho \\ &\geq \frac{w(r)}{(1 - r^2)^\gamma} \varphi_{r^2}(t) \int_r^1 \frac{\rho (1 - \rho^2)^\gamma}{(1 - (r\rho)^2)^{t-1}} d\rho. \end{aligned}$$

Now note that

$$\int_{r_0}^r \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho \lesssim \int_r^1 \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho, \quad r \in [r_0, 1),$$

with constants independent on $t \geq \gamma + 3$. Indeed, this estimate holds for $t = \gamma + 3$, and, hence, by monotonicity of the function $\rho \mapsto (1 - (\rho r)^2)^{-1}$, for all $t \geq \gamma + 3$.

Thus, using Lemma 2.1 and the fact that the function $(1-\rho)^{-\gamma}w(\rho)$ is increasing on $[r_0, 1)$, we obtain

$$\begin{aligned} \int_{r_0}^r \rho w(\rho) I(t, r\rho) d\rho &\leq \frac{w(r)}{(1-r^2)^\gamma} \varphi_{r^2}(t-2) \int_{r_0}^r \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho \\ &\leq \kappa_1 \frac{w(r)}{(1-r^2)^\gamma} \varphi_{r^2}(t-2) \int_r^1 \frac{\rho(1-\rho^2)^\gamma}{(1-(r\rho)^2)^{t-1}} d\rho \\ &\leq \kappa_2 \int_r^1 \rho w(\rho) \frac{1}{(1-(r\rho)^2)^{t-1}} \varphi_{r\rho}(t) d\rho, \end{aligned}$$

(where κ_1, κ_2 are positive constants which do not depend on t), which completes the proof. \square

Proof of Theorem 1.4. We need to prove only the lower bound, the upper bound is already proved in Theorem 1.3. Let us prove the minoration with the test function $f(z) = \frac{1}{(1-rz)^n}$. Using (2.1) with $\alpha = r_0$, we need to show that

$$\begin{aligned} \frac{\|f'\|_{L_a^p(w)}^p}{n^p r^p} &= \int_{r_0}^1 \rho w(\rho) I(pn+p, r\rho) d\rho \\ &\geq \frac{C}{(1-r)^p} \int_{r_0}^1 \rho w(\rho) I(pn, r\rho) d\rho = \frac{C}{(1-r)^p} \|f\|_{L_a^p(w)}^p. \end{aligned}$$

Since $r \in [r_0, 1)$ and $n \geq \frac{\gamma+3}{p}$, by Lemma 2.2 applied with $t = pn + p$ and $t = pn$ this means that

$$\int_r^1 \rho w(\rho) I(pn+p, r\rho) d\rho \geq \frac{C}{(1-r)^p} \int_r^1 \rho w(\rho) I(pn, r\rho) d\rho.$$

By Lemma 2.1, this is equivalent to the estimate

$$\begin{aligned} \int_r^1 \rho w(\rho) \frac{\varphi_{r\rho}(pn+p-2)}{(1-(r\rho)^2)^{pn+p-1}} d\rho \\ \geq \frac{C}{(1-r)^p} \int_r^1 \rho w(\rho) \frac{\varphi_{r\rho}(pn-2)}{(1-(r\rho)^2)^{pn-1}} d\rho. \end{aligned}$$

The last statement is obvious since

$$\begin{aligned}
& \int_r^1 \rho w(\rho) \frac{\varphi_{r\rho}(pn+p-2)}{(1-(r\rho)^2)^{pn+p-1}} d\rho \\
& \geq \frac{1}{(1-r^2)^p} \int_r^1 \rho w(\rho) \frac{\varphi_{r\rho}(pn+p-2)}{(1-(r\rho)^2)^{pn-1}} d\rho \\
& \geq \frac{1}{(1-r^2)^p} \int_r^1 \rho w(\rho) \frac{\varphi_{r\rho}(pn-2)}{(1-(r\rho)^2)^{pn-1}} d\rho,
\end{aligned}$$

where the last inequality is due to the fact that $t \mapsto \varphi_u(t)$ is increasing for all $0 \leq u < 1$. \square

3. THE CASE OF SUPER-POLYNOMIALLY DECREASING WEIGHTS. PROOF OF THEOREM 1.5:

For the proof of Theorem 1.5 we will need a definition from the theory of model subspaces of the Hardy space. For a finite subset σ of \mathbb{D} with $\text{card } \sigma = n$, consider the finite Blaschke product

$$B_\sigma = \prod_{\lambda \in \sigma} b_\lambda,$$

where $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$, $\lambda \in \mathbb{D}$. Define the model space K_{B_σ} by

$$K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Consider the family $(e_k)_{1 \leq k \leq n}$ in K_{B_σ} (known as *Malmquist basis*, see [13, p. 117]),

$$e_1(z) = \frac{(1 - |\lambda_1|)^{1/2}}{1 - \bar{\lambda}_1 z} \quad \text{and} \quad e_k(z) = \left(\prod_{j=1}^{k-1} b_{\lambda_j}(z) \right) \frac{(1 - |\lambda_k|)^{1/2}}{1 - \bar{\lambda}_k z}, \quad k \in [2, n],$$

The family $(e_k)_{1 \leq k \leq n}$ associated with σ is an orthonormal basis of the n -dimensional space K_{B_σ} .

In what follows we denote by $L_a^p(w, s\mathbb{D})$ and by $H^p(s\mathbb{D})$, $s > 0$, the weighted Bergman space and the Hardy space in the disc $s\mathbb{D} = \{z : |z| < s\}$, respectively. If $w \equiv 1$, we write simply $L_a^p(s\mathbb{D})$ and we write L_a^p if $s = 1$.

Lemma 3.1. *Let $n \geq 1$, $r, s \in [0, 1)$ and $p \in [1, +\infty]$. We set*

$$M_{p,s}(n, r) = \sup \left\{ |f(\xi)| : \xi \in \mathbb{D}, f \in \mathcal{R}_{n,r}, \|f\|_{L_a^p(s\mathbb{D})} \leq 1 \right\}.$$

Then

$$(3.1) \quad M_{p, \frac{2}{3}}(n, r) \leq d \frac{c^n}{(1-r)^{n+b}},$$

where $d > 0$, $b > 0$, $c > 1$ are some absolute positive constants (may be, depending on p).

Remark 3.2. Lemma 3.1 is valid not only for $s = \frac{2}{3}$, but for every $s \in (0, 1)$, with constants $d > 0$, $b > 0$, $c > 1$ depending both on s and p .

Proof. For every $f \in \mathcal{R}_{n,r}$ and $\xi \in \mathbb{D}$, we have

$$\left| f\left(\frac{1}{2}\xi\right) \right| = \left| f_{\frac{2}{3}}\left(\frac{3}{4}\xi\right) \right| = \left| \int_{\mathbb{D}} f_{\frac{2}{3}}(u) \left(\overline{k_{\frac{2}{3}}\xi}(u)\right)^2 dA(u) \right|,$$

where $k_\lambda(z) = \frac{1}{1-\lambda z}$ is the standard Cauchy kernel associated with $\lambda \in \mathbb{D}$, and A is the normalized area measure on \mathbb{D} . Applying Hölder's inequality we obtain

$$\left| f\left(\frac{1}{2}\xi\right) \right| \leq \|f_{\frac{2}{3}}\|_{L_a^p} \left\| \left(k_{\frac{3}{4}}\xi\right)^2 \right\|_{L_a^{p'}} = \left(\frac{3}{2}\right)^{1/p} \|f\|_{L_a^p(\frac{2}{3}\mathbb{D})} \left\| \left(k_{\frac{3}{4}}\xi\right)^2 \right\|_{L_a^{p'}}, \quad \xi \in \mathbb{T},$$

where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Now, note that

$$\left\| \left(k_{\frac{3}{4}}\xi\right)^2 \right\|_{L_a^{p'}} \leq \left\| \left(k_{\frac{3}{4}}\xi\right)^2 \right\|_{H^\infty} = \left(\frac{1}{1-\frac{3}{4}}\right)^2 = 16.$$

Finally, supposing $\|f\|_{L_a^p(\frac{2}{3}\mathbb{D})} \leq 1$, we obtain

$$\|f\|_{L_a^p(\frac{1}{2}\mathbb{D})} \leq \|f\|_{H^\infty(\frac{1}{2}\mathbb{D})} \leq 16 \left(\frac{3}{2}\right)^{1/p} \leq 24,$$

which gives

$$(3.2) \quad M_{p, \frac{2}{3}}(n, r) \leq 24M_{2, \frac{1}{2}}(n, r).$$

It remains to obtain a suitable upper bound for $M_{2, \frac{1}{2}}(n, r)$. Let us prove that

$$(3.3) \quad M_{2, \frac{1}{2}}(n, r) \leq 2\sqrt{n} \left(\frac{2}{1-r}\right)^{n+\frac{1}{2}}.$$

For every $f \in \mathcal{R}_{n, r}$, we have $f_{\frac{1}{2}} \in \mathcal{R}_{n, \frac{1}{2}r} \subset \mathcal{R}_{n, r}$. If $\{1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n\}$ is the set of the poles of f (thus, $|\lambda_j| \leq r$, $j = 1, \dots, n$), then $f \in K_{B_\sigma}$ with $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset r\mathbb{D}$, whereas the set $\{2/\bar{\lambda}_1, \dots, 2/\bar{\lambda}_n\}$ is the set of the poles of the function $f_{\frac{1}{2}}$ and $f_{\frac{1}{2}} \in K_{B_{\sigma'}}$ with $\sigma' = \{\frac{1}{2}\lambda_1, \dots, \frac{1}{2}\lambda_n\} \subset \frac{r}{2}\mathbb{D}$. Hence, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that

$$(3.4) \quad f_{\frac{1}{2}} = \sum_{k=1}^n \alpha_k e_k,$$

on \mathbb{D} , where $(e_k)_{k=1}^n$ is the Malmquist basis associated with the set σ' . Since both $f_{\frac{1}{2}}$ and $\sum_{k=1}^n \alpha_k e_k$ are meromorphic in \mathbb{C} the equality (3.4) is in fact valid everywhere in \mathbb{C} . Thus,

$$f(\xi) = \sum_{k=1}^n \alpha_k \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{\left(1 - \frac{1}{4}|\lambda_k|^2\right)^{1/2}}{1 - \bar{\lambda}_k \xi}, \quad \xi \in \mathbb{D},$$

and by the Cauchy-Schwarz inequality,

$$(3.5) \quad |f(\xi)| \leq \left(\sum_{k=1}^n |\alpha_k|^2 \right)^{1/2} \left(\sum_{k=1}^n \left| \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \bar{\lambda}_j \xi} \right) \frac{\left(1 - \frac{1}{4}|\lambda_k|^2\right)^{1/2}}{1 - \bar{\lambda}_k \xi} \right|^2 \right)^{1/2}.$$

for any $\xi \in \mathbb{D}$. Now, if $\lambda \in r\mathbb{D}$ and $\xi \in \mathbb{D}$,

$$\frac{\frac{\lambda}{2} - 2\xi}{1 - \bar{\lambda}\xi} = \frac{2\left(\frac{\lambda}{4} - \xi\right)}{1 - \bar{\lambda}\xi} \frac{1 - \frac{\bar{\lambda}\xi}{4}}{1 - \bar{\lambda}\xi} = 2b_{\frac{\lambda}{4}}(\xi) \left(1 + \frac{3\bar{\lambda}}{4(1 - \bar{\lambda}\xi)}\right),$$

which gives

$$\left| \frac{\frac{\lambda}{2} - 2\xi}{1 - \lambda\xi} \right| \leq 2 \left(1 + \frac{3r}{4} \frac{1}{1-r} \right) = \frac{4-r}{2(1-r)} \leq \frac{2}{1-r}.$$

We get

$$(3.6) \quad \sum_{k=1}^n \left| \left(\prod_{j=1}^{k-1} \frac{\frac{\lambda_j}{2} - 2\xi}{1 - \lambda_j \xi} \right) \frac{(1 - \frac{1}{4} |\lambda_k|^2)^{1/2}}{1 - \lambda_k \xi} \right|^2 \leq \frac{1}{(1-r)^2} \sum_{k=1}^n 2^{2(k-1)} \left(\frac{1}{1-r} \right)^{2(k-1)} \\ \leq \frac{1}{4} \left(\frac{2}{1-r} \right)^{2n+1}.$$

Now we first notice that

$$\left(\sum_{k=1}^n |\alpha_k|^2 \right)^{1/2} = \|f_{\frac{1}{2}}\|_{H^2}.$$

For any function $\varphi(z) = \sum_{k \geq 0} \widehat{\varphi}(k) z^k$ in H^2 , one has

$$\|\varphi\|_{H^2}^2 = \sum_{k \geq 0} \frac{|\widehat{\varphi}(k)|}{\sqrt{k+1}} \sqrt{k+1} |\widehat{\varphi}(k)| \leq \|\varphi\|_{L_a^2} \|\varphi\|_{B_{2,2}^{1/2}},$$

We now use the upper bound of [21, Theorem A, (4)]: for $\varphi \in \mathcal{R}_{n,\rho}$ one has

$$\begin{aligned} \|\varphi\|_{B_{2,2}^{1/2}}^2 &= \|\varphi'\|_{L_a^2}^2 + \|\varphi\|_{H^2}^2 \\ &\leq (2+r) \frac{n}{1-r} \|\varphi\|_{H^2}^2 + \|\varphi\|_{H^2}^2 \\ &\leq \frac{4n}{1-r} \|\varphi\|_{H^2}^2, \end{aligned}$$

which gives

$$\|\varphi\|_{H^2} \leq 2 \sqrt{\frac{n}{1-\rho}} \|\varphi\|_{L_a^2}.$$

In particular, with $\varphi = f_{\frac{1}{2}}$ we get $\varphi \in \mathcal{R}_{n, \frac{1}{2}r}$ and

$$(3.7) \quad \|f_{\frac{1}{2}}\|_{H^2} \leq 2 \sqrt{\frac{n}{1-r/2}} \|f_{\frac{1}{2}}\|_{L_a^2} \leq 2\sqrt{2n} \|f_{\frac{1}{2}}\|_{L_a^2}.$$

We conclude from (3.5), (3.6) and (3.7) that for any $\xi \in \mathbb{D}$,

$$|f(\xi)| \leq \|f_{\frac{1}{2}}\|_{H^2} \left(\frac{1}{4} \frac{2^{2n+1}}{(1-r)^{2n+1}} \right)^{\frac{1}{2}} \leq \frac{1}{2} \left(\frac{2}{1-r} \right)^{n+\frac{1}{2}} 2\sqrt{2n} \|f_{\frac{1}{2}}\|_{L_a^2},$$

that is,

$$|f(\xi)| \leq \sqrt{2n} \left(\frac{2}{1-r} \right)^{n+\frac{1}{2}} \|f\|_{L_a^2(\frac{1}{2}\mathbb{D})}, \quad \xi \in \mathbb{D}.$$

Taking the supremum over $\xi \in \mathbb{D}$ and $f \in \mathcal{R}_{n,r}$ we obtain (3.3).

Combining (3.2) and (3.3) and choosing $d = 48$, $b = \frac{1}{2}$ and $c > 2$ such that $2^n \sqrt{n} \leq c^n$ for any $n \geq 1$, we complete the proof and obtain (3.1). \square

Proof of Theorem 1.5. Take $r \in (0, 1)$ and $R \in (0, r)$ and let us represent the norm $\|f'\|_{L_a^p(w)}$ of a function $f \in \mathcal{R}_{n,r}$ as $I_1 + I_2$,

$$I_1 = \int_0^R \|(f_\rho)'\|_p^p w(\rho) d\rho, \quad I_2 = \int_R^1 \|(f_\rho)'\|_p^p w(\rho) d\rho.$$

Here and everywhere below in this proof, C_i , $i = 1, \dots, 5$, are positive constants, depending, may be, only on p and w (but not on n and r). By (1.2), we have for the first integral

$$I_1 \leq C_1 \left(\frac{n}{1-R} \right)^p \int_0^R \|f_\rho\|_p^p w(\rho) d\rho \leq C_2 \left(\frac{n}{1-R} \right)^p \|f\|_{L_a^p(w)}^p.$$

Note that $f_\rho \in \mathcal{R}_{n,\rho r} \subset \mathcal{R}_{n,r}$, and, thus, $\|f_\rho\|_\infty \leq M_{p, \frac{2}{3}}(n, r) \|f_\rho\|_{L_a^p(\frac{2}{3}\mathbb{D})}$. Applying (1.2) once again together with an obvious inequality $\|f_\rho\|_p \leq \|f_\rho\|_\infty$, we get

$$\begin{aligned} I_2 &\leq C_3 \left(\frac{n}{1-r} \right)^p \int_R^1 \|f_\rho\|_\infty^p w(\rho) d\rho \\ &\leq C_3 \|f\|_{L_a^p(\frac{2}{3}\mathbb{D})}^p \left(\frac{n}{1-r} \right)^p \int_R^1 M_{p, 2/3}^p(n, r) w(\rho) d\rho \\ &\leq C_3 \|f\|_{L_a^p(\frac{2}{3}\mathbb{D})}^p \left(\frac{n}{1-r} \right)^p \frac{c^{pn}}{(1-r)^{pn+pb}} w(R), \end{aligned}$$

where the last inequality follows from Lemma 3.1. Note that

$$\|f\|_{L_a^p(\frac{2}{3}\mathbb{D})}^p \leq (w(2/3))^{-1} \|f\|_{L_a^p(w)}^p.$$

Hence,

$$I_2 \leq C_4 \left(\frac{n}{1-r} \right)^p \frac{c^{pn}}{(1-r)^{pn+pb}} w(R) \|f\|_{L_a^p(w)}^p.$$

Now, choose a positive increasing sequence $(\gamma_n)_{n \in \mathbb{N}}$ such that $n = o(\gamma_n)$, as $n \rightarrow +\infty$. For any n we fix r_n° such that the function $w(r)(1-r)^{-\gamma_n}$ decreases on $[r_n^\circ, 1)$. Now for a fixed n take r, R so that $r_n^\circ < R < r < 1$ and

$$1-R = (1-r)^{1/2}, \quad 1-r_n^\circ = (1-r)^{1/4}.$$

We have

$$w(R) \leq w(r_n^\circ) \frac{(1-R)^{\gamma_n}}{(1-r_n^\circ)^{\gamma_n}} = w(r_n^\circ) (1-r)^{\gamma_n/4}.$$

Hence, using the fact that w is bounded on $[r_1^\circ, 1)$, we obtain

$$I_2 \leq C_4 \left(\frac{n}{1-r} \right)^p \|f\|_{L_a^p(w)}^p \cdot c^{pn} \frac{(1-r)^{\gamma_n/4}}{(1-r)^{pn+pb}}.$$

Let us show that for sufficiently large n ,

$$c^{pn} \frac{(1-r)^{\gamma_n/4}}{(1-r)^{pn+pb}} \rightarrow 0, \quad r \rightarrow 1-.$$

Indeed, choosing r so that $c < (1-r)^{-1}$, we get

$$c^{pn} \frac{(1-r)^{\gamma_n/4}}{(1-r)^{pn+pb}} \leq (1-r)^{\frac{\gamma_n}{4} - 2pn - pb} \rightarrow 0, \quad r \rightarrow 1-,$$

since $n = o(\gamma_n)$, $n \rightarrow \infty$. Hence, there exists a sequence (r_n) , $r_n \rightarrow 1-$, such that

$$\frac{I_2^{1/p}}{n \|f\|_{L_a^p(w)}} = o\left(\frac{1}{1-r_n} \right), \quad n \rightarrow \infty.$$

The corresponding estimate for I_1 is obvious since $1-R_n = (1-r_n)^{1/2}$. \square

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