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SHORT COMMUNICATIONS

Attractors of Foliations with Transversal Parabolic Geometry of Rank One

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The aim of the paper is to present a unified method for studying foliations with transversal parabolic geometry of rank one. We develop Frances' ideas [1] and the ideas that I applied to conformal foliations [2].

1. PARABOLIC GEOMETRIES OF RANK ONE

Let G be a simple Lie group of real rank one with finite center, and let P be a parabolic subgroup. We use the notation from [1]. Let $\mathbf{X} = G/P$. Such homogeneous spaces $\mathbf{X}$ are the boundaries $\mathbf{X} = \partial\mathbf{H}^d_{\mathbb{K}}$ of various d-dimensional hyperbolic spaces $\mathbf{H}^d_{\mathbb{K}}$. Here $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or the octonion algebra $\mathbb{O}$. We assume that $d \geq 2$ if $\mathbb{K} = \mathbb{R}$ and $d = 2$ if $\mathbb{K} = \mathbb{O}$; in all other cases, $d \geq 1$.

Thus, $\mathbf{X} = G/P = \partial\mathbf{H}^d_{\mathbb{K}}$, where $G = \mathrm{Iso}(\mathbf{H}^d_{\mathbb{K}})$ and P is the stationary subgroup of G at some point of $\partial\mathbf{H}^d_{\mathbb{K}}$. There is a well-known complete list of these groups and the corresponding homogeneous spaces:

- $G = SO(1,d)$ and $d \geq 2$ for $\mathbb{K} = \mathbb{R}$, and moreover $\mathbf{X} = \partial\mathbf{H}^d_{\mathbb{R}}$ is the sphere $\mathbf{S}^{d-1}$.
- $G = SU(1,d)$ and $d \geq 1$ for $\mathbb{K} = \mathbb{C}$, and moreover $\mathbf{X} = \partial\mathbf{H}^d_{\mathbb{C}}$ is the sphere $\mathbf{S}^{2d-1}$.
- $G = \mathrm{Sp}(1,d)$ and $d \geq 1$ for $\mathbb{K} = \mathbb{H}$, and moreover $\mathbf{X} = \partial\mathbf{H}^d_{\mathbb{H}}$ is the sphere $\mathbf{S}^{4d-1}$.
- $G = F_4^{-20}$ for $\mathbb{K} = \mathbb{O}$, and moreover $\mathbf{X} = \partial\mathbf{H}^d_{\mathbb{K}}$ is the sphere $\mathbf{S}^{15}$.

Definitions and facts concerning Cartan and parabolic geometries can be found, e.g., in the book [3]. Let G be a Lie group, let P be a closed subgroup, and let $\mathfrak{g}$ and $\mathfrak{p}$ be their Lie algebras. Let $B(N,P)$ be a P-bundle with projection $p\colon B \to N$. A nondegenerate P-equivariant $\mathfrak{g}$-valued 1-form ω on B is called a *Cartan connection*. The pair $\xi = (B(N,P),\omega)$ is called a *Cartan geometry* of type (G,P), and (N,ξ) is called a *Cartan manifold*.

A Cartan geometry $\xi = (B(N,P),\omega)$ of type (G,P), where G is a simple Lie group of real rank one with finite center and P is a parabolic subgroup is called a *parabolic geometry of rank one*, or a P_1-*geometry* for short. A parabolic geometry is said to be *regular* if its curvature is consistent with the grading of the Lie algebra $\mathfrak{g}$ in the sense of [1]. The Cartan geometry $(G(\mathbf{X},P),\omega_G)$, where ω_G is the Maurer–Cartan form on G, is denoted by ξ^0.

Let $\xi = (B(N,P),\omega)$ and $\xi' = (B'(N',P),\omega')$ be Cartan geometries of the same type (G,P). An *isomorphism* of ξ and ξ' is a diffeomorphism $\widehat{f}\colon B \to B'$ such that $f^*\omega' = \omega$ and $\widehat{f} \circ R_a = R_a \circ \widehat{f}$ for all $a \in P$, where R_a is the right action of an element $a \in P$ on B and B', respectively. The projection $f\colon N \to N'$ of $\widehat{f}$ is called an *isomorphism of the Cartan manifolds* (N,ξ) and (N',ξ'). Let $\mathrm{Aut}(N,\xi)$ be the automorphism group of (N,ξ).

Throughout the following, $\xi = (B(N,P),\omega)$ is a regular parabolic geometry of rank one.

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2. ESSENTIAL LOCAL ISOMORPHISMS

Let $\gamma\colon U \to V$ be a local isomorphism of a parabolic manifold (N, ξ) such that $\gamma(v) = v \in U \cap V$, and let $\widehat{\gamma}$ be (some) local isomorphism of the geometry ξ with projection γ. Fix a point $\widehat{v} \in p^{-1}(v)$. There exists a unique element $a \in P$ satisfying $\widehat{\gamma}(\widehat{v}) = \widehat{v}a$. Consider the subgroup $\Gamma_{\widehat{v}} := \langle a \rangle$ of P generated by a. One can readily see that the following is meaningful.

Definition 1. If a local isomorphism γ of a geometry ξ has a fixed point v such that the induced group $\Gamma_{\widehat{v}}$, where $p(\widehat{v}) = v$, is not a relatively compact subgroup of the Lie group P, the isomorphism γ is said to be *essential* (at the point v). Otherwise, γ is said to be *inessential*.

By applying the developing mapping for the parabolic geometry ξ, we show that the technique used in [1] for the automorphism group $\mathrm{Aut}(N, \xi)$ may be applied to the pseudogroup $\mathcal{H}(N, \xi)$ of local isomorphisms of the geometry (N, ξ). In view of this, we obtain the following sufficient local condition for the Lichnerowicz conjecture to hold for an arbitrary parabolic geometry of rank one.

Theorem 1. *Let ξ be an arbitrary regular parabolic geometry of rank one on N. If there exists an essential local isomorphism of the parabolic manifold (N, ξ), then (N, ξ) is isomorphic to either the parabolic manifold $(\mathbf{X}, \xi^0)$ if N is compact or the parabolic manifold $(\mathbf{X} \setminus \{\nu\}, \xi^0|_{\mathbf{X}\setminus\{\nu\}})$, where $\nu \in \mathbf{X}$, otherwise.*

We prove the following criterion for the action of the automorphism group $\mathrm{Aut}(N, \xi)$ of the parabolic manifold (N, ξ) to be proper.

Theorem 2. *The full automorphism group $\mathrm{Aut}(N, \xi)$ acts properly on N if and only if it does not contain essential automorphisms.*

Theorems 1 and 2 imply Frances' main theorem in [1], which claims that if the automorphism group of a regular parabolic geometry (N, ξ) of rank one acts improperly on N, then the assertion of Theorem 1 holds.

Another criterion for the action on N of the automorphism group of a P_1-geometry ξ to be proper was obtained in [4] in terms of Weyl structures.

3. CRITERION FOR A P_1-FOLIATION TO BE RIEMANNIAN

The foliation (M, F) specified by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ is said to have *transversal geometry* ξ and is called a *P_1-foliation* if ξ is a regular parabolic geometry of rank one on N such that each γ_{ij} is a local isomorphism of (N, ξ).

Recall that the local isomorphism pseudogroup generated by the local isomorphisms γ_{ij}, $i, j \in J$, is called the *holonomy pseudogroup* of the foliation (M, F) and is denoted by $\mathcal{H}(M, F)$. Let L be an arbitrary leaf of (M, F); take an $x \in L \cap U_i$, $i \in J$. The group of germs of local diffeomorphisms in $\mathcal{H}(M, F)$ at the point $v = f_i(x)$ is called the (*germ*) *holonomy group* of the leaf L. The following definition is meaningful.

Definition 2. The holonomy group of a leaf L is said to be *essential* if it contains an essential local isomorphism germ at a point $v = f_i(x)$, where $x \in L \cap U_i$.

By applying Proposition 2 in my paper [2], we obtain the following criterion for the foliations in question to be Riemannian.

Theorem 3. *A P_1-foliation (M, F) is Riemannian if and only if it does not have leaves with essential holonomy group.*

4. EXISTENCE OF ATTRACTORS

Recall that an arbitrary union of fibers of a foliation (M, F) is called a *saturated subset* of M.

Definition 3. A nonempty closed saturated subset $\mathcal{M} \subset M$ is called an *attractor* of the foliation (M, F) if there exists a saturated open neighborhood $\mathrm{Attr}(\mathcal{M})$ of $\mathcal{M}$ such that the closure of every leaf in $\mathrm{Attr}(\mathcal{M}) \setminus \mathcal{M}$ contains $\mathcal{M}$. The set $\mathrm{Attr}(\mathcal{M})$ is called the *basin* of the attractor $\mathcal{M}$.

A *minimal set* of a foliation (M, F) is a nonempty closed saturated subset of M that does not have proper subsets with the same properties. In contrast to the case of compact manifolds, there exist foliations without minimal sets on noncompact manifolds. An application of Theorems 1 and 3 permits us to prove the following theorem without assuming that the foliated manifold is compact.

Theorem 4. *One of the following two claims holds for each P_1-foliation (M, F):*

(1) *The foliation (M, F) is Riemannian.*

(2) *There exists a leaf with essential holonomy group. The closure $\mathcal{M} = \overline{L}$ of every leaf L with essential holonomy group is an attractor and a minimal set, and the restriction of the foliation to the basin $\mathrm{Attr}(\mathcal{M})$ of the attractor $\mathcal{M}$ is a $(G, \mathbf{X})$-foliation. The union $\mathcal{K}$ of closures of all leaves with essential holonomy group is a closed subset of M, and the restriction (M_0, F_{M_0}) of this foliation to the open set $M_0 := M \setminus \mathcal{K}$ is a Riemannian foliation.*

For the case in which (M, F) is a conformal foliation of codimension $q \geq 3$, a similar result was obtained by the author in [2].

5. STRUCTURE OF P_1-FOLIATIONS ON COMPACT MANIFOLDS

Using Theorem 4 and my argument in [2] for conformal foliations, we prove the following theorem.

Theorem 5. *If (M, F) is a P_1-foliation on a compact manifold M, then one of the following two assertions holds*:

(1) *The foliation (M, F) is a complete Riemannian foliation.*

(2) *The foliation (M, F) is transversally homogeneous and has finitely many* (*at least one*) *minimal sets $\mathcal{M}_j$, $j = 1, \dots, m$, which are attractors, and moreover, every leaf belongs to the basin of at least one of these attractors.*

Theorem 5 can be viewed as a confirmation of the *analog of Lichnerowicz's conjecture* for P_1-foliations.

For conformal foliations of codimension $q \geq 3$, this result was obtained by the author in [2], and the transversal homogeneity was proved by Tarquini [5] under the additional assumption that the conformal foliation is transversally analytic.

Corollary. *Let (M, F) be a proper non-Riemannian P_1-foliation on a compact manifold M. Then (M, F) is a $(G, \mathbf{X})$-foliation with finitely many* (*at least one*) *compact leaves L_j, $j = 1, \dots, m \geq 1$. Moreover, each L_j is an attractor, and every leaf of this foliation belongs to the basin of at least one of these attractors.*

Examples of conformal foliations with various attractors whose minimal sets were constructed by the author in [6].

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